The beauty inside of Inequality

Preface:

Inequality is one of the nicest and hardest major parts of Mathematics, because of its appearance in Math contests; and also it requires a solid knowledge to solve a inequality problem. But because of the elegance, Inequality attracts lots of student generations joining in solving and creating more. To maintain the passion and creativeness, many sites and forums nowadays have shared lots of topics and documents of inequality for everybody, but it's still intricate and not arranged entiringly.

Maybe many people will consider that studying Inequality is boring, since its large amount of knowledges and complex presentation of that amount of knowledges, especially upper secondary high school and higher. But I will demonstrate carefully and clearly for everyone through this document file, so everyone can realize how beautiful inequality is and its application.

This file that I make is based on documents of many teacher in Vietnam and around the world, such as: Tran Quoc Anh, Nguyen Van Mau, Pham Kim Hung, Vo Quoc Ba Can, Vasile Cîrtoaje,… I spent lots of time to complie the file so if there's a mistake, hope everybody can understand and give me a feedback because my knowledge is limited. All feedbacks can be sent to the following email: thinh06032001@gmail.com or Facebook.

Thank you very much! :)

--- Do Huu Duc Thinh ---

Season 1: Some old and modern techniques of Inequalities in Math contests

In this season, I will state again some inequalities and basic techniques that are useful to find the solution of proving the inequality and solving Min-Max problems. Also I will add more lemmas with inferences and developments (specifically the pink one) so everyone, especially students, can find a nice way to solve the inequality.

I. Inequalities can be proved by equivalence:

1) $a^2+b^2 \ge \frac{(a+b)^2}{2} \ge 2ab$ (*Basic BCS inequality for 2 numbers*)

$\rightarrow \sqrt{\frac{a^2+b^2}{2}} \ge \frac{a+b}{2} \ge \sqrt{ab} \ge \frac{2ab}{a+b} = \frac{2}{\frac{1}{a}+\frac{1}{b}}$ (*Basic RMS - AM - GM - HM inequality for $a,b>0$*)

$\rightarrow \frac{a+b}{2} \le \sqrt{\frac{a^2+b^2}{2}} \le a+b-\frac{2ab}{a+b}$ *with* $a,b\ge 0$ *such that* $a+b>0$

2) $a^2+b^2+c^2 \ge \frac{(a+b+c)^2}{3} \ge ab+bc+ca$

$\rightarrow \left\langle \begin{array}{l} *3(a^2b^2+b^2c^2+c^2a^2) \ge (ab+bc+ca)^2 \ge 3abc(a+b+c) \\ *\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c} \ge \sqrt{3(a^2+b^2+c^2)} \ge a+b+c \ge \sqrt{3(ab+bc+ca)} \text{ for } a,b,c>0 \end{array}\right.$

3) $a^2+b^2+c^2 \ge 2ab+2ac-2bc$

4) $a^3+b^3+c^3 \ge 3abc \ \forall a,b,c$ *such that* $a+b+c\ge 0$ (*special case: $a,b,c\ge 0$ - Cauchy's inequality for 3 non-negative numbers*)

5) $(a^2+b^2)(c^2+d^2) \ge (ac+bd)^2$ (*Basic Bunyakovsky's inequality*)

6) $a^3+b^3 \ge ab(a+b) \ \forall a,b$ *such that* $a+b\ge 0$

$\rightarrow 4(a^3+b^3) \ge (a+b)^3 \ge 4ab(a+b)$ *for* $a+b\ge 0$

7) $2(a^2+b^2-ab)^2 \ge a^4+b^4 \ge 2ab(2a^2-3ab+2b^2) \ \forall a,b \rightarrow \sqrt{a^2-ab+b^2} \ge \frac{a^2+b^2}{a+b} = a+b-\frac{2ab}{a+b}$ *with* $a+b\ne 0$

8) $\frac{1}{a^2}+\frac{1}{b^2} \ge \frac{8}{(a+b)^2}$ *with real numbers* a,b *such that* $ab>0$ (*special case: $a,b>0$*)

9) $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \ge \frac{3}{2}$ *with* $a,b,c>0$ (*Nesbitt's inequality*) $\rightarrow$ *This is even true for real numbers* $a;b;c$ *such that* $ab+bc+ca>0$.

10) $(a+b)(b+c)(c+a) \ge \frac{8}{9}(a+b+c)(ab+bc+ca) \ge 8abc$ *for* $a,b,c\ge 0$.

$\rightarrow \left\langle \begin{array}{l} *abc \ge (b+c-a)(c+a-b)(a+b-c) \text{ for } a,b,c \text{ are sides of a triangle.} \\ *(a+b+c)^3 \ge 27abc \text{ for } a,b,c\ge 0 \text{ (Cauchy's inequality for 3 non-negative variables)} \end{array}\right.$

11) $\frac{a^2}{x}+\frac{b^2}{y} \ge \frac{(a+b)^2}{x+y}$ *with* $x,y>0$ (*Basic Bunyakovsky Cauchy - Schwarz inequality*)

$*$*Also, from this inequality we have the chain:* $\frac{a^2}{b}+\frac{b^2}{a} \ge \sqrt{2(a^2+b^2)} \ge a+b \ge 2\sqrt{ab}$ *for* $a,b>0$.

12) $\left\langle \begin{array}{l} *2(ax+by) \ge (a+b)(x+y) \ge 2(ay+bx) \text{ with } a\ge b,\ x\ge y \\ *3(ax+by+cz) \ge (a+b+c)(x+y+z) \ge 3(az+by+cx) \text{ with } a\ge b\ge c,\ x\ge y\ge z \end{array}\right.$ (*Basic Chebyshev's inequality*)

13) $a^2b+b^2c+c^2a \ge ab^2+bc^2+ca^2$ *with* $a\ge b\ge c$

$*$*More general:* $a^nb+b^nc+c^na \ge ab^n+bc^n+ca^n$ *for* $a\ge b\ge c\ge 0,\ n\in Z^+$

14) $2(a^n+b^n) \ge (a^x+b^x)(a^y+b^y)$ *with* $a,b>0;\ x,y,n\in Z^+: x+y=n \rightarrow a^{m+n}+b^{m+n} \ge a^m.b^n+a^n.b^m$ *for* $a,b\ge 0,\ m,n\in Z^+$

15) $a^2+ab+b^2 \ge \frac{3}{4}(a+b)^2;\ a^2-ab+b^2 \ge \frac{1}{4}(a+b)^2 \ \forall a,b \rightarrow 3 \ge \frac{a^2-ab+b^2}{a^2+ab+b^2} \ge \frac{1}{3}$ *with real numbers* $a,b: a^2+b^2>0$.

16) $\sqrt{a^2+b^2}+\sqrt{c^2+d^2} \ge \sqrt{(a+c)^2+(b+d)^2}$ (*Basic Minkovsky's inequality*)

17*) $\frac{1}{(a+1)^2}+\frac{1}{(b+1)^2} \ge \frac{1}{ab+1}$ *with* $a,b\ge 0$ (*The equality happens iff* $a=b=1$)

18) Consider $f(a;b)=\frac{1}{1+a^2}+\frac{1}{1+b^2}-\frac{2}{1+ab}$ with $a,b>0$. If $ab\ge 1$ then $f(a;b)\ge 0$; if $ab\le 1$ then $f(a;b)\le 0$.

19) $\frac{1}{a^2-1}+\frac{1}{b^2-1}\ge\frac{2}{ab-1}$ with $a,b>1$

20) $(a^2+b^2+c^2)(a+b+c)\ge 3.\max\left\{a^2b+b^2c+c^2a;\ ab^2+bc^2+ca^2\right\}$ with $a,b,c\ge 0$

21) $(a^2+b^2+c^2)^2\ge(a+b+c)(a^2b+b^2c+c^2a)\ \forall a,b,c>0$

22) $\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}\ge\frac{3}{2}\ge\frac{b}{a+b}+\frac{c}{b+c}+\frac{a}{c+a}$ with $a\ge b\ge c>0$

23*) $(a^2+b^2+c^2)^2\ge 3(a^3b+b^3c+c^3a)$ (Vasile's inequality)

24) $\frac{1}{1+x^2}\ge 1-\frac{x}{2}$ with $x\ge 0\to\frac{1}{1+a^2}+\frac{1}{1+b^2}+\frac{1}{1+c^2}\ge\frac{3}{2}\ \forall a,b,c\ge 0: a+b+c=3.$

25) $(a+b+c)^3\ge a^3+b^3+c^3+24abc\ \forall a,b,c>0$

II. Some familiar Inequalities, lemmas and techniques: (ascending by higher level)

a) For junior – early-senior:

1) $\sqrt{\frac{a_1^2+a_2^2+...+a_n^2}{n}}\ge\frac{a_1+a_2+...+a_n}{n}\ge\sqrt[n]{a_1a_2...a_n}\ge\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}}$ with $a_i>0\ \ \forall i=1;2;...;n$ and $n\in Z^+,\ n\ge 2$

