APPLICATIONS OF TSINTSIFAS' INEQUALITY

D.M. BĂTINEŢU-GIURGIU, DANIEL SITARU, NECULAI STANCIU

Abstract. In this article are created a few applications of Tsintsifas' inequality published in American Mathematical Monthly in 1989.

Tsintsifas inequality:
If $x, y, z > 0$ then in any triangle ABC:

$$\frac{xa^2}{y+z} + \frac{yb^2}{z+x} + \frac{zc^2}{x+y} \geq 2\sqrt{3} \cdot [ABC]$$

Equality holds for $a = b = c; x = y = z$.

Proof. We use Doucet's inequality: $4R + r \geq \sqrt{3}s$ and well known:

(1) $$a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr)$$

$$\sum \frac{xa^2}{y+z} + 2(s^2 - r^2 - 4Rr) =$$

$$= \sum \frac{xa^2}{y+z} + \sum a^2 = (x+y+z) \sum \frac{a^2}{y+z} \geq$$

$$\overbrace{\geq}^{BERGSTROM} (x+y+z) \cdot \frac{(a+b+c)^2}{\sum(y+z)} =$$

$$= (x+y+z) \cdot \frac{(2s)^2}{2(x+y+z)} = 2s^2$$

$$\sum \frac{xa^2}{y+z} \geq 2s^2 - 2(s^2 - r^2 - 4Rr) =$$

$$= 2(4R+r)r \overbrace{\geq}^{DOUCET} 2s\sqrt{3}r =$$

$$= 2\sqrt{3}sr = 2\sqrt{3}[ABC]$$

□

APPLICATION 1
If ΔABC its an equilateral one and $x, y, z > 0$ then $a = b = c; [ABC] = \frac{a^2\sqrt{3}}{4}$.
By Tsintsifas' inequality:

$$\left(\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}\right)a^2 \geq 2\sqrt{3}[ABC] =$$

$$= 2\sqrt{3} \cdot \frac{a^2\sqrt{3}}{4} = \frac{3a^2}{2} \Rightarrow$$

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geq \frac{3}{2}$$

which is Nesbitt - Ionescu inequality.
Equality holds for $x = y = z$.

APPLICATION 2

If $x = y = z > 0$ then by Tsintsifas' inequality:

$$\frac{1}{2}(a^2 + b^2 + c^2) \geq 2\sqrt{3}[ABC]$$

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}[ABC]$$

which is Ionescu - Weitzenbock's inequality.
Equality holds for $a = b = c$.

APPLICATION 3

If $x, y, z > 0; a = \sqrt{3}; b = \sqrt{2}; c = 1$ then by Tsintsifas' inequality:

$$\frac{3x}{y+z} + \frac{2y}{z+x} + \frac{1}{x+y} > \sqrt{6}$$

Proof.

$$b^2 + c^2 = 2 + 1 = 3 = a^2;$$

$$[ABC] = \frac{1}{2}AB \cdot AC = \frac{1}{2}bc = \frac{1}{2} \cdot \sqrt{2} \cdot 1 = \frac{\sqrt{2}}{2}$$

$$\frac{3x}{y+z} + \frac{2y}{z+x} + \frac{1}{x+y} > 2 \cdot \sqrt{3} \cdot \frac{\sqrt{2}}{2} = \sqrt{6}$$

□

APPLICATION 4

If $(F_n)_{n\geq 0}; F_0 = 0; F_1 = 1; F_{n+2} = F_{n+1} + F_n; n \in \mathbb{N}$ (Fibonacci's sequence) then:

$$\frac{3F_n^2}{F_{2n+3}} + \frac{2F_{n+1}^2}{F_n^2 + F_{n+2}^2} + \frac{F_{n+2}^2}{F_{2n+1}} > \sqrt{6}; n \in \mathbb{N}^*$$

Proof. In Tsintsifas' inequality we take:

$$x = F_n^2; y = F_{n+1}^2; z = F_{n+2}^2; n \in \mathbb{N}^*$$

and use: $F_n^2 + F_{n+1}^2 = F_{2n+1}; F_{n+1}^2 + F_{n+2}^2 = F_{2n+3}, n \in \mathbb{N}^*$ □

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Mathematics Department, Colegiul National Economic Theodor Costescu, Drobeta Turnu - Severin, Mehedinti, Romania

E-mail address: dansitaru63@yahoo.com