PROPOSED PROBLEM
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Problem 1. Let ABC be a triangle with the centroid G and denote by S_{ABC} its area. Prove that for any point P in the plane

$$\frac{PA \cdot GA^2}{BC} + \frac{PB \cdot GB^2}{CA} + \frac{PC \cdot GC^2}{AB} \geq \frac{4}{3}S_{ABC}.$$

The first, we will recall without proof two known results below

Lemma 1. *For any triangle ABC and all positive real numbers x, y, z then*

$$xa^2 + yb^2 + zc^2 \geq 4S_{ABC}\sqrt{xy + yz + zx}.$$

Remark 1. We have known that there exists a triangle whose side-lengths are m_a, m_b, m_c and its area is $S' = \frac{3}{4}S_{ABC}$. Applying lemma 1 for this triangle yields

$$x \cdot m_a^2 + y \cdot m_b^2 + z \cdot m_c^2 \geq 3S_{ABC}\sqrt{xy + yz + zx}. \tag{1}$$

Lemma 2. If ABC is a triangle and P is any point in its plane, then

$$\frac{PB \cdot PC}{bc} + \frac{PC \cdot PA}{ca} + \frac{PA \cdot PB}{ab} \geq 1.$$

(Hayashi's inequality)

Come back the main problem

Solution. Applying inequality (1) for $(x, y, z) = \left(\frac{PA}{a}, \frac{PB}{b}, \frac{PC}{c}\right)$ and using lamma 2, we obtain

$$\frac{PA}{a}m_a^2 + \frac{PB}{b}m_b^2 + \frac{PC}{c}m_c^2 \geq 3S_{ABC}.$$

Note that

$$m_a = \frac{3}{2}GA,\ m_b = \frac{3}{2}GB,\ m_c = \frac{3}{2}GC$$

The inequality above may be rewritten as

$$\frac{PA \cdot GA^2}{BC} + \frac{PB \cdot GB^2}{CA} + \frac{PC \cdot GC^2}{AB} \geq \frac{4}{3}S_{ABC}.$$

The proof is complete. □