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ROMANIAN MATHEMATICAL MAGAZINE
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SP.134. *Let $ABCD$ be a cyclic quadrilateral with perimeter 2.*

Denote $AB = a, BC = b, CD = c, DA = d$.

Prove that:

$$4 \le \sum \tan\frac{A}{2} < \frac{2(a+c)(b+d)}{\sqrt{\prod(1-a)}}.$$

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Solution by proposer

The function $f:(0,1)\to(0,\infty), f(x) = \sqrt{\frac{1-x}{(1-a)(1-b)(1-c)(1-d)}}$ *is concave.*

We know that $\tan\frac{A}{2} = \sqrt{\frac{(1-a)(1-d)}{(1-c)(1-b)}} = f(1-[(1-a)(1-d)]^2)$*. Applying Jensen* $\to$

$$\sum \tan\frac{A}{2} \le 4f\left(\frac{4-\sum[(1-a)(1-d)]^2}{4}\right) = 2\sqrt{\frac{\sum[(1-a)(1-d)]^2}{(1-a)(1-b)(1-c)(1-d)}}$$

But $\sum[(1-a)(1-d)]^2 < [\sum(1-a)(1-d)]^2 = [(a+c)(b+d)]^2$.

So,

$$\sum \tan\frac{A}{2} < \frac{2(a+c)(b+d)}{\sqrt{\prod(1-a)}}.$$

Now,

$$\sum \tan\frac{A}{2} = \sqrt{\frac{(1-a)(1-d)}{(1-c)(1-b)}} + \sqrt{\frac{(1-b)(1-c)}{(1-a)(1-d)}} + \sqrt{\frac{(1-c)(1-d)}{(1-a)(1-b)}} + \sqrt{\frac{(1-a)(1-b)}{(1-c)(1-d)}},$$

Which is bigger or equal than 4*. Using this* $\to$ *q.e.d.*