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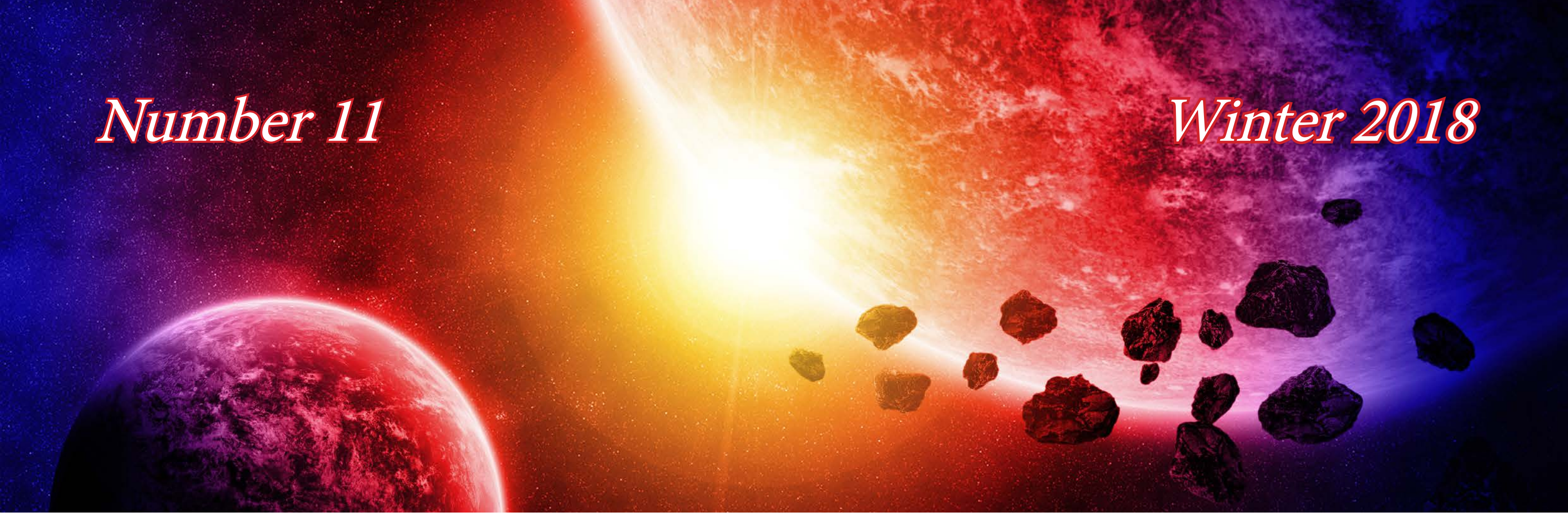

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PROBLEMS FOR JUNIORS

JP.151. Let a, b be positive real numbers. Find the maximum of k such that inequality is true:

$$\frac{1}{a^4}+\frac{1}{b^4}+\frac{k}{a^4+b^4}\geq\frac{8k+32}{(a+b)^4}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

JP.152. Let $ABC; h_a, h_b, h_c$ denote the lengths of altitudes from $A, B, C; l_a, l_b, l_c$ are the lengths of the symmetric divergence lines from $A, B, C; r_a, r_b, r_c$ are radii of the circle next to the corners A, B, C. Prove that:

$$\frac{h_a r_a}{l_a^2}+\frac{h_b r_b}{l_b^2}+\frac{h_c r_c}{l_c^2}\geq 3$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

JP.153. In ΔABC the lengths BC, CA, AB are a, b, c. Let l_a, l_b, l_c be the length of the bisectors from the vertices A, B, C in triangle ABC. Prove that:

$$\frac{l_a l_b}{\sin\frac{C}{2}}+\frac{l_b l_c}{\sin\frac{A}{2}}+\frac{l_c l_a}{\sin\frac{B}{2}}\leq\frac{3}{2}\sqrt{3abc(a+b+c)}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

JP.154. In triangle ABC with sides $BC = a, CA = b,$ $AB = c.r_a, r_b, r_c$ are exradii, h_a, h_b, h_c are distances from A, B, C to BC, CA, AB. Prove that:

$$\frac{r_a r_b}{\sin\frac{A}{2}\sin\frac{B}{2}}+\frac{r_b r_c}{\sin\frac{B}{2}\sin\frac{C}{2}}+\frac{r_c r_a}{\sin\frac{C}{2}\sin\frac{A}{2}}\geq\frac{h_a h_b}{\sin^2\frac{C}{2}}+\frac{h_b h_c}{\sin^2\frac{A}{2}}+\frac{h_c h_a}{\sin^2\frac{B}{2}}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

JP.155. Let a, b, c be positive real numbers such that: $a+b+c = 3$. Prove that:

$$\frac{a^2}{b^4c\sqrt[3]{4(b^6+1)}}+\frac{b^2}{c^4a\sqrt[3]{4(c^6+1)}}+\frac{c^2}{a^4b\sqrt[3]{4(a^6+1)}}\geq\frac{3}{2}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

JP.156. Let ABC be a triangle having the area S. Let be $A' \in (BC)$ such that the incircles of $\Delta AA'B, \Delta AA'C$ have the same radius. Analogous, we obtain the points $B' \in (AC)$, $C' \in (AB)$. Prove that:

$$S = \frac{AA' \cdot BB' \cdot CC'}{s}$$

where s is the semiperimeter of ΔABC.