(RMS - AM - GM - HM inequality for n positive numbers)

$\to$ If $x_1;x_2;...;x_n$ are positive real numbers that $x_1+x_2+...+x_n=1$ then with same condition for a_i we have:

$a_1x_1+a_2x_2+...+a_nx_n\ge a_1^{x_1}.a_2^{x_2}...a_n^{x_n}$ (Weighted AM - GM inequality)

2) For real numbers $a_i;\ b_i\ (i=1;2;...;n)$ we have: $\left(a_1^2+a_2^2+...+a_n^2\right)\left(b_1^2+b_2^2+...+b_n^2\right)\ge\left(a_1b_1+a_2b_2+...+a_nb_n\right)^2$

(Cauchy - Schwarz inequality) $\to\frac{a_1^2}{b_1}+\frac{a_2^2}{b_2}+...+\frac{a_n^2}{b_n}\ge\frac{\left(a_1+a_2+...+a_n\right)^2}{b_1+b_2+...+b_n}\ \forall b_i>0$

$\to\frac{x_1^2}{x_2}+\frac{x_2^2}{x_3}+...+\frac{x_{n-1}^2}{x_n}+\frac{x_n^2}{x_1}\ge\sqrt{n\left(x_1^2+x_2^2+...+x_n^2\right)}\ge x_1+x_2+...+x_n\ \ \forall x_i>0$

3) Let $a_1\ge a_2\ge...\ge a_n$. $\begin{cases}*\text{If } b_1\ge b_2\ge...\ge b_n \text{ then}: n(a_1b_1+a_2b_2+...+a_nb_n)\ge(a_1+a_2+...+a_n)(b_1+b_2+...+b_n)\\ *\text{If } b_1\le b_2\le...\le b_n \text{ then}: n(a_1b_1+a_2b_2+...+a_nb_n)\le(a_1+a_2+...+a_n)(b_1+b_2+...+b_n)\end{cases}$

(Chebyshev's inequality)

4) If $x_{i.j}\ge 0\ (i=1;2;...;n,\ j=1;2;...;m)$ then we have:

$(x_{1.1}+x_{1.2}+...+x_{1.n})(x_{2.1}+x_{2.2}+...+x_{2.n})...(x_{m.1}+x_{m.2}+...+x_{m.n})\ge\left(\sqrt[m]{x_{1.1}x_{2.1}...x_{m.1}}+\sqrt[m]{x_{1.2}x_{2.2}...x_{m.2}}+...+\sqrt[m]{x_{1.n}x_{2.n}...x_{m.n}}\right)^m$

(Holder's inequality)

Ex: $*(a+b)(c+d)\ge\left(\sqrt{ac}+\sqrt{bd}\right)^2\ \ \forall a,b,c,d\ge 0$

$*(a+b+c)(m+n+p)(x+y+z)\ge\left(\sqrt[3]{amx}+\sqrt[3]{bny}+\sqrt[3]{cpz}\right)^3\ \ \forall a,b,c,m,n,p,x,y,z\ge 0$

$\to\begin{cases}*\ \frac{a_1^m}{a_2^m}+\frac{a_2^m}{a_3^m}+...+\frac{a_n^m}{a_1^m}\ge\frac{a_1}{a_2}+\frac{a_2}{a_3}+...+\frac{a_n}{a_1} \text{ for } a_i>0\,(i=1;2;...;n);\ m\in Z^+,\ m\ge 2\\ *\ \sqrt[n]{\frac{a_1^n+a_2^n+...+a_n^n}{n}}\ge\frac{a_1+a_2+...+a_n}{n} \text{ with same condition (Power Mean Inequality)}\\ *\ \frac{1}{a_1^2}+\frac{1}{a_2^2}+...+\frac{1}{a_n^2}\ge\frac{n^3}{(a_1+a_2+...+a_n)^2} \text{ for } a_i>0\ (i=1;2;..;n)\to\frac{1}{a_1^n}+\frac{1}{a_2^n}+...+\frac{1}{a_n^n}\ge\frac{n^{n+1}}{(a_1+a_2+...+a_n)^n}\end{cases}$

$\to$ Let a_i and $b_i>0\ (i=1;2;...;n)$ and real $p,q>0$ such that $p+q=pq$. Then we have:

$\left(a_1^p+a_2^p+...+a_n^p\right)^{\frac{1}{p}}.\left(b_1^q+b_2^q+...+b_n^q\right)^{\frac{1}{q}}\ge a_1b_1+a_2b_2+...+a_nb_n$ (General Holder's inequality)

5) Let $a_i,\ b_i>0\ (i=1;2;..;n)$ and any $k>1$. Then we have: $(a_1^k+b_1^k)(a_2^k+b_2^k)...(a_n^k+b_n^k)\ge\sqrt[k]{(a_1+a_2+...+a_n)^k+(b_1+b_2+...+b_n)^k}$

(Minkovsky's inequality) $\to$ similarly for 3 variables $a_i,b_i,c_i>0$.

6) For any $x\ge -1$ we have: $\begin{cases}*(1+x)^r\ge 1+xr \text{ for } r\ge 1 \text{ and } r\le 0\\ *(1+x)^r\le 1+xr \text{ for } 0\le r\le 1\end{cases}$ (Bernoulli's inequality)

7) For any positive integer m and $a,b,c\ge 0$ we have: $a^m(a-b)(a-c)+b^m(b-c)(b-a)+c^m(c-a)(c-b)\ge 0$

(Schur's inequality) $\to$ This is also true for real $m\ge 1$ and equality happens iff $a=b=c$ or $(a;b;c)\sim(0;k;k)$ with $k>0$.

Δ *Case $m=1$ - Schur deg 3: All forms: (p,q,r will be discussed in later part)*

$\ast\ a^3+b^3+c^3+3abc \ge ab(a+b)+bc(b+c)+ca(c+a) \to (a+b+c)^3+9abc \ge 4(a+b+c)(ab+bc+ca)$

$\to\ a^2+b^2+c^2+\dfrac{9abc}{a+b+c} \ge 2(ab+bc+ca) \to (a+b+c)(a^2+b^2+c^2+ab+bc+ca) \ge 3\left[ab(a+b)+bc(b+c)+ca(c+a)\right]$

$\ast\ abc \ge (b+c-a)(c+a-b)(a+b-c)$ *(Well-known result)*

$\ast\ (b-c)^2(b+c-a)+(c-a)^2(c+a-b)+(a-b)^2(a+b-c) \ge 0$

$\ast\ 3\left(a^3+b^3+c^3\right) \ge (a+b+c)\left[2\left(a^2+b^2+c^2\right)-ab-bc-ca\right]$

$\ast\ 4(a^3+b^3+c^3)+15abc \ge (a+b+c)^3$

$\ast\ \dfrac{a}{b+c}+\dfrac{b}{c+a}+\dfrac{c}{a+b}+\dfrac{4abc}{(a+b)(b+c)(c+a)} \ge 2$

$\ast\ \dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{a}+3 \ge a+b+c+\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$ *for* $a,b,c>0: abc=1.$

$\ast\ \left(a^2b+b^2c+c^2a\right)\left(ab^2+bc^2+ca^2\right) \ge \left(a^2b^2+b^2c^2+c^2a^2\right)(ab+bc+ca)$

Δ *Case $m=2$ - Schur deg 4: All forms:*

$\ast\ a^4+b^4+c^4+abc(a+b+c) \ge ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2) \to a^4+b^4+c^4+2abc(a+b+c) \ge (a^2+b^2+c^2)(ab+bc+ca)$

$\ast\ 2(ab+bc+ca)-(a^2+b^2+c^2) \le \dfrac{6abc(a+b+c)}{a^2+b^2+c^2+ab+bc+ca} \le \dfrac{9abc}{a+b+c}$

$\ast\left[(b-c)(b+c-a)\right]^2+\left[(c-a)(c+a-b)\right]^2+\left[(a-b)(a+b-c)\right]^2 \ge 0$

$\to$ *Let* $a,b,c,x,y,z \ge 0$. *Then we have:* $x(a-b)(a-c)+y(b-c)(b-a)+z(c-a)(c-b) \ge 0$ *iff* $a \ge b \ge c$ *and:*

$\ast\ x \ge y \vee z \ge y$ $\qquad$ $\ast\ ax+cz \ge by$

$\ast\ x+z \ge y$ $\qquad$ $\ast\sqrt{x}+\sqrt{z} \ge \sqrt{y}$

$\ast\ ax \ge by \vee cz \ge by$

(General Vornicu - Schur inequality)

8) $\dfrac{1}{(a+b)^2}+\dfrac{1}{(b+c)^2}+\dfrac{1}{(c+a)^2} \ge \dfrac{9}{4(ab+bc+ca)}$ *for* $a,b,c \ge 0$, *no 2 of which are* 0. *(Iran 96 inequality)*

