

*RMM - Triangle Marathon 801 - 900*

R M M

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801 – 900



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801. In  $\Delta ABC$ ,  $a \leq b, b = c, R = 6, r = 2$ . Find  $a, b, c$ .

*Proposed by Daniel Sitaru – Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

In  $\Delta ABC$ ,  $a \leq b, b = c, R = 6, r = 2, a, b, c = ?$

$\Delta ABC$  is isosceles with  $AB = AC$  ( $\because b = c$ )

$\therefore$  the altitude from A to BC bisects BC  $\Rightarrow h_a = m_a \Rightarrow h_a^2 = m_a^2$

$$\Rightarrow \frac{b^2c^2}{4R^2} = \frac{2b^2 + 2c^2 - a^2}{4} \Rightarrow \frac{b^4}{36} = 4b^2 - a^2 (\because R = 6 \text{ and } b = c) \Rightarrow$$

$$\Rightarrow a = \frac{(1)}{6} b \sqrt{144 - b^2}. \text{ Now, } \Delta = \frac{abc}{4R} \stackrel{(1)}{=} \frac{b^3 \sqrt{144 - b^2}}{144}. \text{ Also, } \Delta = rs = 2 \left( \frac{a+b+c}{2} \right)$$

$$(\because r = 2) \stackrel{(1)}{=} 2b + \frac{b}{6} \sqrt{144 - b^2} \stackrel{(3)}{=} \frac{b}{6} (12 + \sqrt{144 - b^2})$$

$$(2), (3) \Rightarrow \frac{b^3 \sqrt{144 - b^2}}{144} = \frac{b}{6} (12 + \sqrt{144 - b^2}) \Rightarrow \sqrt{144 - b^2}(b^2 - 24) = 288$$

$$\Rightarrow (x - 24)^2(144 - x) = 288^2 (x = b^2) \Rightarrow x^2 - 192x + 7488 = 0$$

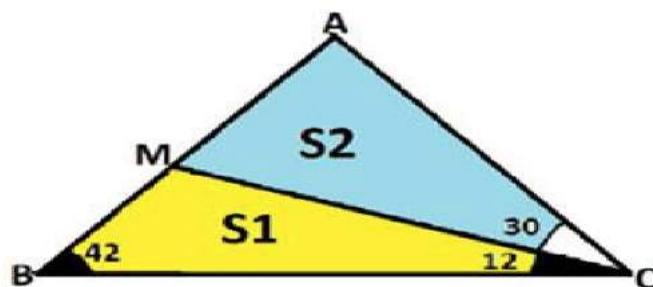
$$\Rightarrow x = 96 + 24\sqrt{3} \text{ or, } x = 96 - 24\sqrt{3} \because b \geq a \therefore b \geq \frac{b}{6} \sqrt{144 - b^2} \Rightarrow b^2 = x \geq 108$$

$$\therefore x = b^2 = 96 + 24\sqrt{3} \Rightarrow b = 2\sqrt{24 + 6\sqrt{3}}$$

$$\text{Using (1) and } b = 2\sqrt{24 + 6\sqrt{3}}, \text{ we get } a = \frac{\sqrt{2880 - 1152\sqrt{3}}}{6} = 4\sqrt{5 - 2\sqrt{3}} \Rightarrow$$

$$\Rightarrow a = 4\sqrt{5 - 2\sqrt{3}} \therefore a = 4\sqrt{5 - 2\sqrt{3}}, b = c = 2\sqrt{24 + 6\sqrt{3}} \text{ (answer)}$$

802.





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**Prove that:  $\frac{S_2}{S_1} = \phi$**

*Proposed by Muhammad Ozcelik-Turkey*

*Solution by Serban George Florin-Romania*

$$\frac{S_2}{S_1} = \frac{\frac{AC \cdot MC \cdot \sin 30^\circ}{2}}{\frac{MC \cdot BC \cdot \sin 12^\circ}{2}} = \frac{AC \cdot \frac{1}{2}}{BC \cdot \sin 12^\circ} = \frac{AC}{2 \cdot BC \cdot \sin 12^\circ} = \frac{\sin 42^\circ}{2 \sin 96^\circ \cdot \sin 12^\circ}$$

$$\Delta ABC, \text{sine theorem} \Rightarrow \frac{AC}{\sin 42^\circ} = \frac{BC}{\sin A}, \frac{AC}{\sin 42^\circ} = \frac{BC}{\sin 96^\circ}, \frac{AC}{BC} = \frac{\sin 42^\circ}{\sin 96^\circ}$$

$$\frac{S_2}{S_1} = \frac{\sin 42^\circ}{2 \sin 96^\circ \cdot \sin 12^\circ} = \frac{\cos 48^\circ}{2 \cdot 2 \sin 48^\circ \cos 48^\circ \sin 12^\circ} = \frac{1}{4 \sin 48^\circ \sin 12^\circ}$$

$$x = 18^\circ, 5x = 90^\circ, 2x = 90^\circ - 3x, \sin 2x = \sin(90^\circ - 3x), \sin 2x = \cos 3x$$

$$2 \sin x \cos x = 4 \cos^3 x - 3 \cos x, \cos x (2 \sin x - 4 \cos^2 x + 3) = 0$$

$$\cos x \neq 0 \Rightarrow 2 \sin x - 4(1 - \sin^2 x) + 3 = 0, 4 \sin^2 x + 2 \sin x - 1 = 0 \Rightarrow$$

$$\Rightarrow \sin A = -\frac{1 + \sqrt{5}}{4} = \sin 18^\circ, \cos^2 18^\circ = 1 - \sin^2 18^\circ = 1 - \frac{1 - 2\sqrt{5} + 5}{16}$$

$$\sin 48^\circ = \sin(30^\circ + 18^\circ) = \sin 30^\circ \cos 18^\circ + \sin 18^\circ \cos 30^\circ =$$