Proposed by Marian Ursărescu - Romania

JP.157. Prove that in ΔABC, the following inequality holds:

$$\sum_{cyc}\left(a\cot\frac{A}{2} \cdot \sin^2 B \cdot \sin^2 C\right) \geq \frac{81r^4}{2R^2(2R-r)}$$

Proposed by Marian Ursărescu - Romania

JP.158. Let $a, b, c > 0$. Prove that:

$$\sum_{cyc}\frac{1}{a} + \sum\frac{a}{b^2+c^2} \geq 3\sum_{cyc}\frac{1}{b+c}$$

Proposed by Andrei Ştefan Mihalcea - Romania

JP.159. Prove that in any ΔABC the following inequality holds:

$$a^2h_bh_c + b^2h_ah_c + c^2h_ah_b \leq 4(R+r)^4$$

Proposed by Marian Ursărescu - Romania

JP.160. Prove that for all non-negative real numbers x, y, z the following inequality holds:

$$\frac{y+z}{(1+x)^2} + \frac{z+x}{(1+y)^2} + \frac{x+y}{(1+z)^2} \geq \frac{x}{1+yz} + \frac{y}{1+zx} + \frac{z}{1+xy}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.161. Let a, b, c, d be positive real numbers such that $abcd \geq 1$. Prove that:

$$\frac{a^2+b^2+c^2+1}{a^3+b^3+c^3+1} + \frac{b^2+c^2+d^2+1}{b^3+c^3+d^3+1} + \frac{c^2+d^2+a^2+1}{c^3+d^3+a^3+1} +$$

$$+\frac{d^2+a^2+b^2+1}{d^3+a^3+b^3+1} \leq 4$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.162. Let x, y, z be non-negative real numbers such that $x + y + z = 1$. Prove that:
(a) $\frac{3}{\sqrt{10}} \le \frac{x}{\sqrt{1+yz}} + \frac{y}{\sqrt{1+zx}} + \frac{z}{\sqrt{1+xy}} \le 1$.
(b) $\sqrt{\frac{3}{5}} \le \frac{x}{\sqrt{1+y+z}} + \frac{y}{\sqrt{1+z+x}} + \frac{z}{\sqrt{1+x+y}} \le 1$.
When do the equalities occur?

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.163. Let ABC be an acute triangle with standard notations. Prove that:

$$\frac{m_a}{a} + \frac{m_b}{b} + \frac{m_c}{c} \le \frac{a+b+c}{4r}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.164. If $a, b \ge 0$ then:

$$(a + b + \sqrt{a^2 + b^2})^2 \ge 6\sqrt{3}ab$$

Proposed by Daniel Sitaru - Romania

JP.165. If $a, b, c \ge 0$ then:

$$4(a + b + c) \le (3\sqrt{3} - 2)(\sqrt{a^2 + b^2} + \sqrt{b^2 + c^2} + \sqrt{c^2 + a^2})$$

Proposed by Daniel Sitaru - Romania

PROBLEMS FOR SENIORS

SP.151. Let a, b, c be positive real numbers such that: $a+b+c = 3$. Prove that:

$$\frac{a}{\sqrt{2(b^4 + c^4)} + 7bc} + \frac{b}{\sqrt{2(c^4 + a^4)} + 7ca} + \frac{c}{\sqrt{2(a^4 + b^4)} + 7ab} \ge \frac{1}{3}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

SP.152. Let a, b, c be positive real numbers. Find the minimum of value:

$$P = \frac{1}{\sqrt{2(a^4 + b^4)}} + \frac{1}{\sqrt{2(b^4 + c^4)}} + \frac{1}{\sqrt{2(c^4 + a^4)}} + \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

SP.153. Solve the system of equations:

$$\begin{cases} 2\left(\frac{x^3}{y^2} + \frac{y^3}{x^2}\right) = \sqrt[4]{8(x^4 + y^4)} + 2\sqrt{xy} \\ 16x^5 - 20x^3 + 5\sqrt{xy} = \sqrt{\frac{y+1}{2}} \end{cases}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

SP.154. Let x, y, z be positive real numbers such that: $x+y+z=3$. Find the minimum of value:

$$P=\frac{\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}+1}{4(\sqrt{x}+\sqrt{y}+\sqrt{z})}+\frac{8}{3}(x^2+y^2+z^2)$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

SP.155. Let x, y, z be positive real numbers such that: $xyz=1$. Find the minimum of value:

$$P=\frac{x^3}{(2y^2-yz+2z^2)^2}+\frac{y^3}{(2z^2-zx+2x^2)^2}+\frac{z^3}{(2x^2-xy+2y^2)^2}+$$
$$+\frac{xy+yz+zx}{3}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

SP.156. Find all the polynomials $P\in\mathbb{R}[x]$ having the property $P(x)=P(x+\sqrt{x^2+1}), \forall x\in\mathbb{R}$.