9) $(a^2+b^2+c^2)^2 \ge 3(a^3b+b^3c+c^3a)$ *(Vasile's inequality)* $\to$ *The equality happens iff* $a=b=c$ *and also for*

$(a;b;c)=\left(k.\sin^2\dfrac{4\pi}{7};k.\sin^2\dfrac{2\pi}{7};k.\sin^2\dfrac{\pi}{7}\right)$ *or any cyclic permutation.*

10) *Let* a_i *and* b_i $(i=1;2;...;n)$ *such that:*

$\ast\ a_1 \ge a_2 \ge ... \ge a_n \ge 0,\ b_1 \ge b_2 \ge ... \ge b_n \ge 0$

$\ast\ a_1 \ge b_1;\ a_1+a_2 \ge b_1+b_2;...;\ a_1+a_2+...+a_{n-1} \ge b_1+b_2+...+b_{n-1}$

$\ast\ a_1+a_2+...+a_{n-1}+a_n = b_1+b_2+...+b_{n-1}+b_n$

For $x_i \ge 0$ *we have:* $\sum\limits_{sym} x_{t_1}^{a_1}x_{t_2}^{a_2}...x_{t_n}^{a_n} \ge \sum\limits_{sym} x_{t_1}^{b_1}x_{t_2}^{b_2}...x_{t_n}^{b_n}$, *where* $(t_1;t_2;...;t_n)$ *are all the permutations of* $(1;2;...;n)$

(Muirhead's inequality)

E.g: $a^3+b^3 \ge ab(a+b) \to a^3b^0+b^3a^0 \ge a^2b^1+a^1b^2$

$a^2+b^2+c^2 \ge ab+bc+ca \to a^2b^0+b^2c^0+c^2a^0 \ge a^1b^1+b^1c^1+c^1a^1$

$a^4+b^4+c^4 \ge abc(a+b+c) \to a^4b^0c^0+b^4c^0a^0+c^4a^0b^0 \ge a^2b^1c^1+b^2c^1a^1+c^2a^1b^1$

b) For senior and higher classes: (it's very hard to express the real form of these inequalities so I will try my best.)

1) $\ast$*If* $\left[\begin{array}{l} a_1 \ge a_2 \ge ... \ge a_n;\ b_1 \ge b_2 \ge ... \ge b_n \\ a_1 \le a_2 \le ... \le a_n;\ b_1 \le b_2 \le ... \le b_n \end{array}\right.$ *and* $(k_1;k_2;...;k_n)$ *is an arbitrary permutation of* $(1;2;...;n)$

then: $a_1b_1+a_2b_2+...+a_nb_n \ge a_1b_{k_1}+a_2b_{k_2}+...+a_nb_{k_n}$.

$\ast$*If* $a_1 \ge a_2 \ge ... \ge a_n$ *and* $b_1 \le b_2 \le ... \le b_n$ *then:* $a_1b_1+a_2b_2+...+a_nb_n \le a_1b_{k_1}+a_2b_{k_2}+...+a_nb_{k_n}$;

$n\left(a_1b_1+a_2b_2+...+a_nb_n\right) \le \left(a_1+a_2+...+a_n\right)\left(b_1+b_2+...+b_n\right)$

(Rearrangement inequality)

2) $\ast$*Convex function: If* $a,b \ge 0$ *such that* $a+b=1$ *then* $f(x)$ *is called a convex function on* $I(a;b) \subset R$ *iff* $\forall x_1;x_2 \in I$ *we have:* $f\left(ax_1+bx_2\right) \le a.f\left(x_1\right)+b.f\left(x_2\right)$

$\ast$*Concave function: If* $a,b \ge 0$ *such that* $a+b=1$ *then* $f(x)$ *is called a concave function on* $I(a;b) \subset R$ *iff* $\forall x_1;x_2 \in I$ *we have:* $f\left(ax_1+bx_2\right) \ge a.f\left(x_1\right)+b.f\left(x_2\right)$

$\ast$ If $f(x)$ is a convex function on interval $I\subset R$ then for any $x_i\in I$ $(i=1;2;..;n)$ we have:

$$f\left(\frac{x_1+x_2+...+x_n}{n}\right)\le\frac{f(x_1)+f(x_2)+...+f(x_n)}{n}$$ (Classic Jensen's inequality)

$\ast$ If $f(x)$ is a convex function on interval $I\subset R$ then for any $x_i\in I$ $(i=1;2;..;n)$ and $p_i>0$ we have:

$$\frac{p_1f(x_1)+p_2f(x_2)+...+p_nf(x_n)}{p_1+p_2+...+p_n}\ge f\left(\frac{p_1x_1+p_2x_2+...+p_nx_n}{p_1+p_2+...+p_n}\right)$$

And if $f(x)$ is a concave function then the inequality is reversed. (General Jensen's inequality)

$\rightarrow$ $\ast$ If $f(x)$ is a convex and continuous function on interval $I\subset R$ then for any $x_i\in I$ $(i=1;2;..;n)$ and $p_i\in(0;1)$ such that $p_1+p_2+...+p_n=1$ we have: $p_1f(x_1)+p_2f(x_2)+...+p_nf(x_n)\ge f(p_1x_1+p_2x_2+...+p_nx_n)$

And if $f(x)$ is a concave function then the inequality is reversed.

*The classic inequality is a special case from the general one with $p_1=p_2=...=p_n$

$\ast$ Let a_i and b_i $(i=1;2;...;n)\in I$ $(I\subset R)$ such that:

_ $a_1\ge a_2\ge...\ge a_n;\ b_1\ge b_2\ge...\ge b_n$

_ $a_1\ge b_1;\ a_1+a_2\ge b_1+b_2;...;\ a_1+a_2+...+a_{n-1}\ge b_1+b_2+...+b_{n-1}$

_ $a_1+a_2+...+a_{n-1}+a_n=b_1+b_2+...+b_{n-1}+b_n$

If $f(x)$ is a convex function on I then we have: $f(a_1)+f(a_2)+...+f(a_n)\ge f(b_1)+f(b_2)+...+f(b_n)$

(Karamata's inequality)

$\ast$ If $f(x)$ is a convex function on $I\subset R$ then for $a_i\in I$ $(i=1;2;..;n)$ we have:

$$f(a_1)+f(a_2)+...+f(a_n)+n(n-2)f\left(\frac{a_1+a_2+...+a_n}{n}\right)\ge(n-1)\left[f(b_1)+f(b_2)+...+f(b_n)\right]$$

where $b_i+\dfrac{a_i}{n-1}=\dfrac{a_1+a_2+...+a_n}{n-1}$ $(i=1;2;..;n)$ (Popoviciu's inequality)

3) Define $p=a+b+c,\ q=ab+bc+ca,\ r=abc$ with a,b,c are any real numbers. If $k=\sqrt{p^2-3q}$ then we have:

$$\frac{p^3-3pk^2-2k^3}{27}\le r\le\frac{p^3-3pk^2+2k^3}{27}$$

$\rightarrow$ The minimum and maximum happens iff 2 of 3 variables $a;b;c$ are equal.

4) $\ast$ Let $f(a;b;c)$ be a symmetric polynomial of degree 3 with $a,b,c\ge0$. Then: $f(a;b;c)\ge0\Leftrightarrow$ $f(1;1;1);\ f(1;1;0);\ f(1;0;0)\ge0$ (SD3 theorem)

$\ast$ Let $f(a;b;c)$ be a cyclic homogeneous polynomial of degree 3 with $a,b,c\ge0$. Then: $f(a;b;c)\ge0$ $\Leftrightarrow$ $f(1;1;1)\ge0;\ f(a;b;0)\ge0$ (CD3 theorem)

$\rightarrow$ Let $f_n(a;b;c)$ be a cyclic homogeneous polynomial of degree n $(n=3;4;5)$ with $a,b,c\ge0$. Then $f_n(a;b;c)\ge0\Leftrightarrow f_n(a;1;1)\ge0$ and $f_n(0;b;c)\ge0$.

5) (S.O.S technique) Define $S=S_a.(b-c)^2+S_b.(c-a)^2+S_c.(a-b)^2$, where $S_a;S_b;S_c$ are functions with variables $a;b;c$.

Then $S\ge0$ iff:

$\ast$ $S_a;\ S_b;\ S_c\ge0$

$\ast$ $a\ge b\ge c;\ S_b\ge0;\ S_b+S_a\ge0; S_b+S_c\ge0$

$\ast$ $a\ge b\ge c;\ S_a\ge0; S_c\ge0;\ S_a+2S_b\ge0; S_c+2S_b\ge0$

$\ast$ $a\ge b\ge c;\ S_b\ge0;\ S_c\ge0;\ a^2.S_b+b^2.S_a\ge0$

$\ast$ $S_a+S_b+S_c\ge0;\ S_aS_b+S_bS_c+S_cS_a\ge0$

$\rightarrow$ Consider $f(a;b;c)=P(a-b)^2+Q(a-c)(b-c)\ge0$ (*)

$\ast$ If $f(a;b;c)$ is symmetric then to prove (*) is true, we assume that $a\ge b\ge c$ or $c=\min\{a;b;c\}$ or $c=\max\{a;b;c\}$ and prove that $P,Q\ge0$.