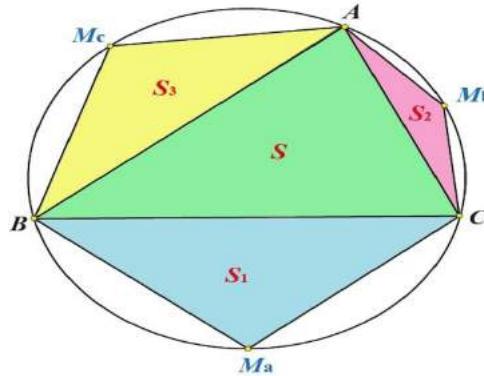
$$= \frac{\sqrt{10 + 2\sqrt{5}}}{8} + \frac{-1 + \sqrt{5}}{4} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{10 + 2\sqrt{5}}}{8} + \frac{\sqrt{15} - \sqrt{18}}{8}$$

$$\sin 12^\circ = \sin(30^\circ - 18^\circ) = \sin 30^\circ \cos 18^\circ - \sin 18^\circ \cos 30^\circ = \frac{\sqrt{10 + 2\sqrt{5}}}{8} - \frac{\sqrt{15} - \sqrt{3}}{8}$$

$$\begin{aligned} \sin 48^\circ \sin 12^\circ &= \frac{10 + 2\sqrt{5}}{64} - \frac{(\sqrt{15} - \sqrt{3})^2}{64} = \frac{10 + 2\sqrt{5} - 18 + 2\sqrt{45}}{64} = \frac{-8 + 8\sqrt{5}}{64} = \\ &= -\frac{1 + \sqrt{5}}{8}, 4 \sin 48^\circ \sin 12^\circ = \frac{-1 + \sqrt{5}}{2} \end{aligned}$$

$$\frac{S_2}{S_1} = \frac{1}{4 \sin 48^\circ \sin 12^\circ} = \frac{1}{-\frac{1 + \sqrt{5}}{8}} = \frac{2}{-1 + \sqrt{5}} = \frac{2(\sqrt{5} + 1)}{5 - 1} = \frac{2(\sqrt{5} + 1)}{4} = \frac{\sqrt{5} + 1}{2} = \phi$$

803.

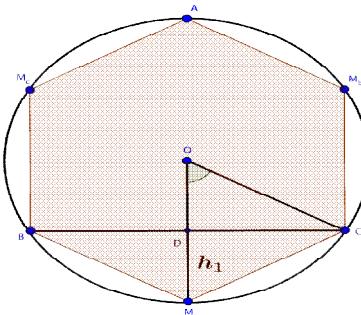


If  $R$  – circumradius of  $ABC$ ,  $r$  – inradius of  $ABC$ ,  $M_a, M_b, M_c$  – mid-points of

$$\text{arcs then: } \frac{S_1 + S_2 + S_3}{S} = \frac{R}{r} - 1$$

*Proposed by Abdilkadir Altintas-Afyonkarashisar-Turkey*

*Solution by Marian Ursarescu-Romania*



Let  $M_a D \perp BC$ . Because  $m(\widehat{A}) = \frac{1}{2}m(\widehat{BC}) \Rightarrow m(\widehat{COD}) = m(\widehat{A}) \Rightarrow$

$$\Rightarrow \cos A = \frac{OD}{R} \Rightarrow OD = R \cos A \Rightarrow R - h_1 = R \cos A \Rightarrow h_1 = R(1 - \cos A) = 2R \sin^2 \frac{A}{2}$$

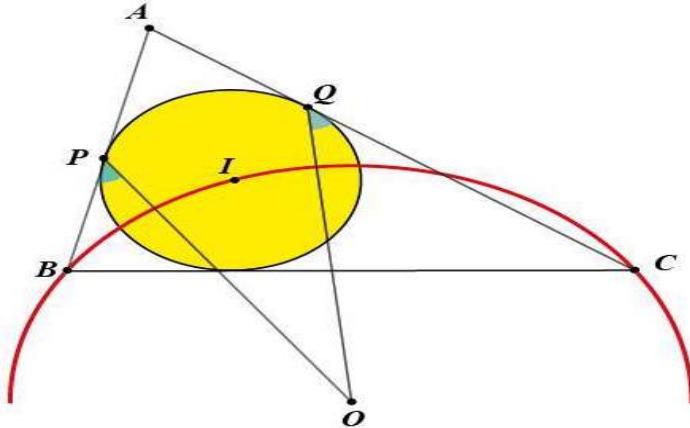
$$\Rightarrow S_1 = \frac{ah_1}{2} = Ra \sin^2 \frac{A}{2}. \text{ Similarly: } S_2 = Ra \sin^2 \frac{B}{2}, S_3 = R_c \sin^2 \frac{C}{2} \Rightarrow$$

$$\frac{S_1 + S_2 + S_3}{3} = \frac{R(a \sin^2 \frac{A}{2} + b \sin^2 \frac{B}{2} + c \sin^2 \frac{C}{2})}{S} \quad (1)$$

$$\text{But in any } \Delta ABC \Rightarrow \sum a \sin^2 \frac{A}{2} = \frac{s(R-r)}{R} \quad (2)$$

$$\text{From (1) + (2)} \Rightarrow \frac{S_1 + S_2 + S_3}{S} = \frac{s(R-r)}{S} = \frac{s(R-r)}{sr} = \frac{R-r}{c} = \frac{R}{r} - 1$$

804.



