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Problem.

$$if\ \ a,b,c \in (0;+\infty) \wedge abc = 1 \ and \ \ x \in \mathbb{R} \ \text{then}$$

$$\sum_{cyc(a,b,c)} \frac{1}{(a^{2x}+b^{2x})(a^x+b^x)+2} \le \frac{1}{2}.$$

Proof.

$$\oplus \ \ a^{2x}+b^{2x} \overset{Cauchy}{\ge} 2(ab)^x$$

Similar:

$$b^{2x}+c^{2x} \ge 2(bc)^x$$

$$c^{2x}+a^{2x} \ge 2(ca)^x$$

$$\oplus \ LHS = \sum_{cyc(a,b,c)} \frac{1}{(a^{2x}+b^{2x})(a^x+b^x)+2} \le \frac{1}{2} \sum_{cyc(a,b,c)} \frac{1}{(ab)^x(a^x+b^x)+1}$$

$$\sum_{cyc(a,b,c)} \frac{1}{(ab)^x(a^x+b^x)+1} = \sum_{cyc(a,b,c)} \frac{1}{(\frac{1}{c})^x(a^x+b^x)+1}$$

because $abc = 1$

$$\oplus \ \text{LHS} \le \frac{1}{2} \sum_{cyc(a,b,c)} \frac{1}{(\frac{1}{c})^x(a^x+b^x)+1} = \frac{1}{2} \sum_{cyc(a,b,c)} \frac{c^x}{a^x+b^x+c^x} = \frac{1}{2} = RHS$$

(done)

$$\oplus \ \text{Equality} \leftrightarrow x = 0 \vee a = b = c = 1.$$