$\ast$ If $f(a;b;c)$ is cyclic then to prove (*) is true, we assume that $c=\min\{a;b;c\}$ or $c=\max\{a;b;c\}$ and prove that $P,Q\ge0$.

(S.S technique)

c) **Some identities:**

(*) Some useful identities in inequality that can be proved by S.O.S, S.S technique:

$$1)\ a^2+b^2-2ab=(a-b)^2; \frac{a}{b}+\frac{b}{a}-2=\frac{(a-b)^2}{ab}$$

$$\to \sqrt{n\left(a_1^2+a_2^2+...+a_n^2\right)}-\left(a_1+a_2+...+a_n\right)=\frac{\sum\limits_{1\le i<j\le n}\left(a_i-a_j\right)^2}{\sqrt{n\left(a_1^2+a_2^2+...+a_n^2\right)}+\left(a_1+a_2+...+a_n\right)}$$

$$For\ n=2: \sqrt{2(a^2+b^2)}-(a+b)=\frac{(a-b)^2}{\sqrt{2(a^2+b^2)}+a+b}$$

$$For\ n=3: \sqrt{3(a^2+b^2+c^2)}-(a+b+c)=\frac{(a-b)^2+(b-c)^2+(c-a)^2}{\sqrt{3(a^2+b^2+c^2)}+(a+b+c)}=2\frac{(a-b)^2+(a-c)(b-c)}{\sqrt{3(a^2+b^2+c^2)}+(a+b+c)}$$

$$\to (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-9=\frac{(a-b)^2}{ab}+\frac{(b-c)^2}{bc}+\frac{(c-a)^2}{ca}=\frac{2(a-b)^2}{ab}+\left(\frac{1}{ac}+\frac{1}{bc}\right)(a-c)(b-c)$$

$$2)\ (a+b+c)^2-3(ab+bc+ca)=a^2+b^2+c^2-(ab+bc+ca)=\frac{1}{2}\left[(a-b)^2+(b-c)^2+(c-a)^2\right]=(a-b)^2+(a-c)(b-c)$$

$$3)\ a^3+b^3-ab(a+b)=(a+b)(a-b)^2$$

$$\to \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2}=\frac{(b-c)^2}{2(a+b)(a+c)}+\frac{(c-a)^2}{2(b+c)(b+a)}+\frac{(a-b)^2}{2(c+a)(c+b)}=\frac{(a-b)^2}{(c+a)(c+b)}+\frac{(a+b+2c)(a-c)(b-c)}{2(a+b)(b+c)(c+a)}$$

$$\to 3(a^3+b^3+c^3)-(a+b+c)(a^2+b^2+c^2)=(a+b)(a-b)^2+(b+c)(b-c)^2+(c+a)(c-a)^2$$

$$=2(a+b)(a-b)^2+(a+b+2c)(a-c)(b-c)$$

$$4)\ (a+b+c)(ab+bc+ca)-9abc=(a+b)(b+c)(c+a)-8abc=a(b-c)^2+b(c-a)^2+c(a-b)^2$$

$$=2c(a-b)^2+(a+b)(a-c)(b-c)$$

$$5)\ a^3+b^3+c^3-3abc=\frac{1}{2}(a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right]=(a+b+c)\left[(a-b)^2+(a-c)(b-c)\right]$$

$$6)\ \frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3=\frac{1}{6abc}\left[(a-b)^2(3c+a-b)+(b-c)^2(3a+b-c)+(c-a)^2(3b+c-a)\right]=\frac{(a-b)^2}{ab}+\frac{(a-c)(b-c)}{bc}$$

$$7)\ \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}-(a+b+c)=\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}+\frac{(c-a)^2}{a}=\left(\frac{1}{a}+\frac{1}{b}\right)(a-b)^2+\frac{b+c}{ac}(a-c)(b-c)$$

$$8)\ a^3+b^3+c^3+3abc-ab(a+b)-bc(b+c)-ca(c+a)=\frac{1}{2}\left[(b+c-a)(b-c)^2+(c+a-b)(c-a)^2+(a+b-c)(a-b)^2\right]$$

$$=(a+b-c)(a-b)^2+c(a-c)(b-c)$$

$$9)\ \frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}-(a+b+c)=\frac{1}{2abc}\left\{(b-c)^2\left[(b+c)^2+a^2\right]+(c-a)^2\left[(c+a)^2+b^2\right]+(a-b)^2\left[(a+b)^2+c^2\right]\right\}$$

$$=\frac{(a+b)^2+c^2}{abc}(a-b)^2+\left[\frac{1}{c}+\frac{(a+c)(b+c)}{abc}\right](a-c)(b-c)$$

$$10)\ \frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}-(a^2+b^2+c^2)=\left(\frac{a}{b}+\frac{1}{2}\right)(a-b)^2+\left(\frac{b}{c}+\frac{1}{2}\right)(b-c)^2+\left(\frac{c}{a}+\frac{1}{2}\right)(c-a)^2$$

$$11)\ a^4+b^4+c^4+a^2b^2+b^2c^2+c^2a^2-a(b^3+c^3)-b(c^3+a^3)-c(a^3+b^3)=\frac{1}{2}\left[(a^2+b^2)(a-b)^2+(b^2+c^2)(b-c)^2+(c^2+a^2)(c-a)^2\right]$$

$$=(a^2+b^2)(a-b)^2+(a^2+ab+b^2+c^2)(a-c)(b-c)$$

<u>(**) More identities:</u> Actually the first nine identities are rare, so just consider the identity 10 onwards:

$$1)\ *\ \frac{a+bc}{b-c}.\frac{b+ca}{c-a}+\frac{b+ca}{c-a}.\frac{c+ab}{a-b}+\frac{c+ab}{a-b}.\frac{a+bc}{b-c}=a+b+c-1\ with\ a\ne b\ne c$$

$$*\ \frac{a-bc}{b-c}.\frac{b-ca}{c-a}+\frac{b-ca}{c-a}.\frac{c-ab}{a-b}+\frac{c-ab}{a-b}.\frac{a-bc}{b-c}=-a-b-c-1\ with\ a\ne b\ne c$$

$$2)\ *\ \frac{a-b}{a+b}+\frac{b-c}{b+c}+\frac{c-a}{c+a}=\frac{(a-b)(b-c)(a-c)}{(a+b)(b+c)(a+c)}$$

$$*\ \frac{a+b}{a-b}+\frac{b+c}{b-c}+\frac{c+a}{c-a}=\frac{a(b-c)^2+b(c-a)^2+c(a-b)^2}{(a-b)(b-c)(a-c)}\ with\ a\ne b\ne c$$

$$3)\ *\ \frac{a+b}{a-b}.\frac{b+c}{b-c}+\frac{b+c}{b-c}.\frac{c+a}{c-a}+\frac{c+a}{c-a}.\frac{a+b}{a-b}=-1\ with\ a\ne b\ne c$$

$$*\ \frac{a-b}{a+b}.\frac{b-c}{b+c}+\frac{b-c}{b+c}.\frac{c-a}{c+a}+\frac{c-a}{c+a}.\frac{a-b}{a+b}=\frac{-\left[a(b-c)^2+b(c-a)^2+c(a-b)^2\right]}{(a+b)(b+c)(c+a)}$$

4) $\ast \frac{a^2+bc}{b+c}.\frac{b^2+ca}{c+a}+\frac{b^2+ca}{c+a}.\frac{c^2+ab}{a+b}+\frac{c^2+ab}{a+b}.\frac{a^2+bc}{b+c}=a^2+b^2+c^2$

$\ast \frac{a^2-bc}{b-c}.\frac{b^2-ca}{c-a}+\frac{b^2-ca}{c-a}.\frac{c^2-ab}{a-b}+\frac{c^2-ab}{a-b}.\frac{a^2-bc}{b-c}=-(a+b+c)^2 \ with\ a\neq b\neq c$

5) $(a^2-bc)(b+c)+(b^2-ca)(c+a)+(c^2-ab)(a+b)=0$

$(a^2+bc)(b-c)+(b^2+ca)(c-a)+(c^2+ab)(a-b)=-2(a-b)(b-c)(c-a)$

6) $\frac{(b+c)^2}{(a-b)(a-c)}+\frac{(c+a)^2}{(b-c)(b-a)}+\frac{(a+b)^2}{(c-a)(c-b)}=1 \ with\ a\neq b\neq c$

7) $\frac{1-ab}{a-b}.\frac{1-bc}{b-c}+\frac{1-bc}{b-c}.\frac{1-ca}{c-a}+\frac{1-ca}{c-a}.\frac{1-ab}{a-b}=-1 \ with\ a\neq b\neq c$