*I – incenter of  $ABC$ ,  $O$  – center of circle passing through  $B, I$  and  $C$*

*P and Q tangency points of incircle. Prove:  $\angle BPO = \angle OQC$*

*Designed by Abdulkadir Altintas-Afyonkarhisar-Turkey*

*Solution by Omran Kouba-Damascus-Syria*

*Clearly, we have  $\angle BOI = 2\angle BCI = \angle BCA$ , and  $\angle COI = 2\angle CBI = \angle CBA$  hence*

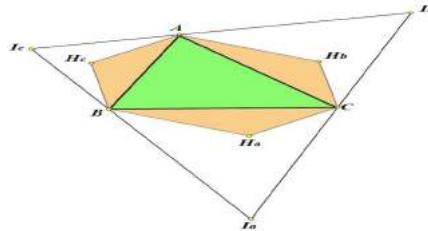
$$\angle BOC = \angle BCA + \angle CBA = \pi - \angle BAC$$

*Thus,  $O$  belongs to the circumcircle  $\omega$  of  $ABC$ , and to the perpendicular bisector of the segment  $BC$ , so  $O$  is the midpoint of the arc opposit  $A$  on  $\omega$ . This proves that  $AO$  is the angle bisector of  $\angle BAC$ , and in particular,  $A, I$  and  $O$  are collinear.*

*Now, triangles  $\Delta AOP$  and  $\Delta AOQ$  are congruent (SAS) which implies that*

$$\angle APO = \angle AQO, \text{ and the desired conclusion follows.}$$

805.





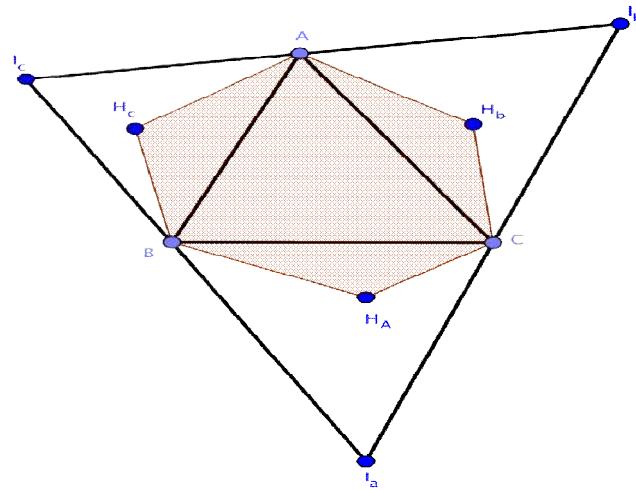
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$I_a, I_b, I_c$  – excenters of  $ABC$ ,  $H_a, H_b, H_c$  – orthocenters of  $I_aBC, I_bCA, I_cAB$

$r, R$  inradius and circumradius respectively. Prove:  $\frac{[AH_cBH_aCH_bA]}{[I_aI_bI_c]} = \frac{r}{R}$

*Proposed by Abdilkadir Altintas-Afyonkarashisar-Turkey*

*Solution by Marian Ursarescu-Romania*



$$\Delta AI_bC \sim \Delta CI_aB \sim \Delta BI_cA \sim \Delta I_aI_bI_c \text{ with angles: } \frac{\pi}{2} - \frac{A}{2}, \frac{\pi}{2} - \frac{B}{2}, \frac{\pi}{2} - \frac{C}{2} \Rightarrow$$

$$\frac{A[I_aBC]}{A[I_aI_bI_c]} = \left( \frac{I_aC}{I_aI_c} \right)^2 \quad (1)$$

$$I_cC \perp I_aI_b \Rightarrow \frac{I_aC}{I_aI_c} = \cos(\widehat{BI_aC}) = \cos\left(\frac{\pi}{2} - \frac{A}{2}\right) = \sin\frac{A}{2} \quad (2)$$

$$\text{From (1)+(2)} \Rightarrow A[I_aBC] = A[I_aI_bI_c] \sin^2 \frac{A}{2}$$

$$\text{Similarly: } A[I_bAC] = A[I_aI_bI_c] \sin^2 \frac{B}{2} \quad (3), \quad A[I_cAB] = A[I_aI_bI_c] \cdot \sin^2 \frac{C}{3}$$

$$\text{From (3)} \Rightarrow A[I_aI_bI_c] = A[ABC] + A[I_aI_bI_c] \left( \sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} \right)$$

$$\Rightarrow A[I_aI_bI_c] \left( 1 - \sin^2 \frac{A}{2} - \sin^2 \frac{B}{2} - \sin^2 \frac{C}{2} \right) = A[ABC] = S \Rightarrow$$

$$\Rightarrow A[I_aI_bI_c] = \frac{S}{1 - \sin^2 \frac{A}{2} - \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2}} = \frac{2S}{\cos A + \cos B + \cos C - 1} \quad (4)$$

$$\text{But in any } \Delta ABC \text{ we have: } \cos A + \cos B + \cos C = 1 + \frac{r}{R} \quad (5)$$



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$$\text{From (4)+(5)} \Rightarrow A[I_a I_b I_c] = \frac{2S}{r} = \frac{2RS}{R} = \frac{2Rs}{r} = 2Rs \Rightarrow A[I_a I_b I_c] = 2Rs \quad (6)$$

