R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

SP.169. ***Prove that for all non-negative real numbers a, b, c***

$$\sqrt{\frac{a^2+2}{b+c+1}}+\sqrt{\frac{b^2+2}{c+a+1}}+\sqrt{\frac{c^2+2}{a+b+1}}\geq 3$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solutions 1,2 by Tran Hong-Vietnam, Solution 3 by Remus Florin Stanca-Romania, Solution 4 by Soumava Chakraborty-Kolkata-India, Solution 5 by Sanong Huayrerai-Nakon Pathom-Thailand, Solution 6 by Soumitra Mandal-Chandar Nagore-India

Solution 1 by Tran Hong-Vietnam

Suppose: $a+b+c=3\Rightarrow 0<a,b,c<3$

Inequality $\Leftrightarrow \sqrt{\frac{a^2+2}{4-a}}+\sqrt{\frac{b^2+2}{4-b}}+\sqrt{\frac{c^2+2}{4-c}}\geq 3.$ ***(1)***

For all $0<x<3$ ***we have:***

$$\sqrt{\frac{x^2+2}{4-x}}\geq\frac{1}{2}(x+1)\quad (*)$$

$$\Leftrightarrow \frac{x^2+2}{4-x}\geq\frac{1}{2}(x+1)^2$$

$\Leftrightarrow \frac{(x-1)^2(x+4)}{4-x}\geq 0$ ***. (True because*** $0<x<3$***)***

Using (*) with $0<a,b,c<3$ ***we have***

$$\sqrt{\frac{a^2+2}{4-a}}+\sqrt{\frac{b^2+2}{4-b}}+\sqrt{\frac{c^2+2}{4-c}}\geq\frac{1}{2}(a+1)+\frac{1}{2}(b+1)+\frac{1}{2}(c+1)$$

$$=\tfrac{1}{2}(a+b+c+3)=\tfrac{6}{2}=3.\ \textit{Proved.}$$

Equality $\Leftrightarrow a=b=c=1.$

Solution 2 by Tran Hong-Vietnam

$$LHS=\frac{\sqrt{(a^2+2)(2+1)}}{\sqrt{(b+c+1)(2+1)}}+\sqrt{\frac{(b^2+2)(2+1)}{(c+a+1)(2+1)}}+\sqrt{\frac{(c^2+2)(2+1)}{(a+b+1)(2+1)}}$$

$$\geq\frac{\sqrt{(a+2)^2}}{\frac{b+c+1+3}{2}}+\frac{\sqrt{(b+2)^2}}{\frac{c+a+1+3}{2}}+\frac{\sqrt{(c+2)^2}}{\frac{a+b+1+3}{2}}$$

$$=\frac{2a+4}{b+c+4}+\frac{2b+4}{c+a+4}+\frac{2c+4}{a+b+4}$$

$$=\frac{2a+4}{b+c+4}+2+\frac{2b+4}{c+a+4}+2+\frac{2c+4}{a+b+4}+2+6$$

$$=2(a+b+c+6)\left(\frac{1^2}{b+c+4}+\frac{1^2}{c+a+4}+\frac{1^2}{a+b+4}\right)-6$$

$$\geq 2(a+b+c+6)\cdot\frac{(1+1+1)^2}{2(a+b+c+6)}-6=9-6=3$$

Proved. Equality $\Leftrightarrow a=b=c.$

Solution 3 by Remus Florin Stanca-Romania

We know that for any real numbers $x,y,z>0$ ***we have that*** $\sqrt{\frac{x^2+y^2+z^2}{3}}\geq$

$$\geq\frac{x+y+z}{3}\Leftrightarrow\sqrt{\frac{a^2+1+1}{3}}\geq\frac{a+2}{3}\Rightarrow\sqrt{a^2+2}\geq\frac{a+2}{\sqrt{3}}\Rightarrow$$

$$\succ\sqrt{\frac{a^2+2}{b+c+1}}\geq\frac{a+2}{\sqrt{3(b+c+1)}}$$

$$\sqrt{\frac{b^2+2}{a+c+1}}\geq\frac{b+2}{\sqrt{3(a+c+1)}}$$

$$\sqrt{\frac{c^2+2}{a+b+1}} \ge \frac{c+2}{\sqrt{3(a+b+1)}}$$

------------------------------------- +

$$\sqrt{\frac{a^2+2}{b+c+1}} + \sqrt{\frac{b^2+2}{a+c+1}} + \sqrt{\frac{c^2+2}{a+b+1}} \ge \sum \frac{a+2}{\sqrt{3(b+c+1)}} \quad \textit{(1)}$$

$$\sqrt{3(b+c+1)} \le \frac{b+c+4}{2} \succ \frac{a+2}{\sqrt{3(b+c+1)}} \ge \frac{2a+4}{b+c+4} \succ$$

$\Rightarrow \sum \frac{a+2}{\sqrt{3(b+c+1)}} \ge 2\sum \frac{a+2}{b+2+c+2}$ ***(2), we know also, that***

$\sum \frac{x}{y+z} \ge \frac{3}{2}$, ***we put*** $x = a+2, y = b+2, z = c+2 \succ 2\sum \frac{a+2}{b+2+c+2} \ge 3$ ***(3)***

$$(1)\overset{\succ}{(2)}(3)\sqrt{\frac{a^2+2}{b+c+1}} + \sqrt{\frac{b^2+2}{a+c+1}} + \sqrt{\frac{c^2+2}{a+b+1}} \ge 3. \quad \textit{(Q.E.D.)}$$