8) $\ast \frac{a^2+bc}{b-c}.\frac{b^2+ca}{c-a}+\frac{b^2+ca}{c-a}.\frac{c^2+ab}{a-b}+\frac{c^2+ab}{a-b}.\frac{a^2+bc}{b-c}=a^2+b^2+c^2 \ with\ a\neq b\neq c$

$\ast \frac{a^2-bc}{b+c}.\frac{b^2-ca}{c+a}+\frac{b^2-ca}{c+a}.\frac{c^2-ab}{a+b}+\frac{c^2-ab}{a+b}.\frac{a^2-bc}{b+c}=-(a+b+c)\left[ab(a-b)^2+bc(b-c)^2+ca(c-a)^2\right]$

9) $\frac{a^2+bc}{a^2-bc}.\frac{b^2+ca}{b^2-ca}+\frac{b^2+ca}{b^2-ca}.\frac{c^2+ab}{c^2-ab}+\frac{c^2+ab}{c^2-ab}.\frac{a^2+bc}{a^2-bc}=-1$

10) $(a+b+c)^3=a^3+b^3+c^3+3(a+b)(b+c)(c+a)$

11) $2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)=(a+b+c)(b+c-a)(c+a-b)(a+b-c)$

12) $\left(ab^n+bc^n+ca^n\right)-\left(a^nb+b^nc+c^na\right)=(a-b)(b-c)(c-a)\sum_{sym}a^pb^qc^r \ with\ p,q,r\in N\ and\ n\geq 2\ such\ that\ p+q+r=n-2$ (?)

E.g: $for\ n=2:\left(ab^2+bc^2+ca^2\right)-\left(a^2b+b^2c+c^2a\right)=(a-b)(b-c)(c-a)$

$n=3:\left(ab^3+bc^3+ca^3\right)-\left(a^3b+b^3c+c^3a\right)=(a-b)(b-c)(c-a)(a+b+c)$

$n=4:\left(ab^3+bc^3+ca^3\right)-\left(a^3b+b^3c+c^3a\right)=(a-b)(b-c)(c-a)(a^2+b^2+c^2+ab+bc+ca)$

13) $a^n(a-b)(a-c)+b^n(b-c)(b-a)+c^n(c-a)(c-b)=\frac{1}{2}\left[\left(a^n+b^n-c^n\right)(a-b)^2+\left(b^n+c^n-a^n\right)(b-c)^2+\left(c^n+a^n-b^n\right)(c-a)^2\right]$

14) If $a,b,c\neq 0$ such that $abc=1$ then: $\frac{1}{ab+b+1}+\frac{1}{bc+c+1}+\frac{1}{ca+a+1}=1$

15) $a^2+b^2+c^2+abc=4\Leftrightarrow 2a+bc=\sqrt{(4-a^2)(4-b^2)},\ etc\Leftrightarrow \frac{2a+bc}{(2+b)(2+c)}+\frac{2b+ca}{(2+c)(2+a)}+\frac{2c+ab}{(2+a)(2+b)}$

$=\frac{(2-b)(2-c)}{2a+bc}+\frac{(2-c)(2-a)}{2b+ca}+\frac{(2-a)(2-b)}{2c+ab}=\frac{a}{2a+bc}+\frac{b}{2b+ca}+\frac{c}{2c+ab}=\frac{bc}{2a+bc}+\frac{ca}{2b+ca}+\frac{ab}{2c+ab}=1$

$\Leftrightarrow \frac{1}{2a+bc}+\frac{1}{2b+ca}+\frac{1}{2c+ab}=\frac{1}{a+b+c-2}=\frac{a+b+c+2}{2(ab+bc+ca)-abc}\Leftrightarrow (a+b+c-2)^2=(2-a)(2-b)(2-c)$

$\rightarrow$ From the identity, there exists $x,y,z>0$ such that: $a=2\sqrt{\frac{xy}{(z+x)(z+y)}};\ b=2\sqrt{\frac{yz}{(x+y)(x+z)}};c=2\sqrt{\frac{zx}{(y+z)(y+x)}}$.

And there exists triangle ABC such that: $a=2\cos A;\ b=2\cos B;\ c=2\cos C$.

16) $a^2+b^2+c^2+2abc=1\Leftrightarrow a+bc=\sqrt{(1-a^2)(1-b^2)},\ etc\Leftrightarrow \frac{a+bc}{(1+b)(1+c)}+\frac{b+ca}{(1+c)(1+a)}+\frac{c+ab}{(1+a)(1+b)}=$

$\frac{(1-b)(1-c)}{a+bc}+\frac{(1-c)(1-a)}{b+ca}+\frac{(1-a)(1-b)}{c+ab}=1\Leftrightarrow \frac{a}{a+bc}+\frac{b}{b+ca}+\frac{c}{c+ab}=2$

$\Leftrightarrow \frac{bc}{a+bc}+\frac{ca}{b+ca}+\frac{ab}{c+ab}=1\Leftrightarrow \frac{1}{a+bc}+\frac{1}{b+ca}+\frac{1}{c+ab}=\frac{2}{a+b+c-1}=\frac{a+b+c+1}{ab+bc+ca-abc}\Leftrightarrow (a+b+c-1)^2=2(1-a)(1-b)(1-c)$

$\rightarrow$ If we substitute $a\rightarrow \frac{a}{2};b\rightarrow \frac{b}{2};c\rightarrow \frac{c}{2}$ we will get identity 15, so:

From the identity, there exists $x,y,z>0$ such that: $a=\sqrt{\frac{xy}{(z+x)(z+y)}};\ b=\sqrt{\frac{yz}{(x+y)(x+z)}};\ c=\sqrt{\frac{zx}{(y+z)(y+x)}}$.

And there exists triangle ABC such that: $a=\cos A;\ b=\cos B;\ c=\cos C$.

$\oplus$ Also if we let $a^2=yz;\ b^2=zx;\ c^2=xy$ with $x,y,z>0$ then $abc=xyz$, so we have 2 identities 17 - 18:

17) $xy+yz+zx+xyz=4\Leftrightarrow \frac{1}{x+2}+\frac{1}{y+2}+\frac{1}{z+2}=\frac{x}{x+2}+\frac{y}{y+2}+\frac{z}{z+2}=1\Leftrightarrow \frac{\sqrt{x}}{2+x}+\frac{\sqrt{y}}{2+y}+\frac{\sqrt{z}}{2+z}=\frac{\sqrt{xyz}}{\sqrt{xy}+\sqrt{yz}+\sqrt{zx}-2}$

18) $xy+yz+zx+2xyz=1\Leftrightarrow \frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}=2\Leftrightarrow \frac{x}{x+1}+\frac{y}{y+1}+\frac{z}{z+1}=1\Leftrightarrow \frac{\sqrt{x}}{1+x}+\frac{\sqrt{y}}{1+y}+\frac{\sqrt{z}}{1+z}=\frac{2\sqrt{xyz}}{\sqrt{xy}+\sqrt{yz}+\sqrt{zx}-1}$

$\rightarrow$ From identity 17, there also exists $m,n,p>0$ such that $x=\frac{2m}{n+p};\ y=\frac{2n}{p+m};\ z=\frac{2p}{m+n}$, similarly for identity 18.

$19)\ x\left(x^2+xy+y^2\right)+y\left(y^2+yz+z^2\right)+z\left(z^2+zx+x^2\right)=y\left(x^2+xy+y^2\right)+z\left(y^2+yz+z^2\right)+x\left(z^2+zx+x^2\right)=(x+y+z)(x^2+y^2+z$

$z\left(x^2+xy+y^2\right)+x\left(y^2+yz+z^2\right)+y\left(z^2+zx+x^2\right)=(x+y+z)(xy+yz+zx)$

$20)\left(x^2+xy+y^2\right)\left(y^2+yz+z^2\right)\left(z^2+zx+x^2\right)=p^2+pq+q^2,\ where\ p=x^2y+y^2z+z^2x;\ q=xy^2+yz^2+zx^2.$

$=3\left(x^2y+y^2z+z^2x\right)\left(xy^2+yz^2+zx^2\right)+\left[(x-y)(y-z)(z-x)\right]^2$

$=\frac{3}{4}\left[xy(x+y)+yz(y+z)+zx(z+x)\right]^2+\frac{1}{4}\left[(x-y)(y-z)(z-x)\right]^2$

$=\frac{1}{2}\left[(x+y+z)^2\left(x^2y^2+y^2z^2+z^2x^2\right)+(xy+yz+zx)^2\left(x^2+y^2+z^2\right)\right]$

$$\rightarrow\begin{cases}\ast(xy+yz+zx)^2\left(x^2+y^2+z^2\right)=(x^2+2yz)(y^2+2zx)(z^2+2xy)+\left[(x-y)(y-z)(z-x)\right]^2\\ \ast(x+y+z)^2\left(x^2y^2+y^2z^2+z^2x^2\right)=(2x^2+yz)(2y^2+zx)(2z^2+xy)+\left[(x-y)(y-z)(z-x)\right]^2\end{cases}$$

