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INEQUALITY IN TRIANGLE 881
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1. In ΔABC:

$$\frac{bc}{al_a}+\frac{ca}{bl_b}+\frac{ab}{cl_c}\le\frac{9R^2}{2S}$$

Proposed by Mehmet Şahin - Ankara - Turkey

Proof.

Using $l_a=\frac{2bc}{b+c}\cos\frac{A}{2}$ *we obtain:*

$$\frac{1}{al_a}=\frac{1}{a\cdot\frac{2bc}{b+c}\cos\frac{A}{2}}=\frac{b+c}{2abc\cdot\cos\frac{A}{2}}=\frac{2R(\sin B+\sin C)}{2abc\cdot\cos\frac{A}{2}}=\frac{R\cdot 2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{4RS\cdot\frac{A}{2}}=$$

$$=\frac{R\cdot 2\cos\frac{A}{2}\cos\frac{B-C}{2}}{4RS\cdot\cos\frac{A}{2}}=\frac{\cos\frac{B-C}{2}}{2S},\ \textit{wherefrom}\ \frac{bc}{al_a}=\frac{bc\cdot\cos\frac{B-C}{2}}{2S}.$$

Because $\cos\frac{B-C}{2}\le 1$ *and* $\sum bc\le\sum a^2\le 9R^2$ *(Leibniz's inequality), it follows:*

$$\sum\frac{bc}{al_a}=\sum\frac{bc\cdot\cos\frac{B-C}{2}}{2S}\le\sum\frac{bc}{2S}\le\frac{9R^2}{2S}.$$

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's emphasises an inequality having an opposite sense.

2) In ΔABC:

$$\frac{bc}{al_a}+\frac{ca}{bl_b}+\frac{ab}{cl_c}\ge\frac{18r}{s}$$

Proposed by Marin Chirciu - Romania

Proof.

Using $\frac{bc}{al_a}=\frac{bc\cdot\cos\frac{B-C}{2}}{2S}$ *we obtain*

$$\sum\frac{bc}{al_a}=\sum\frac{bc\cdot\cos\frac{B-C}{2}}{2S}=\frac{1}{2S}\sum bc\cdot\cos\frac{B-C}{2}\tag{1}$$

With means inequality and $abc=4RS,\prod\cos\frac{B-C}{2}=\frac{s^2+r^2+2Rr}{8R^2}$ *we obtain:*

$$\sum bc\cdot\cos\frac{B-C}{2}\ge 3\sqrt[3]{\prod bc\cdot\cos\frac{B-C}{2}}=3\sqrt[3]{(abc)^2\prod\cos\frac{B-C}{2}}=$$

(2)

$$= 3\sqrt[3]{(4RS)^2 \cdot \frac{s^2+r^2+2Rr}{8R^2}} = 3\sqrt[3]{2s^2r^2(s^2+r^2+2Rr)} \geq 3\sqrt[3]{(12r^2)^3} = 3\cdot 12r^2 = 36r^2$$

We've used above $s^2 \geq 16Rr - 5r^2$ (Gerretsen) $s \geq 3r\sqrt{3}$ (Mitrinovic) and $R \geq 2r$ (Euler). From (1) and (2) it follows the conclusion. Equality holds if and only if the triangle is equilateral.

□

Remark.

We can write the double inequality:

3) In ΔABC:

$$\frac{18r}{s} \leq \frac{bc}{al_a} + \frac{ca}{bl_b} + \frac{ab}{cl_c} \leq \frac{9R^2}{2S}$$

Proof.

*See inequalities **1)** and **2)**.*

Equality holds if and only if the triangle is equilateral.

Remark.

The double inequality can be strengthened:

4) In ΔABC:

$$2\sqrt{3} \leq \frac{bc}{al_a} + \frac{ca}{bl_b} + \frac{ab}{cl_c} \leq \frac{2(R+r)^2}{S}$$

Proposed by Marin Chirciu - Romania

Proof.

Inequality from the left side: $\frac{bc}{al_a} + \frac{ca}{bl_b} + \frac{ab}{cl_c} \geq 2\sqrt{3}$ *it follows from:*

(1) *The proof of **2)** implies* $\sum \frac{bc}{al_a} \geq \frac{3}{2S}\sqrt[3]{2s^2r^2(s^2+r^2+2Rr)}$

(2) *Then* $\frac{3}{2S}\sqrt[3]{2s^2r^2(s^2+r^2+2Rr)} \geq 2\sqrt{3}$

$$\Leftrightarrow 3\sqrt[3]{2s^2r^2(s^2+r^2+2Rr)} \geq 4rs\sqrt{3} \Leftrightarrow$$

$$\Leftrightarrow 27 \cdot 2s^2r^2(s^2+r^2+2Rr) \geq 64r^3s^3 \cdot 3\sqrt{3} \Leftrightarrow 9(s^2+r^2+2Rr) \geq 32rs\sqrt{3}$$

which follows from Doucet's inequality $4R + r \geq s\sqrt{3}$. It remains to prove that:

$$9(s^2+r^2+2Rr) \geq 32r(4R+r) \Leftrightarrow 9s^2 \geq 110Rr + 23r^2,$$

true from Gerretsen's inequality: $s^2 \geq 16Rr - 5r^2$ and Euler's inequality $R \geq 2r$.

It suffices to prove that:

$$9(16Rr - 5r^2) \geq 110Rr + 23r^2 \Leftrightarrow R \geq 2r.$$

From (1) and (2) we obtain $\frac{bc}{al_a} + \frac{ca}{bl_b} + \frac{ab}{cl_c} \geq 2\sqrt{3}$.

Inequality from the right side: $\frac{bc}{al_a} + \frac{ca}{bl_b} + \frac{ab}{cl_c} \leq \frac{2(R+r)^2}{S}$.

*The proof of **1)** implies:*

(1) $$\sum \frac{bc}{al_a} = \sum \frac{bc \cdot \cos \frac{B-C}{2}}{2S} \leq \sum \frac{bc}{2S}$$

With identity $\sum bc = s^2+r^2+4Rr$ *and Gerretsen's inequality* $s^2 \leq 4R^2+4Rr+3r^2$ *we have:*

(2)
$$\sum bc = s^2+r^2+4Rr \leq 4R^2+4Rr+3r^2+r^2+4Rr = 4R^2+8Rr+4r^2 = 4(R+r)^2$$

From (1) *and* (2) *it follows* $\frac{bc}{al_a} + \frac{ca}{bl_b} + \frac{ab}{cl_c} \leq \frac{1}{2S} \cdot 4(R+r)^2 = \frac{2(R+r)^2}{S}$.

□

Equality holds if and only if the triangle is equilateral. □

Remark.

The double inequality ***4)*** *is stronger than* ***3)****.*

5) In ΔABC:

$$\frac{18r}{s} \leq 2\sqrt{3} \leq \frac{bc}{al_a} + \frac{ca}{bl_b} + \frac{ab}{cl_c} \leq \frac{2(R+r)^2}{S} \leq \frac{9R^2}{2S}$$

Proposed by Mehmet Şahin - Turkey, Marin Chirciu - Romania

Proof.

See ***4)****, Euler's inequality* $R \geq 2r$ *and Mitrinovic's inequality* $s \geq 3r\sqrt{3}$.

Equality holds if and only if the triangle is equilateral.

□

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