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ROMANIAN MATHEMATICAL MAGAZINE

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SP.157. Let $f:[0,+\infty)\to[0,+\infty)$ a derivable function and $a>1$. If $f'(x)(f(x)+x^2+2x+a)=1, \forall x\geq 0$ then: $\lim_{x\to\infty} f(x)$ exists and it is finite.

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Solution by Tran Hong-Vietnam

$$f'(x)(f(x)+x^2+2x+a)=1$$

$$\Rightarrow f'(x)>0\ \forall x\geq 0 \ \ (\textit{because } f(x)\geq 0\ \forall x\geq 0, a>1)$$

$$\Rightarrow f(x)\nearrow \textit{ on } [0,+\infty)$$

$$\Rightarrow \lim_{x\to+\infty} f(x)=l\in[0,+\infty) \textit{ or } \lim_{x\to+\infty} f(x)=+\infty.$$

$$\textit{If } \lim_{x\to+\infty} f(x)=+\infty \Rightarrow \lim_{x\to+\infty} f'(x)=\lim_{x\to+\infty}\frac{1}{f(x)+x^2+2x+a}=0$$

$$\Rightarrow \exists\alpha>0: 0<f'(x)\leq\frac{1}{x^2+1}\ (\forall x\geq\alpha)$$

$$\Rightarrow 0<\int_0^x f'(t)\,dt\leq\int_0^x\frac{1}{t^2+1}dt$$

$$\Rightarrow 0<f(x)-f(\alpha)\leq\tan^{-1}(x)-\tan^{-1}(\alpha)$$

$$\Rightarrow f(x)\leq\tan^{-1}(x)+f(\alpha)-\tan^{-1}(\alpha)$$

$$\Rightarrow \lim_{x\to+\infty} f(x)\leq\frac{\pi}{2}+f(\alpha)-\tan^{-1}(\alpha)$$

which is contrary with $\lim_{x\to+\infty} f(x) = +\infty$***.***

So, we have $\lim_{x\to+\infty} f(x) = l \in [0, +\infty)$***. Proved.***