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JP.177. If $a, b, c \geq 0$ then:

$$2(a+b+c)+\sum_{cyc}\sqrt{a^2+b^2-ab}\geq 3\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kelvin Hong-Rawang-Malaysia, Solution 2 by Amit Dutta-Jamshedpur-India, Solution 3 by Boris Colakovic-Belgrade-Serbia, Solution 4 by Michael Sterghiou-Greece, Solution 5 by Ravi Prakash-New Delhi-India, Solution 6 by Seyran Ibrahimov-Maasilli-Azerbaijan, Solution 7 by Tran Hong-Vietnam, Solution 8 by Soumava Chakraborty-Kolkata-India, Solution 9 by Sanong Huayrerai-Nakon Pathom-Thailand

Solution 1 by Kelvin Hong-Rawang-Malaysia

We have:

$$(a+b+c)(b+c+a)\overset{Cauchy-Schwarz\ Inequality}{\geq}\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)^2$$

$$\therefore a+b+c\geq\sqrt{ab}+\sqrt{bc}+\sqrt{ca}$$

Also, that

$$\sum_{cyc}\sqrt{a^2+b^2-ab}\overset{AM-GM}{\geq}\sum_{cyc}\sqrt{2ab-ab}=\sum_{cyc}\sqrt{ab}$$

Therefore

$$2(a+b+c)+\sum_{cyc}\sqrt{a^2+b^2-ab}\geq 2\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)+\sum_{cyc}\sqrt{ab}=3\sum_{cyc}\sqrt{ab}$$

Solution 2 by Amit Dutta-Jamshedpur-India

$\because$ *We know that:*

$$(a^2+b^2-ab)=\frac{1}{4}(a+b)^2+\frac{3}{4}(a-b)^2$$

$$\Rightarrow\sqrt{a^2+b^2-ab}=\sqrt{\frac{1}{4}(a+b)^2+\frac{3}{4}(a-b)^2}\geq\left(\frac{a+b}{2}\right)$$

$$\Rightarrow \sqrt{a^2+b^2-ab} \ge \left(\frac{a+b}{2}\right)$$

$$\sum_{cyc} \sqrt{a^2+b^2-ab} \ge \sum_{cyc} \left(\frac{a+b}{2}\right) \ge \sum_{cyc} a$$

$$\Rightarrow \sum_{cyc}(a+b) + \sum_{cyc} \sqrt{a^2+b^2-ab} \ge \sum_{cyc}(a+b) + \sum_{cyc} a$$

$$\ge 3(a+b+c) \ge \frac{3}{2}\left(\sum 2a\right)$$

$$\ge \frac{3}{2}\{(a+b)+(b+c)+(c+a)\}$$

$$\overset{AM-GM}{\ge} \frac{3}{2}\left(2\sqrt{ab}+2\sqrt{bc}+2\sqrt{ac}\right)$$

$$2(a+b+c) + \sum_{cyc} \sqrt{a^2+b^2-ab} \ge 3\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ac}\right)$$

(proved)

Solution 3 by Boris Colakovic-Belgrade-Serbia

$$a^2+b^2-ab \ge ab \Leftrightarrow \sqrt{a^2+b^2-ab} \ge \sqrt{ab} \Leftrightarrow \sum_{cyc} \sqrt{a^2+b^2-ab} \ge \sqrt{ab}+\sqrt{bc}+\sqrt{ca} \quad \textbf{(1)}$$

$$2(a+b+c) = (a+b)+(b+c)+(c+a) \overset{AM-GM}{\ge} 2\sqrt{ab}+2\sqrt{bc}+2\sqrt{ca} \quad \textbf{(2)}$$

From (1) and (2) $\Rightarrow$ ***LHS*** $\ge 3\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)$

Solution 4 by Michael Sterghiou-Greece

$$2\sum_{cyc} a + \sum_{cyc} \sqrt{a^2+b^2-ab} \ge 3\sum_{cyc} \sqrt{ab} \quad \textbf{(1)}$$

LHS (1) $= 2\sum_{cyc} a + \sum_{cyc} \sqrt{2ab-ab} = 2\sum_{cyc} a + \sum_{cyc} \sqrt{ab}$

It suffices to show that: $\sum_{cyc} a \ge \sum_{cyc} \sqrt{ab}$ ***or*** $\sum_{cyc} \left(\sqrt{a}\right)^2 \ge \sum_{cyc} \sqrt{ab}$

which holds (rearrangement inequality).