$21)\ 2(x^2+y^2)(y^2+z^2)(z^2+x^2)=\left[xy(x+y)+yz(y+z)+zx(z+x)-2xyz\right]^2+\left[(x-y)(y-z)(z-x)\right]^2$

$22)\ 2\left[x^2(y-z)^4+y^2(z-x)^4+z^2(x-y)^4\right]=\left[x(y-z)^2+y(z-x)^2+z(x-y)^2\right]^2+\left[(x-y)(y-z)(z-x)\right]^2$

$23)\ \frac{a^2}{a^2+bc}+\frac{b^2}{b^2+ca}+\frac{c^2}{c^2+ab}=\frac{3}{2}\Leftrightarrow\left(\frac{ab+bc+ca}{a+b+c}\right)^3=abc\Leftrightarrow a^2=bc\ or\ b^2=ca\ or\ c^2=ab$

d) Useful lemmas: In above parts, I've showed some of it. In this part I will state more lemmas, maybe a lot but worth it :)

- *Inequalities with condition about p,q,r (denote p = a+b+c; q = ab+bc+ca; r = abc) – part 1: In this part, I will state lemma with familiar conditions, about the "unusual" conditions, I will show in later seasons.*

$d.1)\ If\ \ a,b,c>0\ such\ that\ abc=1\ then:$

$1)\ a^m+b^m+c^m\ge a^n+b^n+c^n\ (m,n\in Z^+;\ m>n)$

$2)\ \frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge a+b+c;\ \frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\rightarrow\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge\frac{3}{2}(a+b+c-1)\rightarrow\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+3\ge a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$

$3)\ \frac{1}{a^{2k}+a^k+1}+\frac{1}{b^{2k}+b^k+1}+\frac{1}{c^{2k}+c^k+1}\ge 1\ with\ k\in Z^+$

$4)\ \frac{1}{a^k+b^k+1}+\frac{1}{b^k+c^k+1}+\frac{1}{c^k+a^k+1}\le 1\ with\ k\in Z^+$

$5)\ \ast\ \frac{1}{a(b+1)}+\frac{1}{b(c+1)}+\frac{1}{c(a+1)}\ge\frac{3}{2}$

$\ast\ \frac{1}{a(a+b)}+\frac{1}{b(b+c)}+\frac{1}{c(c+a)}\ge\frac{3}{2}\rightarrow\frac{1}{a(ma+nb)}+\frac{1}{b(mb+nc)}+\frac{1}{c(mc+na)}\ge\frac{3}{m+n}\ with\ m,n>0\ (*)$

$6)\ \frac{1}{a^2+2b^2+3}+\frac{1}{b^2+2c^2+3}+\frac{1}{c^2+2a^2+3}\le\frac{1}{2}$

$7)\ \frac{ab}{a^5+b^5+ab}+\frac{bc}{b^5+c^5+bc}+\frac{ca}{c^5+a^5+ca}\le 1$

$8)\ Let\ \ p=a+b+c;\ q=ab+bc+ca\ then:$

$$\ast\,p^2+3\ge 4q\rightarrow\begin{cases}p+\frac{3}{q}\ge 4\\ p+\frac{3}{q}\ge 4.\frac{q}{p}\end{cases}\qquad\qquad \ast\,pq\ge 5p-6$$

$9)\ \frac{a}{b+c+1}+\frac{b}{c+a+1}+\frac{c}{a+b+1}\ge 1\rightarrow\frac{a}{b+c+1}+\frac{b}{c+a+1}+\frac{c}{a+b+1}\ge\frac{a}{a+2}+\frac{b}{b+2}+\frac{c}{c+2}\ge 1\ge\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}$

$\rightarrow\frac{1}{2a+1}+\frac{1}{2b+1}+\frac{1}{2c+1}\ge 1\ge\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}\ \left(Extra:\frac{1}{3a+1}+\frac{1}{3b+1}+\frac{1}{3c+1}\ge\frac{3}{4}\ge\frac{1}{a+3}+\frac{1}{b+3}+\frac{1}{c+3}\right)$

$10)\ (a^2+1)(b^2+1)(c^2+1)\ge(a+b)(b+c)(c+a)\ge(a+1)(b+1)(c+1)\ge 8$

$11)\ \sqrt{(n^2-1)a^2+1}+\sqrt{(n^2-1)b^2+1}+\sqrt{(n^2-1)c^2+1}\le n(a+b+c)\ with\ a+b+c\ge ab+bc+ca$

$\rightarrow\ Special\ case:a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ and\ without\ condition\ abc=1.$

12) $\frac{1}{(a+1)^2}+\frac{1}{(b+1)^2}+\frac{1}{(c+1)^2}\ge\frac{3}{4}$

13) $(a+b+c)^5\ge 81\left(a^2+b^2+c^2\right)$

d.2) If $a,b,c>0$ such that $a+b+c=3$ then:

1) $a^2+b^2+c^2\ge 3\ge ab+bc+ca$

2) $a^m+b^m+c^m\ge 3$ for $m\in Z^+$

3) $\frac{a^m}{b^n}+\frac{b^m}{c^n}+\frac{c^m}{a^n}\ge 3$ for $m,n\in Z^+$ such that $m\ge n$

4) $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\ge a^2+b^2+c^2$

5) $a^{n-1}+b^{n-1}+c^{n-1}\le\frac{a^n+b^n}{a+b}+\frac{b^n+c^n}{b+c}+\frac{c^n+a^n}{c+a}\le a^n+b^n+c^n$ with all $n\in Z^+$

6) $\frac{1}{a^2+b^2+n}+\frac{1}{b^2+c^2+n}+\frac{1}{c^2+a^2+n}\le\frac{3}{n+2}$ with all $n\ge 2$

7) $\frac{a^m}{b^n+c^n}+\frac{b^m}{c^n+a^n}+\frac{c^m}{a^n+b^n}\ge\frac{3}{2}$ with $m,n\in Z^+$ such that $m>n$

8) $abc\left(a^2+b^2+c^2\right)\le 3\to(abc)^n.\left(a^2+b^2+c^2\right)\le 3$ with $n\in Z^+\to\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\ge\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\ge a^2+b^2+c^2$

9) $\frac{1}{a^2+ab+b^2}+\frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2}\ge 1$

10) $a^2b+b^2c+c^2a+abc\le 4$

11) $a^2+b^2+c^2\ge a^2b+b^2c+c^2a$

12*) $\frac{a}{b^2+pc}+\frac{b}{c^2+pa}+\frac{c}{a^2+pb}\ge\frac{3}{1+p}$ with $p\ge 1$

d.3) If $a,b,c>0: ab+bc+ca=3$ then:

1) $a+b+c\ge 3abc\to(a+b+c)^5\ge 243abc$

2) $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge a+b+c$

3) $\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}\ge\frac{3}{2}\to\frac{1}{(a+1)^2}+\frac{1}{(b+1)^2}+\frac{1}{(c+1)^2}\ge\frac{3}{4}$

4) $a^2b+b^2c+c^2a\le a^2+b^2+c^2$

5) $\frac{1}{(a+b)^2+m}+\frac{1}{(b+c)^2+m}+\frac{1}{(c+a)^2+m}\le\frac{3}{4+m}$ with $m\ge 2$

$\to\frac{1}{(xa+yb)^2+m}+\frac{1}{(xb+yc)^2+m}+\frac{1}{(xc+ya)^2+m}\le\frac{3}{(x+y)^2+m}$ with $x,y>0$ and $m\ge 2\left(x^2-xy+y^2\right)$ (?)

