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$$\texttt{Find}\quad \Omega : \int_0^\infty \frac{\arctan x}{1+x^2+x^4}dx$$

Solution by Burghelea Zaharia and Khalef Ruhemi

$$I=\int_0^1 \frac{\arctan x}{1+x^2+x^4}\mathrm{d}x+\int_1^\infty \frac{\arctan x}{1+x^2+x^4}\mathrm{d}x$$

In the second integral we substitute $x=\frac{1}{y}$ and make use of $\arctan\left(\frac{1}{x}\right)=\frac{\pi}{2}-\arctan x$, which is valid for any $x>0$

$$I=\int_0^1 \frac{\arctan x+x^2\arctan\left(\frac{1}{x}\right)}{1+x^2+x^4}\mathrm{d}x=\int_0^1 \frac{(1-x^2)\arctan x}{1+x^2+x^4}\mathrm{d}x+\frac{\pi}{2}\int_0^1 \frac{x^2}{1+x^2+x^4}\mathrm{d}x$$

We also have:

$$\int \frac{1-x^2}{1+x^2+x^4}\mathrm{d}x=\int \frac{\frac{1}{x^2}-1}{\frac{1}{x^2}+1+x^2}\mathrm{d}x=\int \frac{\left(x+\frac{1}{x}\right)'}{1-\left(x+\frac{1}{x}\right)^2}\mathrm{d}x=\frac{1}{2}\ln\left(\frac{1+x+x^2}{1-x+x^2}\right)+C_2$$

Therefore we integrate by parts the first integral:

$$I=\frac{1}{2}\ln\left(\frac{1+x+x^2}{1-x+x^2}\right)\arctan x\Big|_0^1-\frac{1}{2}\int_0^1\ln\left(\frac{1+x+x^2}{1-x+x^2}\right)\frac{dx}{1+x^2}+\frac{\pi}{2}\int_0^1\frac{x^2}{1+x^2+x^4}\mathrm{d}x$$

$$=\frac{1}{2}\ln 3\frac{\pi}{4}-\frac{1}{2}\int_0^1\frac{\ln(1-x^3)-\ln(1-x)-\ln(1+x^3)+\ln(1+x)}{1+x^2}\mathrm{d}x+\frac{\pi}{2}\frac{\sqrt{3}\pi-3\ln 3}{12}$$

$$=\frac{\pi^2}{8\sqrt{3}}-\frac{1}{2}\left(\frac{\pi}{6}\ln(2+\sqrt{3})+\frac{\pi}{8}\ln 2-\frac{4}{3}G-\frac{\pi}{8}\ln 2+G-\frac{\pi}{3}\ln(2+\sqrt{3})-\frac{\pi}{8}\ln 2+\frac{5}{3}G+\frac{\pi}{8}\ln 2\right)$$

$$\Omega=\int_0^\infty\frac{\arctan x}{1+x^2+x^4}\mathrm{d}x=\frac{\pi^2}{8\sqrt{3}}+\frac{\pi}{12}\ln(2+\sqrt{3})-\frac{2}{3}G$$

Where G is Catalan's constant, $G=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$

Proofs:

$$\int_0^1\frac{x^2}{1+x^2+x^4}dx\ =\frac{1}{2}\int_0^1\frac{2}{x^2+\frac{1}{x^2}+1}dx$$

$$\frac{1}{2}\int_0^1\frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-1}dx+\frac{1}{2}\int_0^1\frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+3}dx$$

$$=\frac{1}{4}\ln\left(\frac{x^2-x+1}{x^2+x+1}\right)\Big|_0^1+\frac{1}{2\sqrt{3}}\arctan\left(\frac{x-\frac{1}{x}}{\sqrt{3}}\right)\Big|_0^1$$

$$\int_0^1\frac{x^2}{1+x^2+x^4}dx=\frac{\sqrt{3}\pi-3\ln 3}{12}$$

Let $K = \int_0^1 \frac{\ln(1-x)}{1+x^2}dx$ then substitute $x = \frac{1}{y}$:

$$K = \int_1^\infty \frac{\ln(x-1)}{1+x^2}dx - \int_1^\infty \frac{\ln x}{1+x^2}dx$$

In the first integral put $x = \tan y$ and then use $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$$K_1 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln(\tan\left(\frac{\pi}{4} + \frac{\pi}{2} - y\right) - 1)dy = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\left(\frac{2}{\tan x - 1}\right) dx$$

$$2K_1 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln 2dx \to K_1 = \frac{\pi}{8}\ln 2$$

Since $\int_1^\infty \frac{\ln x}{1+x^2}dx = G \to K = \frac{\pi}{8}\ln 2 - G$

$$J = \int_0^1 \frac{\ln(1+x)}{1+x^2}dx,\ x = \frac{1-t}{1+t} \to t = \frac{1-x}{1+x} \to dx = -\frac{2}{(1+x)^2}\,dt$$

$$J = \int_0^1 \frac{\ln 2 - \ln(1+t)}{1+t^2}dt \to 2J = \ln 2\int_0^1 \frac{dx}{1+x^2} \to J = \frac{\pi}{8}\ln 2$$

