The derivative of trigonometric inverse functions

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Abstract: As we know from the basic elementary theory, the trigonometric inverse functions are y = arcsin x, y = arccos x, y = arctg x and y = arcctg x. These functions by different authors are also called cyclometric functions, and some are also called arc functions.
So far, in the literature of derivation theory, the derivative of these functions is used to represent the formula that shows the derivative of the inverse function in general. So the formula is used:

$$y'_x = \frac{1}{x'_y}$$

Following this work, we will introduce a new method by which we will find the derivative of the functions in question by using the function derivation, definition in its entire range.

Keyword: Derivate, inverse trigonometric function, Limit.

1. Introduction and preliminaries

Definition: The derivative of the function $y = f(x)$ at the salient point $x \in D_f$ is called the limit: $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$, if this limit exist and it is finite. Thus

$f'(x)$=$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$ (*)

As we know, to find the derivative of the function $y = f(x)$ at the salient point $x \in D_f$ according to the formula (*), these actions should be carried out:

1. Calculate the addition Δy function that matches the Δx add-on argument.

Therefore, is found Δy= $f(x+\Delta x) - f(x)$;

2. The addendum of report function with the addendum of argument are found: $\frac{\Delta y}{\Delta x}$
3. The value of limit of this report is found $\Delta x \to 0$; f'(x)= $\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$

Theorem (1): If it is given: f (x) = arc sin (x), then the value is equal

$f'(x) = \frac{1}{\sqrt{1-x^2}}$, $|x| < 1$

Proof: To proof the theorem (1), we use the equation (*) and we will have:

$f'(x)$=$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$, or for our case:

$f'(x)$=$\lim_{\Delta x\to 0}\frac{arcsin(x+\Delta x)-arcsin(x)}{\Delta x}$, (a_1)

Using the formula:

arc sin A-arc sin B= arc sin(A$\sqrt{1-B^2}$*-B*$\sqrt{1-A^2}$*),* the fraction numerator to the expression under the limit sign of (a_1) shall be:

$arcsin(x+\Delta x) - arcsinx$ = arcsin $((x+\Delta x)\sqrt{1-x^2}$-$x\sqrt{1-(x+\Delta x)^2})$

When the last equation is subordinated to (a_1), we will have

$f'(x)$=$\lim_{\Delta x\to 0}\frac{\text{arcsin }((x+\Delta x)\sqrt{1-x^2}-x\sqrt{1-(x+\Delta x)^2}}{\Delta x}$, (a_2)

The limit on the right side of the equation (a_2) when $\Delta x \to 0$ represent an indeterminant form

$\frac{0}{0}$. to be released from this indeterminant form, we use that $\lim_{x\to a}\frac{arcsin g(x)}{g(x)}$=1, under the condition that $\lim_{x\to a} g(x)$ =0 … (I)

So in the equation (a_2)) we act in this manner:

$$f'(x)=\lim_{\Delta x\to 0}\frac{\arcsin\left((x+\Delta x)\sqrt{1-x^2}-x\sqrt{1-(x+\Delta x)^2}\right)}{(x+\Delta x)\sqrt{1-x^2}-x\sqrt{1-(x+\Delta x)^2}}.\lim_{\Delta x\to 0}\frac{(x+\Delta x)\sqrt{1-x^2}-x\sqrt{1-(x+\Delta x)^2}}{\Delta x}, \qquad (a_3)$$

The first factor on the right hand side of the equation (a_3), according to (I) is equal to 1.

By using this fact, (a_3) takes the form:

$$f'(x)=\lim_{\Delta x\to 0}\frac{(x+\Delta x)\sqrt{1-x^2}-x\sqrt{1-(x+\Delta x)^2}}{\Delta x}, \qquad (a_4)$$

As we know in (a_4), when $\Delta x\to 0$, the right side of the equation will be an indefinite form $\frac{0}{0}$. Since the right side of (a_4) is the limit of irrational expression, we to first eliminate the indeterminant form there, we must first rationalize the fractional numerator under the sign of the limit in question. In this manner we have:

$$f'(x)=\lim_{\Delta x\to 0}\frac{1}{(x+\Delta x)\sqrt{1-x^2}+x\sqrt{1-(x+\Delta x)^2}}.\lim_{\Delta x\to 0}\frac{(x+\Delta x)^2(1-x^2)-x^2(1-(x+\Delta x)^2)}{\Delta x}, \quad (a_5).$$

When $\Delta x\to 0$ in the first limit in (a_5) will be gained $\frac{1}{2x\sqrt{1-x^2}}$, while after adjusting the fractional numerator under the expression of the second limit of (a_5) we will have:

$$f'(x)=\frac{1}{2x\sqrt{1-x^2}}\lim_{\Delta x\to 0}\frac{2x\Delta x+\Delta x^2}{\Delta x}, \qquad (a_6)$$

The limit on the right side of the equation (a_6) as the limit of rational expression will be equal to $2x$, and in this way the equation (a_6) will take the form:

$$f'(x)=\frac{1}{2x\sqrt{1-x^2}}.2x, \qquad (a_7)$$

After adjusting on the right side of the equation (a_7) we will have:

$$f'(x)=\frac{1}{\sqrt{1-x^2}}, \qquad (a_8)$$

This presents the proof of the theorem (1).