6) $\frac{1}{a^2+b^2+k}+\frac{1}{b^2+c^2+k}+\frac{1}{c^2+a^2+k}\le\frac{3}{2+k}$ with $k\ge 1$

7) $\frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2}\le 1$

8*) $\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1}\le 1$

d.4) If $a,b,c>0$ such that $a^2+b^2+c^2=3$ then:

1) $a+b+c\ge ab+bc+ca\to\sqrt[n]{a}+\sqrt[n]{b}+\sqrt[n]{c}\ge ab+bc+ca$ with $n\in Z^+$

2) $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge\frac{9}{a+b+c}\to(a+b+c)^3\ge 9(ab+bc+ca)$

3) $\frac{a}{b+2}+\frac{b}{c+2}+\frac{c}{a+2}\le 1;\ \frac{a}{b+1}+\frac{b}{c+1}+\frac{c}{a+1}\ge\frac{3}{2}$

4) $a^3b+b^3c+c^3a\le 3$

5) $a^2b+b^2c+c^2a\le 2+abc\to$ In case $a,b,c\ge 0$: the equality happens iff $\left[\begin{array}{l}a=b=c=1\\ a=\sqrt{2};\ b=1;\ c=0 \text{ or any cyclic permutation}\end{array}\right.$

d.5) If $a,b,c>0$ such that $a+b+c=ab+bc+ca$ then:

1) $\frac{1}{1+a+b}+\frac{1}{1+b+c}+\frac{1}{1+c+a}\le 1$

2) $a+b+c+abc\ge 4\to\begin{cases}\ast\ a+b+c+1\ge 4abc\\ \ast\ (a+1)(b+1)(c+1)\ge 8\end{cases}$

$3)\ \frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\le\frac{3}{2}$

$4)\ (a+b)(b+c)(c+a)\ge 8$

$5)\ \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\ge a^2+b^2+c^2\rightarrow\ \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+\frac{3n}{a^2+b^2+c^2}\ge 3+n\ \ with\ \ n\le 3.$

$6)\ a+b+c\ge abc+2$

$d.6)\ If\ \ a,b,c>0\ \ such\ that\ \ ab+bc+ca+abc=4\ \ then:$

$1)\ a+b+c\ge ab+bc+ca\ge 3\rightarrow\ \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ge\frac{a+b+c}{2}$

$\rightarrow\ \frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge a+b+c$

$2)\ \frac{a+b+c}{2}\ge\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}\ge\frac{3}{2}\rightarrow\ \frac{a+b+c}{2}\ge\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}$

$3)\ \frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2}\le 1$

$4)\ \sqrt{ab}+\sqrt{bc}+\sqrt{ca}\le 3$

$5)\ \frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge a+b+c$

$6)\ a^2b+b^2c+c^2a\le a^2+b^2+c^2$

$d.7)\ If\ \ a,b,c>0\ \ such\ that\ \ a^2+b^2+c^2+abc=4\ \ then:$

$1)\ ab+bc+ca\le a+b+c\le 3\rightarrow\ ab+bc+ca\le abc+2\le a+b+c\le 3$

$2)\ \frac{3}{2}\le\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\le\frac{9}{2(ab+bc+ca)}$

$3)\ a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)\le 6$

$4)\ \frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}\ge 3\rightarrow\ \frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}\ge a^2+b^2+c^2$

$5)\ a+bc\le a+\left(\frac{b+c}{2}\right)^2\le 2;\ etc$

$6)\ a^2+b^2+c^2+ab+bc+ca\ge 2(a+b+c)\rightarrow\ a+b+c\ge\sqrt{a}+\sqrt{b}+\sqrt{c}\ge ab+bc+ca$

$d.8)\ If\ \ a,b,c>0\ \ such\ that\ \ a^2+b^2+c^2=2(ab+bc+ca)\ \ then:\frac{a+b+c}{3}\ge\sqrt[3]{2abc}$

$\rightarrow\ the\ equality\ happens\ iff\ \ (a;b;c)\sim(k;k;4k)\ \ with\ \ k>0.$

$d.9)\ If\ \ a,b,c>0\ \ such\ that\ \ ab+bc+ca=abc+2\ \ then:$

$1)\ Assume\ that\ \ (b-1)(c-1)\ge 0\ \ then:c+ab\ge 2\rightarrow\ a^2+b^2+c^2+abc\ge 4$

$2)\ Max\{ab;bc;ca\}\ge 1;\ Max\{a;b;c\}\ge 1$

- *Inequalities with classic condition (like a,b,c >0,…) – part 1:*

$d.10)\ If\ \ a,b,c>0\ \ then:\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge\frac{a+b+c}{\sqrt[3]{abc}}$

$d.11)\ If\ \ a_1;a_2;...;a_n>0\ \ then:\frac{1}{1+a_1^n}+\frac{1}{1+a_2^n}+...+\frac{1}{1+a_n^n}\ge\frac{n}{1+a_1a_2...a_n}$

$d.12)\ If\ \ a,b,c\ge 0,\ no\ 2\ of\ \ which\ are\ 0\ then:\frac{a^n+b^n}{a+b}+\frac{b^n+c^n}{b+c}+\frac{c^n+a^n}{c+a}\le 3.\frac{a^n+b^n+c^n}{a+b+c}\ \ with\ \ \forall n\in Z^+$

$d.13)\ If\ \ a,b,c>0\ \ then:\frac{1}{2}+\frac{a^2+b^2+c^2}{ab+bc+ca}\ge\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ge\frac{a^2+b^2+c^2}{2(ab+bc+ca)}+1$

$d.14)\ If\ \ a,b,c>0\ \ then:\frac{a}{(b+c)^2}+\frac{b}{(c+a)^2}+\frac{c}{(a+b)^2}\ge\frac{3}{4}.\frac{a+b+c}{ab+bc+ca}\ge\frac{9}{4(a+b+c)}$

$d.15)\ If\ \ a,b,c,x,y,z>0\ \ then:a(y+z)+b(z+x)+c(x+y)\ge 2\sqrt{(ab+bc+ca)(xy+yz+zx)}$

$d.16)\ If\ \ a,b,c,d\ \ge 0\ \ then:\ a^4+b^4+c^4+d^4+2abcd\ge a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2\ \ (Turkevich's\ inequality)$

$\rightarrow\ The\ equality\ happens\ iff\ \ a=b=c=d\ \ and\ \ a=b=c=k>0,\ d=0\ \ or\ any\ cyclic\ \ permutation.$

$d.17)\ If\ \ a,b,c>0\ \ then:\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}\ge 4\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)$

$d.18)\ If\ \ a,b,c\ are\ sides\ of\ \ a\ triangle\ then:\sqrt{3(a+b+c)}\ge\sqrt{a}+\sqrt{b}+\sqrt{c}\ge\sqrt{b+c-a}+\sqrt{c+a-b}+\sqrt{a+b-c}$

$d.19)$ *If* $a,b,c>0$ *then*:

$$* \frac{a^3}{b^2+bc+c^2}+\frac{b^3}{c^2+ca+a^2}+\frac{c^3}{a^2+ab+b^2}\ge\frac{a^2+b^2+c^2}{a+b+c}\ge\frac{a+b+c}{3}$$

$$* \frac{a^3}{b^2-bc+c^2}+\frac{b^3}{c^2-ca+a^2}+\frac{c^3}{a^2-ab+b^2}\ge a+b+c\ge\frac{3(ab+bc+ca)}{a+b+c}$$

$d.20)$ *If* $a,b,c>0$ *then*: $\frac{a}{b^2+bc+c^2}+\frac{b}{c^2+ca+a^2}+\frac{c}{a^2+ab+b^2}\ge\frac{a+b+c}{ab+bc+ca}\ge\frac{a}{a^2+2bc}+\frac{b}{b^2+2ca}+\frac{c}{c^2+2ab}$

$d.21)$ *If* $a,b,c>0$ *then*: $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge3\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}}$

$d.22)$ *If* $a,b,x,y,z>0$ *then*: $\frac{x}{ay+bz}+\frac{y}{az+bx}+\frac{z}{ax+by}\ge\frac{3}{a+b}$

$d.23*)$ *If* $x\ge0$ *then*: $e^x\ge1+x+\frac{x^2}{2}\ge1+x\to e^x\ge\left(1+\frac{x}{n}\right)^n$

$d.24)$ *If* $a_i>0\ \forall i=1;2;...;n$ *then*: $\sum_{i=1}^{n}\frac{a_i^3}{a_i^2+a_i.a_{i+1}+a_{i+1}^2}\ge\frac{a_1+a_2+...+a_n}{3}$ $(a_{n+1}=a_1)$

$d.25)$ *If* $a,b,c>0$ *then*: $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{2ab}{(a+b)^2}\ge2$

$d.26)$ *If* $a,b,c>0$ *then*: $a^2+b^2+c^2+2abc+1\ge2(ab+bc+ca)\to a^2+b^2+c^2+abc+5\ge3(a+b+c)$

$d.27*)$ *If* a,b,c *are sides of a triangle then*: $a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\ge0$ (*IMO* 1983)

$d.28)$ *If* $a,b,c>0$ *and* $m\in Z^+$ *then*: $\frac{a^m}{b^{m-1}}+\frac{b^m}{c^{m-1}}+\frac{c^m}{a^{m-1}}\ge\frac{a^{m-1}}{b^{m-2}}+\frac{b^{m-1}}{c^{m-2}}+\frac{c^{m-1}}{a^{m-2}}\ge...\ge\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2}\ge\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\ge a+b+c$

$d.29)$ *If* $a,b,c>0$ *then*: $\frac{a}{a+\sqrt{(a+b)(a+c)}}+\frac{b}{b+\sqrt{(b+c)(b+a)}}+\frac{c}{c+\sqrt{(c+a)(c+b)}}\le1$

$d.30)$ *If* $a,b,c>0$ *then*: $\sqrt{\frac{a}{a+b}}+\sqrt{\frac{b}{b+c}}+\sqrt{\frac{c}{c+a}}\le\frac{3}{\sqrt{2}}$