Let $I_2 = \frac{\ln(x^3+1)}{1+x^2}dx$ and substitute $x = \frac{1}{y}$:

$$I_2 = \int_1^\infty \frac{\ln(y^3+1) - \ln(y^3)}{1+y^2}dy = \frac{1}{2}\left(\int_0^\infty \frac{\ln(1+x^3)}{1+x^2}dx - 3\int_1^\infty \frac{\ln x}{1+x^2}dx\right)$$

(1) $\quad I_2 = \frac{1}{2}I(1) - \frac{3}{2}G$ Where:

$$I(t) = \int_0^\infty \frac{\ln(t+x^3)}{1+x^2}dx \to I'(t) = \int_0^\infty \frac{dx}{(t+x^3)(1+x^2)}$$

Using partial fractions:

$$I'(t) = \frac{1}{1+t^2}\left(\int_0^\infty \left(\frac{x}{1+x^2} - \frac{x^2}{t+x^3}\right)dx + t\int_0^\infty \frac{1}{1+x^2}dx - t\int_0^\infty \frac{x}{t+x^3}dx + \int_0^\infty \frac{dx}{t+x^3}\right)$$

In the last two integrals substitute $x^3 = ty$

$$I'(t) = \frac{1}{1+t^2}\left(\ln\left(\frac{\sqrt{1+x^2}}{\sqrt[3]{t+x^3}}\right)\Bigg|_0^\infty + \frac{\pi}{2}t + \frac{1}{3}t^{-2/3}\int_0^\infty \frac{y^{-2/3}}{1+y}dy - \frac{1}{3}t^{2/3}\int_0^\infty \frac{y^{-1/3}}{1+y}dy\right)$$

Using the identity $\int_0^\infty \frac{x^{a-1}}{1+x}dx = \pi\csc(a\pi)$ results in:

$$I'(t) = \frac{1}{1+t^2}\left(\frac{1}{3}\ln t + \frac{\pi}{2}t + \frac{2\pi}{3\sqrt{3}}(t^{-2/3} - t^{2/3})\right)$$

Since $I(0) = 3\int_0^\infty \frac{\ln x}{1+x^2}dx$ with $x = \frac{1}{y}$ gives $I(0) = -3\int_0^\infty \frac{\ln y}{1+y^2}dy \to I(0) = 0$
Therefore $I(1) - I(0) = \int_0^\infty \frac{\ln(1+x^3)}{1+x^2}dx$, but $I(1) - I(0) = \int_0^1 I'(t)dt$

$$I(1) = \frac{1}{3}\int_0^1 \frac{\ln t}{1+t^2}dt + \frac{\pi}{2}\int_0^1 \frac{t}{1+t^2}dt + \frac{2\pi}{3\sqrt{3}}\int_0^1 \frac{t^{-2/3} - t^{2/3}}{1+t^2}dt$$

$$I(1) = -\frac{1}{3}G + \frac{\pi}{4}\ln 2 + \frac{2\pi}{3}\ln(2+\sqrt{3})$$

Combining the result with (1) gives:

$$I_2 = \int_0^1 \frac{\ln(1+x^3)}{1+x^2}dx = \frac{\pi}{3}\ln(2+\sqrt{3}) + \frac{\pi}{8}\ln 2 - \frac{5}{3}G$$

Let $I_3 = \int_0^1 \frac{\ln(1-x^3)}{1+x^2}dx$ making two substitution $x = -\frac{1}{y}$ and $x = -y$

$$2I_3 = \int_{-\infty}^{0} \frac{\ln(1+y^3)}{1+y^2}dy - \int_{-\infty}^{-1} \frac{\ln y^3}{1+y^2}dy$$

$$\begin{aligned} I_3 &= \frac{1}{2}\left(\int_{-\infty}^{\infty} \frac{\ln(1+x^3)}{1+x^2}dx - \int_0^{\infty} \frac{\ln(1+x^3)}{1+x^2}dx - 3\int_1^{\infty} \frac{\ln x}{1+x^2}dx - i\pi\int_1^{\infty} \frac{1}{1+x^2}dx\right) \\ &= \frac{1}{2}\left(X(1) - I(1) - 3G - i\frac{\pi^2}{4}\right) \end{aligned}$$

Where $X(t) = \int_{-\infty}^{\infty} \frac{\ln(1+x^3)}{1+x^2}dx$

$$X'(t) = \frac{1}{1+t^2}\left(\ln\left(\frac{\sqrt{1+x^2}}{\sqrt[3]{t+x^3}}\right)\Bigg|_{-\infty}^{\infty} + \pi t + \frac{2}{3}t^{-2/3}\int_0^{\infty} \frac{t^{-2/3}}{1+t}dt - \frac{2}{3}t^{2/3}\int_0^{\infty} \frac{t^{-1/3}}{1+t}dt\right)$$

With the same approach from $I(1)$ results in:

$$I_3 = \frac{1}{2}\left(i\frac{\pi^2}{4} + \frac{\pi}{2}\ln 2 + \pi\ln(2+\sqrt{3}) + \frac{1}{3}G - \frac{\pi}{4}\ln 2 - \frac{2\pi}{3}\ln(2+\sqrt{3}) - 3G - i\frac{\pi^2}{4}\right)$$

$$I_3 = \int_0^1 \frac{\ln(1-x^3)}{1+x^2}dx = \frac{\pi}{6}\ln(2+\sqrt{3}) + \frac{\pi}{8}\ln 2 - \frac{4}{3}G$$