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Problem: *if* $x_k \geq 1, \forall k = \overline{1;n}$ $(n \in N^* \wedge n \geq 2)$ and $a \in (0;1)$ then:

$$\sum_{k=1}^{n}([x_k])^a \leq n^{1-a} \quad ([\sum_{k=1}^{n} x_k])^a$$

($[x]$ is the integer part of the real number x)

Proof.

$$f(x) = x^a \to f''(x) = a(a-1)x^{a-2} \leq 0, \forall x \geq 0$$

$$\to LHS = \sum_{k=1}^{n} f([x_k]) \overset{Jensen}{\leq} n.f\left(\frac{\sum_{k=1}^{n}[x_k]}{n}\right) = n^{1-a}(\sum_{k=1}^{n}[x_k])^a \to (1)$$

We have: $[x+y] \geq [x]+[y], \forall x, y \in (0;+\infty) \to (*)$

Let: $\sum_{k=1}^{n-1}[x_k] \leq [\sum_{k=1}^{n-1} x_k]$

$$\to \sum_{k=1}^{n}[x_k] = \sum_{k=1}^{n-1}[x_k] + [x_n] \overset{Use(*)}{\leq} [\sum_{k=1}^{n} x_k] \to (2)$$

From (1) *and* (2) $\to LHS \leq n^{1-a} \quad ([\sum_{k=1}^{n} x_k])^a = RHS$ (done)

Note. Some beautiful results from inequality

$+$ *if* $n = 2,\ a = \frac{1}{2}$ *then* $\sqrt{[x_1]} + \sqrt{[x_2]} \leq \sqrt{2}.\sqrt{[x_1 + x_2]}, \forall x_1, x_2 \in [1;+\infty)$.

$+$*if* $n = 3,\ a = \frac{1}{2}$ *then* $\sqrt{[x_1]} + \sqrt{[x_2]} + \sqrt{[x_3]} \leq \sqrt{3}.\sqrt{[x_1 + x_2 + x_3]}, \forall x_1, x_2, x_3 \in [1;+\infty)$.

$+$ *if* $n = 2,\ a = \frac{1}{3}$ *then* $\sqrt[3]{[x_1]} + \sqrt[3]{[x_2]} \leq \sqrt[3]{4}\sqrt[3]{[x_1 + x_2]}, \forall x_1, x_2 \in [1;+\infty)$.

$+$ *if* $n = 3,\ a = \frac{1}{3}$ *then* $\sqrt[3]{[x_1]} + \sqrt[3]{[x_2]} + \sqrt[3]{[x_3]} \leq \sqrt[3]{9}\sqrt[3]{[x_1 + x_2 + x_3]}, \forall x_1, x_2, x_3 \in [1;+\infty)$.