PROPOSED PROBLEM
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Problem 1. Let $x_1, x_2, \ldots, x_n$ be non-negative real numbers satisfying

$$\frac{x_1}{1+x_1} + \frac{2x_2}{1+x_2} + \cdots + \frac{nx_n}{1+x_n} = 1.$$

Find the maximum possible value of

$$P = x_1 x_2^2 \cdots x_n^n.$$

Solution. From the given condition and by the AM-GM inequality, we obtain

$$\begin{aligned}\frac{1}{1+x_1} &= \frac{2x_2}{1+x_2} + \frac{3x_3}{1+x_3} + \cdots + \frac{nx_n}{1+x_n} \\ &\geq (2+3+\cdots+n)\sqrt[2+3+\cdots+n]{\left(\frac{x_2}{1+x_2}\right)^2 \left(\frac{x_3}{1+x_3}\right)^3 \cdots \left(\frac{x_n}{1+x_n}\right)^n}\end{aligned}$$

$$\begin{aligned}\frac{1}{1+x_2} &= \frac{x_1}{1+x_1} + \frac{x_2}{1+x_2} + \frac{3x_3}{1+x_3} + \cdots + \frac{nx_n}{1+x_n} \\ &\geq (2+3+\cdots+n)\sqrt[2+3+\cdots+n]{\left(\frac{x_1}{1+x_1}\right)\left(\frac{x_2}{1+x_2}\right) \left(\frac{x_3}{1+x_3}\right)^3 \cdots \left(\frac{x_n}{1+x_n}\right)^n}\end{aligned}$$

$$\begin{aligned}\frac{1}{1+x_3} &= \frac{x_1}{1+x_1} + \frac{2x_2}{1+x_2} + \frac{2x_3}{1+x_3} + \cdots + \frac{nx_n}{1+x_n} \\ &\geq (2+3+\cdots+n)\sqrt[2+3+\cdots+n]{\left(\frac{x_1}{1+x_1}\right)\left(\frac{x_2}{1+x_2}\right)^2 \left(\frac{x_3}{1+x_3}\right)^2 \cdots \left(\frac{x_n}{1+x_n}\right)^n}\end{aligned}$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$\begin{aligned}\frac{1}{1+x_n} &= \frac{x_1}{1+x_1} + \frac{2x_2}{1+x_2} + \cdots + \frac{(n-1)x_{n-1}}{1+x_{n-1}} + \frac{(n-1)x_n}{1+x_n} \\ &\geq (2+3+\cdots+n)\sqrt[2+3+\cdots+n]{\left(\frac{x_1}{1+x_1}\right)\left(\frac{x_2}{1+x_2}\right)^2 \cdots \left(\frac{x_{n-1}}{1+x_{n-1}}\right)^{n-1} \left(\frac{x_n}{1+x_n}\right)^{n-1}}\end{aligned}$$

From these relations above, we infer that

$$\begin{aligned}&\frac{1}{1+x_1} \cdot \frac{1}{(1+x_2)^2} \cdot \frac{1}{(1+x_3)^3} \cdots \frac{1}{(1+x_n)^n} \geq \\ &(2+3+\cdots+n)^{1+2+\cdots+n} \left(\frac{x_1}{1+x_1}\right)\left(\frac{x_2}{1+x_2}\right)^2 \left(\frac{x_3}{1+x_3}\right)^3 \cdots \left(\frac{x_n}{1+x_n}\right)^n\end{aligned}$$

which implies that

$$x_1 x_2^2 \cdots x_n^n \leq \frac{1}{(2+3+\cdots+n)^{1+2+\cdots+n}}$$

The equality holds if and only if

$$x_1 = x_2 = \cdots = x_n = \frac{1}{2 + 3 + \cdots + n}.$$

Thus

$$\max P = \frac{1}{(2 + 3 + \cdots + n)^{1+2+\cdots+n}}.$$

□