Now, we have:  $m(\widehat{H_a BC}) = \frac{c}{2}$ ,  $m(\widehat{H_a CB}) = \frac{B}{2}$  and  $m(\widehat{BH_a C}) = \frac{\pi}{2} + \frac{A}{2}$ , but in  $\Delta BCI$

$$(I = \text{incenter}), \text{ we have } m(\widehat{IBC}) = \frac{B}{2}, m(\widehat{ICB}) = \frac{C}{2}, m(\widehat{BIC}) = \frac{\pi}{2} + \frac{A}{2} \Rightarrow$$

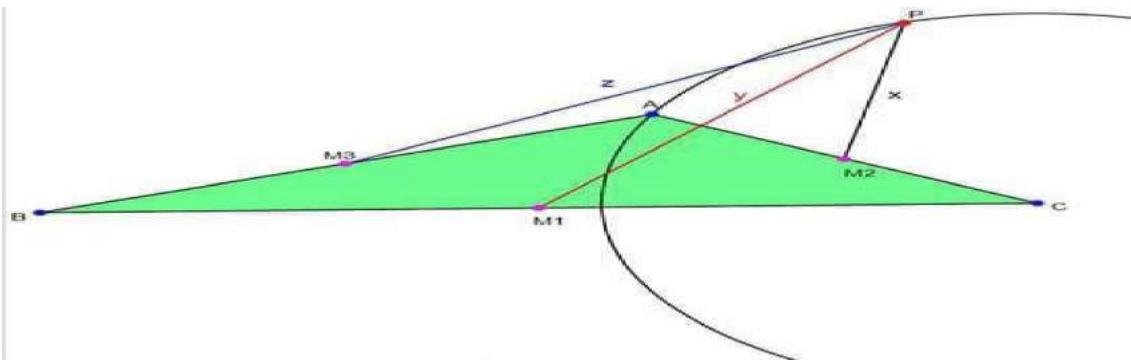
$$\left. \begin{array}{l} \Rightarrow A[BH_c C] = \frac{ar}{2} \text{ and similarly we have} \\ A[H_B AC] = \frac{br}{2}, A[AH_c B] = \frac{cr}{2} \end{array} \right\} \Rightarrow$$

$$A[H_c BC] + A[H_b AC] + A[AH_c B] = sr \quad (7)$$

$$\text{From (7)} \Rightarrow A[AH_c BH_a CH_b A] = S + sr = 2sr \quad (8)$$

$$\text{From (6) + (8)} \Rightarrow \frac{A[AH_c BH_a CH_b A]}{A[I_a I_b I_c]} = \frac{2sr}{2Rs} = \frac{r}{R}$$

806.

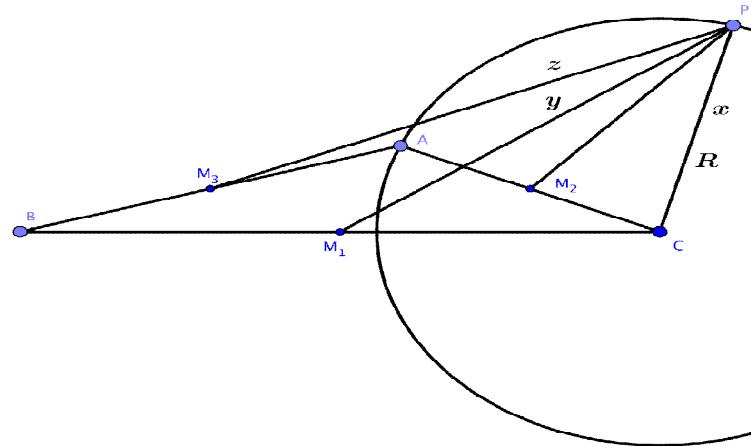


$M_1, M_2, M_3$  - midpoints of sides,  $P$  - any point on circle centered at  $C$  and

passes through  $A$ . Prove:  $c^2 = a^2 - 3b^2 \Rightarrow z^2 = x^2 + y^2$

*Proposed by Abdilkadir Altintas-Afyonkarashisar-Turkey*

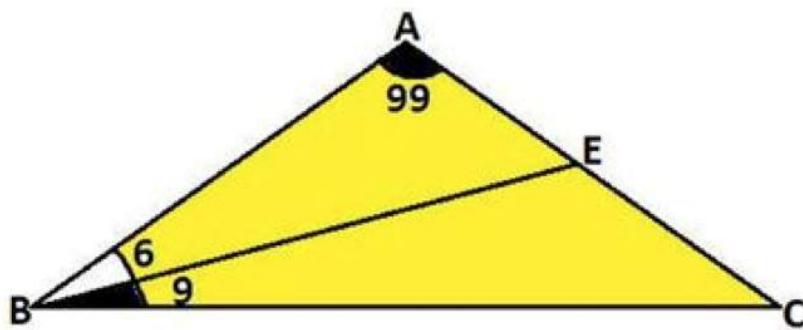
*Solution by Marian Ursarescu-Romania*



From median theorem  $\Rightarrow z^2 = \frac{2(PB^2 + PA^2) - c^2}{4}$ ,  $y^2 = \frac{2(PB^2 + PC^2) - a^2}{4}$ ,  $x^2 = \frac{2(PA^2 + PC^2) - b^2}{4} \Rightarrow$   
 $\Rightarrow$  we must show this:  $\frac{2(PB^2 + PA^2) - c^2}{4} = \frac{2(PB^2 + PC^2) - a^2}{4} + \frac{2(PA^2 + PC^2) - b^2}{4} \Leftrightarrow$   
 $2PB^2 + 2PA^2 - c^2 = 2PB^2 + 2PC^2 - a^2 + 2PA^2 + 2PC^2 - b^2 \Leftrightarrow$   
 $\Leftrightarrow -c^2 = 4PC^2 - a^2 - b^2 \Leftrightarrow 4PC^2 = a^2 + b^2 - c^2 \quad (1)$   
 But  $c^2 = a^2 - 3b^2 \quad (2)$

From (1) + (2) we must show:  $4PC^2 = 4b^2 \Leftrightarrow PC = b$ , true because  $PC = R = b$ .