So, according to theorem (1), it is: $(arcsinx)'=\frac{1}{\sqrt{1-x^2}}$

Theorem (2): If it is given: f (x) =$arccosx$, then the value is equal

f'(x) =$-\frac{1}{\sqrt{1-x^2}}$, $|x| < 1$

Proof: To proof the theorem (2), we use the equation (*) and we will have:

$f'(x)$ =$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$, or in our case

$$f'(x) = \frac{arccos(x+\Delta x)-arccosx}{\Delta x}, \qquad (b_1)$$

Using formula: *arccos A*- *arccos B*= *arccos (AB*+$\sqrt{(1-A^2)(1-B^2)}$)), fraction numerator to the expression under the limit sign of (b_1) will be:

$$arccos(x+\Delta x) - arccosx = \arccos\left(x(x+\Delta x) + \sqrt{(1-x^2)(1-(x+\Delta x)^2}\right).$$

When replacing the last equation in (b_1), we will have:

$$f'(x) = \lim_{\Delta x\to 0}\frac{\arccos\left(x(x+\Delta x)+\sqrt{(1-x^2)(1-(x+\Delta x)^2}\right)}{\Delta x}, \qquad (b_2)$$

Limit on the right side of the equation (b_2) when $\Delta x \to 0$, represent an indeterminant form $\frac{0}{0}$.

To be released from this indeterminant form, we get the substitution:

$$\arccos\left(x(x+\Delta x) + \sqrt{(1-x^2)(1-(x+\Delta x)^2}\right) = t \qquad (b_3)$$

From where we will get it:

$$x(x+\Delta x) + \sqrt{(1-x^2)(1-(x+\Delta x)^2)} = cost \qquad (b_4)$$

From equation (b_4), Δx should be found. That's why we do so

$$\sqrt{(1-x^2)(1-(x+\Delta x)^2} = cost - x^2 - x\Delta x \qquad (b_5)$$

When we equalize (b_5) side by side we rise in square, we will get:

$$(1-x^2)(1-(x+\Delta x)^2) = cos^2t + x^4 + x^2\Delta^2x - 2x^2cost - 2x\Delta xcost + 2x^3\Delta x, \quad (b_6)$$

By performing the necessary elementary actions in the equation(b_6), we get:

$$\Delta^2x + 2x(1-cost)\Delta x + 2x^2(1-cost) - sin^2t = 0 \qquad (b_7).$$

As seen in (b_7) a quadratic equation is obtained according to the Δx variable, whose solutions are:

$$(b_8) \quad \begin{cases} \Delta x = -x + x\text{cost} + \text{sint}\sqrt{1-x^2} \\ \Delta x = -x + x\text{cost} - \text{sint}\sqrt{1-x^2} \end{cases}, \text{ and}$$

From the equation (b_8), we get the second solution for expression with which is equal toΔx.

When $\Delta x \to 0$ from (b_3), t→0 by replacing all of these data in (b_2), us will have:

$$f'(x) = \lim_{\Delta x\to 0} \frac{t}{(-x+x\text{cost})-\text{sint}\sqrt{1-x^2}} \qquad (b_9)$$

Dividing it with t the numerator and the denominator of the fraction that it is in the expression under the right side of the limit (b_9) and by adjusting it, we will get:

$$f'(x) = \lim_{t\to 0} \frac{1}{\frac{cost-1}{x}x - \frac{sint}{t}\sqrt{1-x^2}} \qquad (b_{10})$$

By using the condition of the function limits, the equation(b_{10}), will have the form:

$$f'(x) = \frac{1}{-x\lim\limits_{t\to 0}\frac{1-cost}{t} - \sqrt{1-x^2}\lim\limits_{t\to 0}\frac{sint}{t}} \qquad (b_{11})$$

Since $\lim\limits_{t\to 0}\frac{1-cost}{t} = 0, and \lim\limits_{t\to 0}\frac{sint}{t} = 1$, at the equation (b_{11}) we will get:

$$f'(x) = -\frac{1}{\sqrt{1-x^2}} \qquad (b_{12})$$

Equation (b_{12}) represents proof of the theorem (2).

So according to the theorem (2), is equal to:

$(arccosx)' = -\frac{1}{\sqrt{1-x^2}}$.