Proposed by Marian Ursărescu - Romania

SP.157. Let $f:[0,+\infty)\to[0,+\infty)$ a derivable function and $a>1$. If $f'(x)(f(x)+x^2+2x+a)=1, \forall x\geq 0$ then: $\lim_{x\to\infty} f(x)$ exists and it is finite.

Proposed by Marian Ursărescu - Romania

SP.158. Prove that for any real numbers $a_1, a_2, \ldots, a_n$, the inequality holds:

$$\sum_{i=1}^{n} a_i(a_i+a_{i+1})(a_i^2+a_{i+1}^2)\ldots(a_i^{2^k}+a_{i+1}^{2^k})\geq 0$$

where k is a positive integer and $a_{n+1}=a_1$. When does the equality occur?

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

SP.159. Prove that in any triangle ABC with standard notations, the inequality holds:

$$(a+b+c)(a^2+b^2+c^2)\geq 2(b+c)h_a^2+2(c+a)h_b^2+2(a+b)h_c^2.$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

SP.160. Let a, b, c be positive real numbers such that $a+b+c\geq\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$. Prove that:

$$3(a^3b+b^3c+c^3a)\geq(a+b+c)^2$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

SP.161. Prove that the following inequalities hold for all real numbers $a, b, c \in [0, 1]$
(a) $(1-a+a^2)(1-b+b^2)(1-c+c^2) \leq (1-abc+a^2b^2c^2)$.
(b) $(1-a+a^2)^2(1-b+b^2)^2(1-c+c^2)^2 \leq$
$\leq (1-ab+a^2b^2)(1-bc+b^2c^2)(1-ca+c^2a^2)$. When does the equality occur?

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

SP.162. If $m \in [0, \infty), x, y, z, t \in (0, \infty)$, then in any triangle ABC, with usual notations holds:

$$\sum_{cyclic} \frac{(xa^2 + ym_b^2)^{m+1}}{(zw_b^2 + th_c^2)^m} \geq \frac{(4x+3y)^{m+1}}{3^m(z+t)^m}\sqrt{3}S$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

SP.163. If $m \in [0, \infty), x, y, z, t \in (0, \infty)$, then in any triangle ABC, with usual notations holds:

$$\sum_{cyclic} \frac{(xa^2 + yb^2)^{m+1}}{(zw_b^2 + tw_c^2)^m} \geq \frac{4^{m+1}(x+y)^{m+1}}{3^{m-\frac{1}{2}}(z+t)^m}S$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

SP.164. If $a, b, c > 0$ then:

$$(a+b)\sqrt{a^2+b^2-ab}+(b+c)\sqrt{b^2+c^2-bc}+(c+a)\sqrt{c^2+a^2-ca} \geq$$
$$\geq 2(ab+bc+ca)$$

Proposed by Daniel Sitaru - Romania

SP.165. If $a, b, c \geq 0$ then:

$$(a+b)\sqrt{a^2+b^2}+(b+c)\sqrt{b^2+c^2}+(c+a)\sqrt{c^2+a^2} \geq$$
$$\geq (2\sqrt{3}-1)(ab+bc+ca)$$

Proposed by Daniel Sitaru - Romania

UNDERGRADUATE PROBLEMS

UP.151. Given real numbers $a_1, a_2, \ldots, a_n \in [0, 1]$. Find the maximum and minimum possible value of

$$\Omega = a_1 + a_2 + \ldots + a_n + (1-a_1)(1-a_2)\ldots(1-a_n).$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

UP.152. Let $a_1, a_2, \ldots, a_n (n \geq 3)$ be positive real numbers such that:

$$\sum_{1 \leq i < j \leq n} a_i a_j \geq \frac{n(n-1)}{2}.$$

Prove that:

$$\sum_{i=1}^{n} \frac{1}{a_i^2 + n - 1} \leq 1.$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

UP.153. Find:

$$\Omega = \int_0^1 \int_0^\infty \left(\frac{x^{100}}{100^x}\right)(y \ln y)^{100} dxdy$$

Proposed by Ekpo Samuel - Nigeria

UP.154. If $a_n \in (0, \infty); n \in \mathbb{N}; n \geq 1; \lim_{n\to\infty} \frac{a_{n+1}}{n^2 a_n} = a > 0$ then find:

$$\Omega = \lim_{n\to\infty} \left(\frac{1}{\sqrt[n]{a_n}} \cdot \sum_{k=1}^{n} \left[\left(\sqrt[2k]{k!} + \sqrt[2k+1]{(k+1)!} \right)^2 \right] \right)$$

$[*]$ - great integer function.