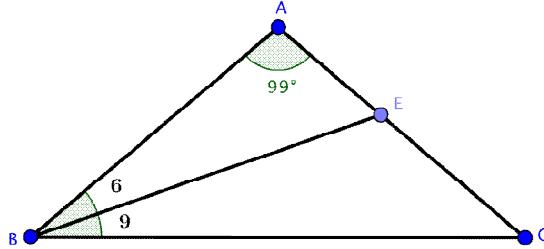
807.



Prove that:  $\frac{EC}{AE} = \phi$  (Golden ratio)

*Proposed by Muhammed Ozcelik-Turkey*

*Solution by Daniel Sitaru-Romania*



$$\begin{aligned}
 \frac{AE}{\sin 6^\circ} &= \frac{BE}{\sin 99^\circ} \cdot \frac{EC}{\sin 9^\circ} = \frac{BE}{\sin 66^\circ} \rightarrow \frac{EC}{AE} = \frac{\frac{BE \cdot \sin 9^\circ}{\sin 66^\circ}}{\frac{BE \cdot \sin 6^\circ}{\sin 99^\circ}} = \frac{\sin 9^\circ \cdot \sin 99^\circ}{\sin 6^\circ \cdot \sin 66^\circ} = \\
 &= \frac{\frac{1}{2}(\cos 90^\circ - \cos 108^\circ)}{\frac{1}{2}(\cos 60^\circ - \cos 72^\circ)} = \frac{0 + \sin 18^\circ}{\frac{1}{2} - \sin 18^\circ} = \frac{\frac{\sqrt{5}-1}{4}}{\frac{1}{2} - \frac{\sqrt{5}-1}{4}} = \frac{\sqrt{5}-1}{3-\sqrt{5}} = \\
 &= \frac{(\sqrt{5}-1)(3+\sqrt{5})}{9-5} = \frac{3\sqrt{5}+5-3-\sqrt{5}}{4} = \frac{1+\sqrt{5}}{2} = \phi
 \end{aligned}$$

808. In  $\triangle ABC$  the following relationship holds:

$$\sum \frac{r_b + r_c}{r_a} = s \sum \frac{a}{r_a^2}$$

*Mehmet Sahin-Ankara-Turkey*

*Solution by Daniel Sitaru-Romania*

$$\begin{aligned}
 \sum \frac{r_b + r_c}{r_a} &= \sum \frac{\frac{S}{s-b} + \frac{S}{s-c}}{\frac{S}{s-a}} = \sum \frac{(s-b+s-c)(s-a)}{(s-b)(s-c)} = \sum \frac{a(s-a)}{(s-b)(s-c)} = \\
 &= \sum \frac{as(s-a)^2}{s^2} = \frac{s}{s^2} \sum a(s-a)^2 = s \sum \frac{a}{\left(\frac{s}{s-a}\right)^2} = s \sum \frac{a}{r_a^2}
 \end{aligned}$$

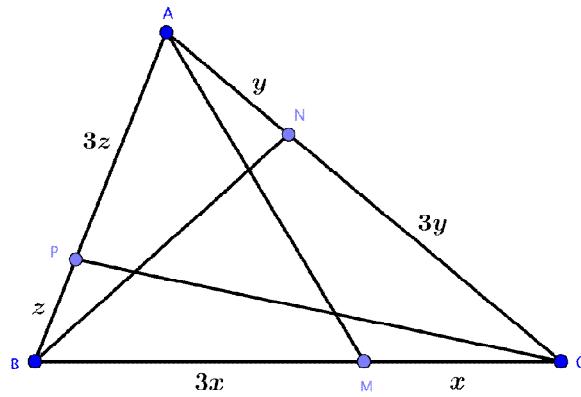
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**809. If in  $\Delta ABC$ ,  $M \in (BC)$ ,  $N \in (CA)$ ,  $P \in (AB)$ ,  $\frac{MB}{MC} = \frac{NC}{NA} = \frac{PA}{PB} = 3$  then**

**find:**

$$\Omega = \frac{AM^2 + BN^2 + CP^2}{a^2 + b^2 + c^2}$$



*Proposed by Daniel Sitaru – Romania*

**Solution 1 by Soumava Chakraborty-Kolkata-India**

Let  $\vec{a}, \vec{b}, \vec{c}$  be position vectors of  $A, B, C$  respectively. Then position vectors of  $M$  is

$$\begin{aligned} & \frac{1}{4}(\vec{b} + 3\vec{c}). \text{ Now, } AM^2 = \left| \frac{\vec{b}+3\vec{c}}{4} - \vec{a} \right|^2 = \frac{1}{16} |(\vec{b} - \vec{a}) + 3(\vec{c} - \vec{a})|^2 = \\ & = \frac{1}{16} \left[ |\vec{b} - \vec{a}|^2 + 9|\vec{c} - \vec{a}|^2 + 6(\vec{b} - \vec{a}) \cdot (\vec{c} - \vec{a}) \right] = \frac{1}{16} [AB^2 + 9AC^2 + 6\vec{AB} \cdot \vec{AC}] \end{aligned}$$

$$\begin{aligned} \text{Similarly, } BN^2 &= \frac{1}{16} [BC^2 + 9AB^2 + 6\vec{BC} \cdot \vec{BA}]; CP^2 = \frac{1}{16} [AC^2 + 9BC^2 + 6\vec{CA} \cdot \vec{CB}] \Rightarrow \\ &\Rightarrow AM^2 + BN^2 + CP^2 = \frac{10}{16} (AB^2 + BC^2 + CA^2) + \frac{6}{16} E \end{aligned}$$

