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SP.015. Prove that if $a, b, c \in [0, \infty)$ then:

$$25\sum a^2\sqrt[5]{a} + 11\sum ab^6 \geq 33\sum a^2b$$

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Solution by proposer

By Young's inequality:

$$px^q + qx^p \geq pqxy; p > 1; \frac{1}{p} + \frac{1}{q} = 1; x, y \geq 0$$

$$\frac{1}{q} = 1 - \frac{1}{p} \Rightarrow \frac{1}{q} = \frac{p-1}{p} \Rightarrow q = \frac{p}{p-1}$$

$$px^{\frac{p}{p-1}} + \frac{p}{p-1}y^p \geq \frac{p^2}{p-1}xy$$

$$p\int_0^x x^{\frac{p}{p-1}}dx + \frac{p}{p-1}y^p\int_0^x dx \geq \frac{p^2}{p-1}y\int_0^x x\,dx$$

$$p\frac{x^{\frac{p}{p-1}+1}}{\frac{p}{p-1}+1} + \frac{p}{p-1}y^px \geq \frac{p^2}{p-1}y \cdot \frac{x^2}{2}$$

$$\frac{x^{\frac{2p-1}{p-1}}}{\frac{2p-1}{p-1}} + \frac{1}{p-1}xy^p \geq \frac{p}{2(p-1)}x^2y$$

For $p = 6, x = a, y = b$:

$$\frac{x^{\frac{11}{5}}}{\frac{11}{5}} + \frac{1}{5}xy^6 \geq \frac{6}{10}x^2y \to \frac{5}{11}a^{\frac{11}{5}} + \frac{1}{5}ab^6 \geq \frac{3}{5}a^2b$$

$$\frac{5}{11}\sum a^{\frac{11}{5}} + \frac{1}{5}\sum ab^6 \geq \frac{3}{5}\sum a^2b$$

$$25\sum a^2\sqrt[5]{a} + 11\sum ab^6 \geq 33\sum a^2b$$