Proposed by D.M. Bătineţu - Giurgiu, Daniel Sitaru - Romania

UP.155. If $m \in \mathbb{N}; x, y, z > 0$ then in ΔABC the following relationship holds:

$$3m + \left(\frac{a^2x^2}{yz}\right)^{m+1} + \left(\frac{b^2y^2}{zx}\right)^{m+1} + \left(\frac{c^2z^2}{xy}\right)^{m+1} \geq 4(m+1)\sqrt{3}S$$

Proposed by D.M. Bătineţu - Giurgiu, Daniel Sitaru - Romania

UP.156. If $a, b, c > 0; b < c; b(a+c) \in \mathbb{N}; b(a+c) \geq 1; n \in \mathbb{N}; n \geq 2;$ $x_k \in [b, c]; t_k > 0; k \in \overline{1, n}$ then:

$$\left(\sum_{k=1}^{n} t_k x_k\right) \left(\sum_{k=1}^{n} \frac{t_k}{x_k}\right)^{b(a+c)} \leq \left(\frac{a+b+c}{1+b(a+c)}\right)^{1+ab+bc} \cdot \left(\sum_{k=1}^{n} t_k\right)^{1+ab+bc}$$

Proposed by D.M. Bătineţu - Giurgiu, Daniel Sitaru - Romania

UP.157. If $m \geq 0; x, y > 0$ then in ΔABC the following relationship holds:

$$\frac{h_a}{h_b^m h_c^m (xh_b + yh_c)^m} + \frac{h_b}{h_c^m h_a^m (xh_c + yh_a)^m} + \frac{h_c}{h_a^m h_b^m (xh_a + yh_b)^m} \geq$$

$$\geq \frac{9}{(x+y)^m \cdot s^{2m} \cdot r^{m-1}}$$

Proposed by D.M. Bătineţu - Giurgiu, Daniel Sitaru - Romania

UP.158. Find:

$$\Omega=\int_0^1\left(x^2\log(x)\cot\left(\frac{\pi x}{2}\right)\right)dx$$

Proposed by Arafat Rahman Akib - Dhaka - Bangladesh

UP.159. Find:

$$\Omega=\int_{-1}^{+1}(x^2+2x+1)^{\frac{3}{2}}(1-x)^{\frac{1}{2}}dx$$

Proposed by Ekpo Samuel - Nigeria

UP.160. Let $x,y,z\in(0;+\infty)$ and $a\geq 1$. Prove:

$$\frac{1}{([x]+\{y\}+1)^a}+\frac{1}{([y]+\{z\}+1)^a}+\frac{1}{([z]+\{x\}+1)^a}\geq\frac{3^{a+1}}{(x+y+z+3)^a}$$

$[x]$ is the integer part of the real number x;
$\{x\}$ is the fraction of real numbers x.

Proposed by Nguyen Van Nho - Nghe An - Vietnam

UP.161. Find:

$$\Omega=\int_0^\infty\int_0^\infty\frac{dxdy}{(x^4+2x^2+1)(y^2+25)^4}$$

Proposed by Ekpo Samuel - Nigeria

UP.162. If $ABCD$ tetrahedron; $AB=a_1;AC=a_2;AD=a_3;$ $BC=a_4;BD=a_5;CD=a_6$ then:

$$\sum_{1\leq i<j\leq 6}(a_i+a_j)^2\geq 4\sqrt{3}S[ABCD]$$

$S[ABCD]$ - total area of tetrahedron $ABCD$.

Proposed by Daniel Sitaru - Romania

UP.163. Let a,b,c be positive real numbers such that: $(a^2+b^2)(b^2+c^2)(c^2+a^2)=8$. Find the minimum value of:

$$P=\frac{a}{b(b+2c+1)(a+3c)^2}+\frac{b}{c(c+2a+1)(b+3a)^2}+\frac{c}{a(a+2b+1)(c+3b)^2}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

UP.164. Solve the equation: $\sqrt{2(x^4+1)}+2\sqrt{3x-2x^4}=7-3x$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

UP.165. Solve the system of equations:

$$\begin{cases} \frac{1}{\sqrt{a^3}}+\frac{1}{\sqrt{b^3}}+\frac{1}{\sqrt{c^3}}=3 \\ \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}=\frac{(a^3+b^3+c^3)^2}{3} \end{cases}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

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