Solution 4 by Soumava Chakraborty-Kolkata-India

$\because a, b, c \ge 0, \frac{a^2+2}{b+c+1}$, ***etc*** > 0

$\therefore$ ***by A-G, LHS*** $\ge 3\sqrt[3]{\sqrt{\frac{(a^2+2)(b^2+2)(c^2+2)}{(b+c+1)(c+a+1)(a+b+1)}}} \overset{?}{\ge} 3$

$$\Leftrightarrow (a^2+2)(b^2+2)(c^2+2) \overset{?}{\ge} (b+c+1)(c+a+1)(a+b+1)$$

$$\Leftrightarrow a^2b^2c^2 + 2\sum a^2b^2 + 3\sum a^2 + 7 \underset{(1)}{\overset{?}{\ge}} \sum a^2 b + \sum ab^2 + 2abc + 3\sum ab + 2\sum a$$

Now, $\frac{1}{2}a^2(b-1)^2 \ge 0 \Rightarrow \frac{1}{2}(a^2b^2 + a^2) \overset{(i)}{\ge} a^2b$

$$\frac{1}{2}b^2(c-1)^2 \ge 0 \Rightarrow \frac{1}{2}(b^2c^2 + b^2) \overset{(ii)}{\ge} b^2c$$

$$\frac{1}{2}c^2(a-1)^2 \ge 0 \Rightarrow \frac{1}{2}(c^2a^2 + c^2) \overset{(iii)}{\ge} c^2a$$

$$\frac{1}{2}b^2(a-1)^2 \ge 0 \Rightarrow \frac{1}{2}(a^2b^2 + b^2) \overset{(iv)}{\ge} ab^2$$

$$\frac{1}{2}c^2(b-1)^2 \ge 0 \Rightarrow \frac{1}{2}(b^2c^2 + c^2) \overset{(v)}{\ge} bc^2$$

$$\frac{1}{2}a^2(c-1)^2 \ge 0 \Rightarrow \frac{1}{2}(c^2a^2 + a^2) \overset{(vi)}{\ge} ca^2$$

Also, $\because \frac{1}{2}[\sum(a-b)^2] \ge 0, \therefore \sum a^2 \overset{(vii)}{\ge} \sum ab$

$$\because \sum(a-1)^2 \ge 0, \therefore \sum a^2 + 3 \overset{(viii)}{\ge} 2\sum a$$

$$\because (abc-1)^2 \ge 0, \therefore a^2b^2c^2 + 1 \overset{(ix)}{\ge} 2abc$$

$$\because \sum(ab-1)^2 \ge 0, \therefore \sum a^2b^2 + 3 \overset{(x)}{\ge} 2\sum ab$$

(i)+(ii)+(iii)+(iv)+(v)+**$(vi) + (vii) + (viii) + (ix) + (x) \Rightarrow$ **(1) is true (proved)

Solution 5 by Sanong Huayrerai-Nakon Pathom-Thailand

For** $a, b, c \ge 0$, **we have

$$(a^2+1+1)(b^2+1+1) \ge (a+b+1)^2$$

$$(b^2+1+1)(c^2+1+1) \ge (b+c+1)^2$$

$$(c^2+1+1)(a^2+1+1) \ge (c+a+1)^2$$

$$\Rightarrow (a^2+2)(b^2+2)(c^2+2) \ge (a+b+1)(b+c+1)(c+a+1)$$

Hence $\sqrt{\frac{a^2+2}{b+c+1}} + \sqrt{\frac{b^2+2}{c+a+1}} + \sqrt{\frac{c^2+2}{a+b+1}} \ge 3\sqrt[3]{\sqrt[2]{\left(\frac{a^2+2}{b+c+1}\right)\left(\frac{b^2+2}{c+a+1}\right)\left(\frac{c^2+2}{a+b+1}\right)}} \ge 3$

Iff $\sqrt[6]{\left(\frac{a^2+2}{b+c+1}\right)\left(\frac{b^2+2}{c+a+1}\right)\left(\frac{c^2+2}{a+b+1}\right)} \ge 1$ ***and it is to be true.***

Therefore it is to be true

Solution 6 by Soumitra Mandal-Chandar Nagore-India

By Cauchy-Schwarz inequality,

$$(a^2+1+1)(b^2+1+1) \ge \left(\sqrt{a^2\cdot 1} + \sqrt{b^2\cdot 1} + \sqrt{1\cdot 1}\right)^2 = (a+b+1)^2$$

Similarly, $(b^2+2)(c^2+2) \ge (b+c+1)^2$ ***and*** $(c^2+2)(a^2+2) \ge (c+a+1)^2$

Multiplying the above we have $\prod_{cyc}(a^2+2) \ge \prod_{cyc}(a+b+1)$

$$\sum_{cyc}\sqrt{\frac{a^2+2}{b+c+1}} \overset{AM\ge GM}{\ge} 3\sqrt[3]{\prod_{cyc}\sqrt{\frac{a^2+2}{b+c+1}}} = 3$$