$d.31)$ *If* a,b,c *are sides of a triangle then*: $a^4+b^4+c^4\le2\left(a^2b^2+b^2c^2+c^2a^2\right)$ (*the equality happens iff* a,b,c *are sides of a degenerate triangle.*)

$d.32*)$ *If* $a,b,c>0$ *and* $k\ge0$ *then*: $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge\frac{ka+b}{ka+c}+\frac{kb+c}{kb+a}+\frac{kc+a}{kc+b}$

$d.33)$ *If* $a,b,c>0$ *then*: $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{8abc}{(a+b)(b+c)(c+a)}\ge4$

$d.34)$ *If* $a,b,c\ge0$, *no 2 of which are 0 then*: $\frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{8abc}{(a+b)(b+c)(c+a)}\ge2$ (*Jack Garfunkel's inequality*)

- *Inequalities with classic condition – part 2: In this part I will show some inequalities about variables p,q,r (denote p = a+b+c; q = ab+bc+ca; r = abc), even Schur's inequality.*
  Firstly, we have some identities about p,q,r:

1) $a^2+b^2+c^2=p^2-2q$

2) $a^2b^2+b^2c^2+c^2a^2=q^2-2pr$

3) $(a+b)(b+c)(c+a)=pq-r\to ab(a+b)+bc(b+c)+ca(c+a)=pq-3r$

4) $(a+b)(a+c)+(b+c)(b+a)+(c+a)(c+b)=p^2+q$

5) $a^3+b^3+c^3=p^3-3pq+3r$

6) $a^3b^3+b^3c^3+c^3a^3=q^3-3pqr+3r^2$

7) $a^4+b^4+c^4=p^4-4p^2q+2q^2+4pr$

8) $ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)=p^2q-2q^2-pr$

9) *Denote* $S=-4p^3r+p^2q^2+18pqr-4q^3-27r^2$, *then*:

$* \left[(a-b)(b-c)(c-a)\right]^2=S$

$* a^2b+b^2c+c^2a=\frac{pq-3r-\sqrt{S}}{2}$ *if* $(a-b)(b-c)(c-a)<0$; $\frac{pq-3r+\sqrt{S}}{2}$ *if* $(a-b)(b-c)(c-a)\ge0$

$* a^3b+b^3c+c^3a=\frac{p^2q-2q^2-pr-p\sqrt{S}}{2}$ *if* $(a-b)(b-c)(c-a)<0$; $\frac{p^2q-2q^2-pr+p\sqrt{S}}{2}$ *if* $(a-b)(b-c)(c-a)\ge0$

10) $a^2b^2(a^2+b^2)+b^2c^2(b^2+c^2)+c^2a^2(c^2+a^2)=-2p^3r+p^2q^2+4pqr-2q^3-3r^2$

11) $ab(a^4+b^4)+bc(b^4+c^4)+ca(c^4+a^4)=p^4q-p^3r-4p^2q^2+7pqr+2q^3-3r^2$

12) $a^6+b^6+c^6=p^6-6p^4q+6p^3r+9p^2q^2-12pqr-2q^3+3r^2$

- *Some inequalities about the relation of p,q,r:*

1) $p^2 \ge 3q;\ q^2 \ge 3pr$

2) $pq \ge 9r$

$\to\ p^3 \ge \frac{27}{8}(pq - r) \ge 3pq \ge \frac{9q^2}{p} \ge 27r$

3) $p^3 + 9r \ge 4pq \to r \ge \frac{p(4q - p^2)}{9} \to r \ge \max\left\{0; \frac{p(4q - p^2)}{9}\right\}$

4) $2p^3 + 9r \ge 7pq$

5) $p^2q + 3pr \ge 4q^2$

6) $p^4 + 4q^2 + 6pr \ge 5p^2q$

7) $r \ge \frac{(4q - p^2)(p^2 - q)}{6p} \to r \ge \max\left\{0; \frac{(4q - p^2)(p^2 - q)}{6p}\right\}$

8) $r \le \frac{p(5q - p^2)}{18};\ r \le \frac{p^4 - 7p^2q + 13q^2}{9p}$

Combining inequality 3, 6, 8 *and we get*: $\min\left\{\frac{p(5q - p^2)}{18}; \frac{p^4 - 7p^2q + 13q^2}{9p}\right\} \ge r \ge \max\left\{0; \frac{p(4q - p^2)}{9}; \frac{(4q - p^2)(p^2 - q)}{6p}\right\}$

9*) $\frac{9pq - 2p^3 - 2k\sqrt{k}}{27} \le r \le \frac{9pq - 2p^3 + 2k\sqrt{k}}{27}$, *with* $k = p^2 - 3q$. *This result comes from solving the inequality* $S = -4p^3r + p^2q^2 + 18pqr - 4q^3 - 27r^2 \ge 0$ *with variable r.* $\to$ *From this result, by AM - GM we have:*

$$\ast\ 27r \ge 9pq - 2p^3 - 2k\sqrt{k} = 9pq - 2p^3 - \frac{2(p^2 - 3q)(p^2 - 2q)\sqrt{p^2\left(p^2 - 3q\right)}}{p(p^2 - 2q)}$$

$$\ge \frac{p^2(9q - 2p^2)(p^2 - 2q) - (p^2 - 3q)\left[\left(p^2 - 2q\right)^2 + p^2\left(p^2 - 3q\right)\right]}{p(p^2 - 2q)} = \frac{(p^2 - 2q)\left(-3p^4 + 14p^2q - 6q^2\right) - p^2(p^2 - 3q)^2}{p(p^2 - 2q)}$$

$$= \frac{(4q - p^2)\left(4p^4 - 10p^2q + 3q^2\right)}{p(p^2 - 2q)}$$

$$\ast\ 27r \le 9pq - 2p^3 + 2k\sqrt{k} = 9pq - 2p^3 + \frac{2(p^2 - 3q)\left(p^2 - \frac{3}{2}q\right)\sqrt{p^2\left(p^2 - 3q\right)}}{p\left(p^2 - \frac{3}{2}q\right)}$$

$$\le \frac{p\left(9pq - 2p^3\right)\left(2p^2 - 3q\right) + 2(p^2 - 3q)\left[\left(p^2 - \frac{3}{2}q\right)^2 + p^2\left(p^2 - 3q\right)\right]}{p\left(2p^2 - 3q\right)} = \frac{\left(2p^2 - 3q\right)\left[2(9pq - 2p^3) + (p^2 - 3q)(2p^2 - 3q)\right] + 4p^2\left(p^2 - 3q\right)^2}{2p\left(2p^2 - 3q\right)}$$

$$= \frac{\left(2p^2 - 3q\right)\left(-2p^4 + 9p^2q + 9q^2\right) + 4p^2\left(p^4 - 6p^2q + 9q^2\right)}{2p\left(2p^2 - 3q\right)} = \frac{27q^2(p^2 - q)}{2p\left(2p^2 - 3q\right)}$$

So we have the chain: $\frac{q^2}{3p} \ge \frac{q^2(p^2 - q)}{2p\left(2p^2 - 3q\right)} \ge r \ge \frac{(4q - p^2)\left(4p^4 - 10p^2q + 3q^2\right)}{27p(p^2 - 2q)}$, *more interesting, the third inequality is stronger than Schur deg* 3 *and* 4, *since*:

$$\frac{(4q - p^2)\left(4p^4 - 10p^2q + 3q^2\right)}{27p(p^2 - 2q)} - \frac{p(4q - p^2)}{9} = \frac{(4q - p^2)(p^2 - 3q)(p^2 - q)}{27p(p^2 - 2q)} \ge 0 \text{ in case } 4q \ge p^2$$

$$\frac{(4q - p^2)\left(4p^4 - 10p^2q + 3q^2\right)}{27p(p^2 - 2q)} - \frac{(4q - p^2)(p^2 - q)}{6p} = \frac{(4q - p^2)^2(p^2 - 3q)}{54p(p^2 - 2q)} \ge 0$$

Hence we can also conclude that: $r \ge \max\left\{0; \frac{(4q - p^2)\left(4p^4 - 10p^2q + 3q^2\right)}{27p(p^2 - 2q)}\right\}$

Further more, we can write inequality 9 *as*: $\frac{p^3 - 3pk - 2k\sqrt{k}}{27} \le r \le \frac{p^3 - 3pk + 2k\sqrt{k}}{27}$

$$\Leftrightarrow \frac{\left(p + \sqrt{k}\right)^2\left(p - 2\sqrt{k}\right)}{27} \le r \le \frac{\left(p - \sqrt{k}\right)^2\left(p + 2\sqrt{k}\right)}{27}$$

For my thinking, the p,q,r technique is one of the nicest and hardest techniques to use, due to long calculations, time loss and it requires carefulness in calculation. But the nice thing to say is its benefit in solving hard inequalty about symmetric and cyclic expressions. There is a similar technique to p,q,r, that is u,v,w technique (can be found and search in inequality sites and forums)