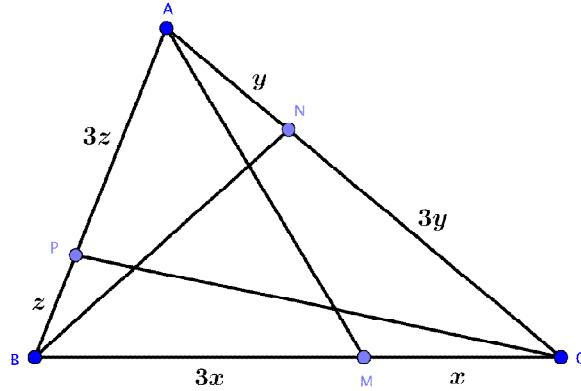
$$\text{Where } E = \vec{AB} \cdot \vec{AC} + \vec{BC} \cdot \vec{BA} + \vec{CA} \cdot \vec{CB} = -(\vec{BC} \cdot \vec{AB} + \vec{AB} \cdot \vec{CA} + \vec{CA} \cdot \vec{BC})$$

$$\text{Also, } \vec{AB} + \vec{BC} + \vec{CA} = \vec{0} \Rightarrow \vec{AB}^2 + \vec{BC}^2 + \vec{CA}^2 - 2E = 0 \Rightarrow E = \frac{1}{2}(AB^2 + BC^2 + CA^2)$$

$$\therefore AM^2 + BN^2 + CP^2 = \frac{10}{16} (AB^2 + BC^2 + CA^2) + \frac{3}{13} (AB^2 + BC^2 + CA^2) =$$

$$= \frac{13}{16} (AB^2 + BC^2 + CA^2) \Rightarrow \frac{AM^2 + BN^2 + CP^2}{a^2 + b^2 + c^2} = \frac{13}{16}$$

*Solution 2 by Ravi Prakash - New Delhi-India*



Let  $MC = x, NA = y, PB = z$ .

Then  $MB = 3x, NC = 3y$  &  $PA = 3z$  &  $BC = 4x, AC = 4y$  &

$AB = 4z$ . Stewart's theorem with cevian  $AM$  gives

$$16z^2x + 16y^2 \cdot 3x = 4x(AM^2 + 3x^2) \Rightarrow AM^2 \stackrel{(1)}{=} 4z^2 + 12y^2 - 3x^2$$

Similarly, Stewart's theorem with cevian

$$BN \Rightarrow 16z^2 \cdot 3y + 16x^2y = 4y(BN^2 + 3y^2) \Rightarrow$$

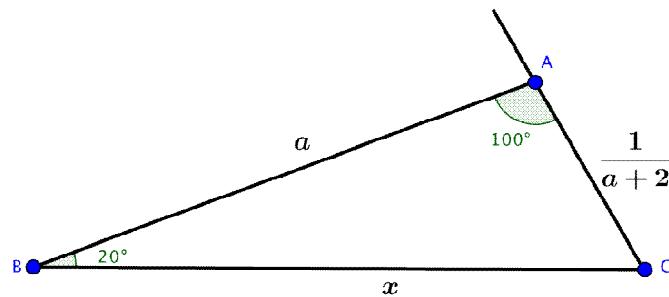
$$\Rightarrow BN^2 \stackrel{(2)}{=} 12z^2 + 4x^2 - 3y^2 \text{ & Stewart's theorem: with cevian } CP \Rightarrow$$

$$16y^2z + 16x^2 \cdot 3z = 4z(CP^2 + 3z^2) \Rightarrow CP^2 \stackrel{(3)}{=} 4y^2 + 12x^2 - 3z^2$$

$$(1) + (2) + (3) \Rightarrow AM^2 + BN^2 + CP^2 = 13(x^2 + y^2 + z^2) \Rightarrow \frac{AM^2 + BN^2 + CP^2}{a^2 + b^2 + c^2} =$$

$$= \frac{13(x^2 + y^2 + z^2)}{16(x^2 + y^2 + z^2)} = \frac{13}{16} \quad (\text{Answer})$$

810.



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$$BC = x = ?$$

*Proposed by Murat Oz-Turkey*

*Solution 1 by Omran Kouba-Damascus-Syria*

**From the sine law we have**  $\frac{1}{(a+2) \sin 20^\circ} = \frac{a}{\sin 60^\circ}$ .

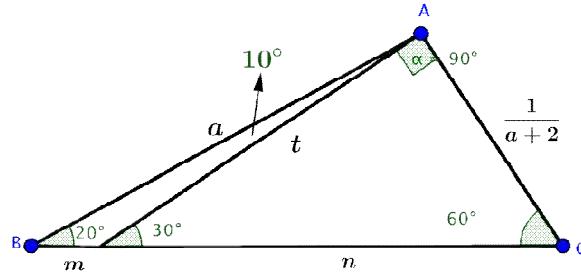
$$\text{Thus } a(a+2) = \frac{\sin(3 \times 20^\circ)}{\sin 20^\circ} = 1 + 2 \cos 40^\circ$$