Solution 5 by Ravi Prakash-New Delhi-India

$$a+b+\sqrt{a^2+b^2-ab} - 3\sqrt{ab} \ge a+b+\sqrt{2ab-ab} - 3\sqrt{ab} =$$

$$= a+b-2\sqrt{ab} = \left(\sqrt{a}-\sqrt{b}\right)^2 \ge 0$$

$$\Rightarrow a+b+\sqrt{a^2+b^2-ab} \ge 3\sqrt{ab} \quad \textbf{(1)}$$

Similarly, $b+c+\sqrt{b^2+c^2-bc}\ge 3\sqrt{bc}$ *(2)*

and $c+a+\sqrt{c^2+a^2-ca}\ge 3\sqrt{ca}$ *(3)*

Adding (1), (2), (3), we get:

$$2(a+b+c)+\sum_{cyc}\sqrt{a^2+b^2-ab}\ge 3\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)$$

Solution 6 by Seyran Ibrahimov-Maasilli-Azerbaijan

$$\sum_{cyc} a+b+\sum_{cyc}\sqrt{a^2+b^2-ab}\ge 3\sum_{cyc}\sqrt{ab}$$

$$a+b+\sqrt{a^2+b^2-ab}\ge 3\sqrt{ab}\Rightarrow \textit{(1)}$$

$$\Rightarrow\left(\sqrt{a}-\sqrt{b}\right)^2+\sqrt{a^2+b^2-ab}-\sqrt{ab}\ge 0\ \ (\forall a,b\ (a-b)^2\ge 0)$$

$$\overset{a^2+b^2\ge 2ab}{\Rightarrow}\left(\sqrt{a}-\sqrt{b}\right)^2+\sqrt{ab}-\sqrt{ab}=\left(\sqrt{a}-\sqrt{b}\right)^2\ge 0\quad (*)$$

$$\overset{(*)}{\Rightarrow} b+c+\sqrt{b^2+c^2-bc}\ge 3\sqrt{bc}\quad \textit{(2)}$$

$$\wedge\ a+c+\sqrt{a^2+c^2-ac}\ge 3\sqrt{ac}\quad \textit{(3)}$$

(1)+(2)+(3)$\Rightarrow$

$$2\sum_{cyc} a+\sum_{cyc}\sqrt{a^2+b^2-ab}\ge 3\sum_{cyc}\sqrt{ab}$$

(Proved)

Solution 7 by Tran Hong-Vietnam

Using Cauchy's inequality, we have: $a+b\ge 2\sqrt{ab};\ b+c\ge 2\sqrt{bc};\ c+a\ge 2\sqrt{ac}$

$$\to 2(a+b+c)\ge 2\left(\sqrt{ab}+\sqrt{ac}+\sqrt{bc}\right)\quad \textit{(1)}$$

$$\sqrt{a^2+b^2-ab}\ge\sqrt{2ab-ab}=\sqrt{ab}\quad \textit{(2)}$$

$$\sqrt{b^2+c^2-bc}\ge\sqrt{2bc-bc}=\sqrt{bc}\quad \textit{(3)}$$

$$\sqrt{a^2+c^2-ac}\ge\sqrt{2ac-ac}=\sqrt{ac}\quad \textit{(4)}$$

$\to$ *(1)+(2)+(3)+(4) we proved. Equality then* $a=b=c$*.*

Solution 8 by Soumava Chakraborty-Kolkata-India

$$2\sum a+\sum\sqrt{a^2+b^2-ab}\overset{(1)}{\ge} 3\left(\sum\sqrt{ab}\right)$$

$$\because a^2+b^2-ab=\frac{1}{4}(a+b)^2+\frac{3}{4}(a-b)^2\geq\frac{(a+b)^2}{4}$$

$$\therefore \sqrt{a^2+b^2-ab}\geq\frac{a+b}{2}(\because a+b\geq 0\ as\ a,b\geq 0)\ \textit{etc.}$$

$$\therefore \textit{LHS of (1)} \overset{(a)}{\geq} 2\sum a+\frac{1}{2}\sum(a+b)=3\sum a$$

$$\textit{Also, RHS of (1)} \overset{CBS}{\leq} 3\sqrt{\sum a}\sqrt{\sum a}=3\sum a$$

$$\overset{by\ (a)}{\leq} \textit{LHS of (1) (Proved)}$$

Solution 9 by Sanong Huayrerai-Nakon Pathom-Thailand

For $x,y\geq 0$*, we have* $x^2-xy+y^2\geq\left(\frac{x+y}{2}\right)^2$

Hence for $a,b,c\geq 0$*, we get*

$$2(a+b+c)+\sqrt{a^2-ab+b^2}+\sqrt{b^2-bc+c^2}+\sqrt{c^2-ca+a^2}$$

$$\geq 2(a+b+c)+\frac{a+b}{2}+\frac{b+c}{2}+\frac{c+a}{2}$$

$$=(a+b)+(b+c)+(c+a)+\frac{a+b}{2}+\frac{b+c}{2}+\frac{c+a}{2}$$

$$\geq 2\sqrt{ab}+2\sqrt{bc}+2\sqrt{ca}+\sqrt{ab}+\sqrt{bc}+\sqrt{ca}$$

$$=3\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)$$

Therefore, it is to be true.