Theorem (3).If it is given f (x) =$arctgx$, then the value is equal

f'(x) =$\frac{1}{1+x^2}$.

Proof: To proof theorem (3), we use the equation (*) and we will have:

$f'(x)$ =$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$, or in our case

$f'(x) = \frac{arctg(x+\Delta x)-arctgx}{\Delta x}$, (c_1)

By using formula$arctgA - arctgB = arctg\frac{A-B}{1+AB}$, fraction numerator to the expression under the limit sign of (c_1) will be:

$$arctg(x+\Delta x) - arctgx = arctg\frac{\Delta x}{1+x(x+\Delta x)}$$

When last equation is substituted in (c_1),we will get:

$f'(x)$ =$\lim_{\Delta x\to 0}\frac{arctg\frac{\Delta x}{1+x(x+\Delta x)}}{\Delta x}$ ...(c_2)

Limit on the right side of the equation (c_2), when $\Delta x \to 0$, represent an indeterminant form$\frac{0}{0}$.

To be released from this indefinite form, we will get the substitution:

$arctg\frac{\Delta x}{1+x(x+\Delta x)} = t$ (c_3)

From where we get it

$\frac{\Delta x}{1+x(x+\Delta x)} = tgt$ (c_4)

Δx from equation (c_4) will be equal to:

$\Delta x = \frac{(1+x^2)tgt}{1-xtgt}$ (c_5)

When $\Delta x \to 0$,from equation (c_3), $t\to 0$ by replacing all of these data in (c_2), us will have:

$f'(x)$ =$\lim_{t\to 0}\frac{t}{\frac{(1+x^2)tgt}{1-xtgt}}$ (c_6)

By adjusting the expression under the sign of the limit on the right side of the equation (c_6) we will get:

$f'(x)$ =$\lim_{t\to 0}\frac{t(1-xtgt)}{(1+x^2)tgt}$ (c_7)

By using the features of the function limits, the equation(c_7), we will have the form:

$f'(x)$=$\frac{1}{1+x^2}\lim_{t\to 0}\frac{t}{tgt}\lim_{t\to 0}(1-xtgt)$ (c_8)

Since $\lim_{t\to 0}\frac{t}{tgt} = 1$ and $\lim_{t\to 0}(1-xtgt) = 1$, equation (c_8) will be transformed to the following: $f'(x) = \frac{1}{1+x^2}$ (c_9)

By (c_9) we have proofed the theorem (3). So according to theorem (3), we have:

$(arctgx)'=\frac{1}{1+x^2}$

Theorem (4): If it is given f (x) =$arcctgx$, then the value is equal

f'(x) =$-\frac{1}{1+x^2}$

Proof: To proof theorem (4), we use the equation (*) and we will have:

$f'(x)$ =$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$, or in our case

$$f'(x)=\frac{arcctg(x+\Delta x)-arcctgx}{\Delta x}, \qquad (d_1)$$

By using formula$arcctgA-arcctgB=arcctg\frac{AB+1}{B-A}$, fraction numerator to the expression under the limit sign of (d_1) will be:

$$arcctg(x+\Delta x)-arcctgx=\ arcctg\frac{x(x+\Delta x)+1}{-\Delta x}$$

When last equation is substituted in(d_1), we will get:

$$f'(x)=\lim_{\Delta x\to 0}\frac{arcctg\frac{x(x+\Delta x)+1}{-\Delta x}}{\Delta x} \qquad (d_2)$$

Or because $arcctg$ (-α) $=-arcctg(\alpha)$, and (d_2) will be

$$f'(x)=-\lim_{\Delta x\to 0}\frac{arcctg\frac{x(x+\Delta x)+1}{\Delta x}}{\Delta x} \qquad (d_3)$$

Limit on the right side of the equation (d_2), when $\Delta x\to 0$, represent an indeterminant form $\frac{0}{0}$.

To be released from this indefinite form, we get the substitution:

$$arcctg\frac{x(x+\Delta x)+1}{\Delta x}=t \qquad (d_4)$$

Thus we have

$$\Delta x=\frac{x^2}{ctgt-x} \qquad (d_5)$$

When $\Delta x\to 0$,from equation (d_4), t→0 by replacing all of these data in (d_3) we will have:

$$f'(x)=\lim_{t\to 0}\frac{t}{\frac{x^2}{ctgt-x}} \qquad (d_6)$$

By adjusting the expression under the sign of the limit on the right side of the equation (d_6) we will get:

$$f'(x)=-\lim_{t\to 0}\frac{t(ctgt-x)}{x^2+1} \qquad (d_7)$$

By using the features of the function limits, the equation(d_7), will have the form:

$f'(x) = -\frac{1}{1+x^2}$ (d_8)

The equation (d_8) shows the proof of the theorem (4). It is obvious that

$$(arcctgx)' = -\frac{1}{1+x^2}$$

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