**So,  $(a+1)^2 = 2 + 2 \cos 40^\circ = 4 \cos^2 20^\circ$ , and consequently**

$$a = 2 \cos 20^\circ - 1 = 4 \cos^2 10^\circ - 3. \text{ Again, from the sine law we have: } \frac{x}{\sin 100^\circ} = \frac{a}{\sin 60^\circ}.$$

$$\text{Thus, } x = \frac{a \cos 10^\circ}{\cos 30^\circ} = \frac{(4 \cos^2 10^\circ - 3) \cos 10^\circ}{\cos 30^\circ} = \frac{4 \cos^3 10^\circ - 3 \cos 10^\circ}{\cos 30^\circ} = \frac{\cos(3 \times 10^\circ)}{\cos 30^\circ} = 1.$$

*Solution 2 by Soumava Chakraborty-Kolkata-India*



$$\sin 30^\circ = \frac{1}{(a+2)n} = \frac{1}{2} \Rightarrow n = \frac{2}{a+2}, \tan 30^\circ = \frac{1}{(a+2)t} = \frac{1}{\sqrt{3}} \Rightarrow t = \frac{\sqrt{3}}{a+2}$$

$$\text{Using Stewart's theorem, } a^2 \cdot \frac{2}{a+2} + \frac{1}{(a+2)^2} m = \left(m + \frac{2}{a+2}\right) \left\{ \frac{3}{(a+2)^2} + \frac{2m}{a+2} \right\} \Rightarrow$$

$$\Rightarrow m^2(a+2)^2 + 3m(a+2) + 3 - a^2(a+2)^2 = 0 \Rightarrow m = \frac{-3 + \sqrt{4a^2(a+2)^2 - 3}}{2(a+2)}$$

$$\text{Again, } \frac{m}{\sin 10^\circ} = \frac{a}{\sin 150^\circ} = 2a \Rightarrow \sin 10^\circ = \frac{m}{2a}. \text{ Also, } \frac{m+n}{\sin 100^\circ} = \frac{a}{\sin 60^\circ} \Rightarrow \cos 10^\circ = \frac{\sqrt{3}(m+n)}{2a}$$

$$\therefore \sin 20^\circ - 2 \sin 10^\circ \cos 10^\circ = \frac{2m}{2a} \cdot \frac{\sqrt{3}(m+n)}{2a} = \frac{\sqrt{3}m(m+n)}{2a^2} \Rightarrow \sin(30^\circ - 10^\circ) =$$

$$= \frac{\sqrt{3}m(m+n)}{2a^2} \Rightarrow \sin 30^\circ \cos 10^\circ - \cos 30^\circ \sin 10^\circ = \frac{\sqrt{3}m(m+n)}{2a^2} \Rightarrow$$

$$\Rightarrow \frac{\cos 10^\circ}{2} - \frac{\sqrt{3} \sin 10^\circ}{2} = \frac{\sqrt{3}m(m+n)}{2a^2} \Rightarrow \frac{\sqrt{3}(m+n)}{2a} - \sqrt{3} \frac{m}{2a} = \frac{\sqrt{3}}{a^2} (m+n) \text{ (using (2), (3))}$$



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$$\begin{aligned}
 & \Rightarrow \frac{n}{2a} = \frac{m(m+n)}{a^2} \Rightarrow a_n = 2m(m+n) \Rightarrow 2m^2 + 2mn - a_n = 0 \\
 & \Rightarrow m = \frac{\sqrt{n^2 + 2a_n - n}}{2} = \frac{\sqrt{\left(\frac{2}{a+2}\right)^2 + 2a\frac{2}{(a+2)} - \frac{2}{a+2}}}{2} \left( \because n = \frac{2}{a+2} \right) \\
 & = \frac{2\sqrt{1 + a(a+2)^2} - 2}{2(a+2)} \stackrel{(1)}{=} \frac{\sqrt{4a^2(a+2)^2 - 3} - 3}{2(a+2)}
 \end{aligned}$$

*Putting  $a(a+2) = \alpha$ , (4) becomes:  $2\sqrt{1+\alpha} + 1 = \sqrt{4a^2 - 3} \Rightarrow$*

$$\begin{aligned}
 & \Rightarrow 4(1+\alpha) + 1 + 4\sqrt{1+\alpha} = 4a^2 - 3 \Rightarrow \sqrt{1+\alpha} = a^2 - \alpha - 2 = (\alpha - 2)(\alpha + 1) \Rightarrow \\
 & \Rightarrow 1 = (\alpha - 2)^2(\alpha + 1) = (a^2 + 2a - 2)^2(a + 1)^2 \Rightarrow (a^2 + 2a - 2)(a + 1) = 1 \Rightarrow \\
 & \Rightarrow a^3 + 3a^2 - 3 = 0 \Rightarrow a^2(a + 3) = 3 \Rightarrow a^2(a + 3)(a + 1) = 3(a + 1) \Rightarrow \\
 & \Rightarrow a^2\{(a + 2)^2 - 1\} = 3(a + 1) \Rightarrow a^2(a + 2)^2 - 3 = 3a + a^2 \Rightarrow 4a^2(a + 2)^2 - 12 = \\
 & = 12a + 4a^2 \Rightarrow 4a^2(a + 2)^2 - 3 \stackrel{(5)}{=} (2a + 3)^2. \text{ Now, } m + n = \frac{\sqrt{4a^2(a+2)^2-3}}{2(a+2)} + \frac{4}{2(a+2)} = \\
 & = \frac{\sqrt{4a^2(a+2)^2-3}+1}{2(a+2)} \stackrel{(5)}{=} \frac{2a+3+1}{2(a+2)} \stackrel{(\text{using (1)})}{=} 1 \Rightarrow x = m + n = 1 \quad (\text{Answer})
 \end{aligned}$$

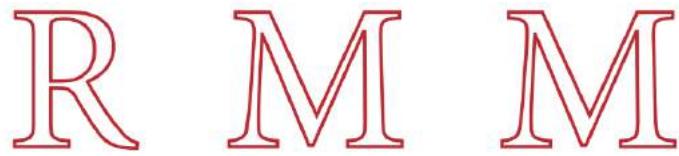
**811. In  $\Delta ABC$  the following relationship holds:**

$$\frac{r_a}{a} + \frac{r_b}{b} + \frac{r_c}{c} = \frac{s^2 + (r_a + r_b + r_c)^2}{2[I_a I_b I_c]}, \quad \Delta I_a I_b I_c - \text{excentral triangle}$$

*Proposed by Mehmet Sahin-Ankara-Turkey*

*Solution by Daniel Sitaru-Romania*

$$\begin{aligned}
 \sum \frac{r_a}{a} &= \sum \frac{bcr_a}{abc} = \frac{S}{abc} \sum \frac{bc}{s-a} = \frac{1}{\frac{abc}{S}} \cdot \frac{s^2 + (4R + r)^2}{s} = \\
 &= \frac{1}{2 \cdot \frac{1}{2} \cdot \frac{abcrs}{r^2 s}} \cdot \frac{s^2 + (r_a + r_b + r_c)^2}{1} = \\
 &= \frac{s^2 + (r_a + r_b + r_c)^2}{2 \cdot \frac{abcS}{2r^2 s}} = \frac{s^2 + (r_a + r_b + r_c)^2}{2[I_a I_b I_c]}
 \end{aligned}$$



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**812. In  $\Delta ABC$  the following relationship holds:**

$$\frac{a^2}{r_b + r_c} + \frac{b^2}{r_c + r_a} + \frac{c^2}{r_a + r_b} = 4R - 2r$$

*Proposed by Mehmet Sahin-Ankara-Turkey*

*Solution by Daniel Sitaru-Romania*

$$\begin{aligned} \sum \frac{a^2}{r_b + r_c} &= \sum \frac{a^2}{\frac{s-b}{s-b} + \frac{s-c}{s-c}} = \frac{1}{s} \sum \frac{a^2}{\frac{s-b+s-c}{(s-b)(s-c)}} = \frac{1}{rs} \sum a(s-b)(s-c) = \\ &= \frac{1}{rs} \cdot 2rs(2R - r) = 4R - 2r \end{aligned}$$

**813. In  $\Delta ABC$  the following relationship holds:**

$$\frac{a}{r_b + r_c} + \frac{b}{r_c + r_a} + \frac{c}{r_a + r_b} = \frac{2(r_a + r_b + r_c)}{a + b + c}$$

*Proposed by Mehmet Sahin-Ankara-Turkey*

*Solution by Daniel Sitaru-Romania*

$$\begin{aligned} \sum \frac{a}{r_b + r_c} &= \sum \frac{a}{\frac{s-b}{s-b} + \frac{s-c}{s-c}} = \frac{1}{s} \sum \frac{a}{\frac{s-b+s-c}{(s-b)(s-c)}} = \frac{1}{rs} \sum (s-b)(s-c) = \\ &= \frac{1}{rs} \cdot r(4R + r) = \frac{1}{a+b+c} (r_a + r_b + r_c) = \frac{2(r_a + r_b + r_c)}{a + b + c} \end{aligned}$$

**814. In  $\Delta ABC$  the following relationship holds:**

$$\frac{r_b + r_c}{r_a} + \frac{r_c + r_a}{r_b} + \frac{r_a + r_b}{r_c} = \frac{2[I_a I_b I_c]}{[ABC]} - 2, \quad \Delta I_a I_b I_c - \text{excentral triangle}$$

*Proposed by Mehmet Sahin-Ankara-Turkey*



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*Solution by Daniel Sitaru-Romania*

$$\begin{aligned}
 \sum \frac{r_b + r_c}{r_a} &= \sum \frac{\frac{S}{s-b} + \frac{S}{s-c}}{\frac{S}{s-a}} = \sum \frac{a(s-a)}{(s-b)(s-c)} = \sum \frac{as(s-a)^2}{S^2} = \\
 &= \frac{s}{S^2} \sum a(s-a)^2 = \frac{s}{Srs} \cdot 2rs(2R-r) = \frac{2s}{S}(2R-r) = \frac{4sR}{S} - \frac{2s}{S} = \\
 &= \frac{\frac{2abc}{2r}}{S} - 2 = \frac{2 \cdot \frac{abcS}{2r^2s}}{S} - 2 = \frac{2[I_a I_b I_c]}{[ABC]} - 2
 \end{aligned}$$

**815. In  $\Delta ABC$  the following relationship holds:**

$$\frac{r + r_a}{h_a} + \frac{r + r_b}{h_b} + \frac{r + r_c}{h_c} = \frac{[I_a I_b I_c]}{[ABC]}, \quad \Delta I_a I_b I_c - \text{excentral triangle}$$

*Proposed by Mehmet Sahin-Ankara-Turkey*

*Solution by Daniel Sitaru-Romania*

$$\begin{aligned}
 \sum \frac{r + r_a}{h_a} &= \sum \frac{\frac{S}{s} + \frac{S}{s-a}}{\frac{2S}{a}} = \frac{1}{2} \sum \frac{a}{s} + \frac{1}{2} \sum \frac{a}{s-a} = \frac{2s}{2s} + \frac{1}{2} \cdot \frac{2(2R-r)}{r} = \\
 &= 1 + \frac{2R-r}{r} = \frac{2R}{r} = \frac{2RS}{S} = \frac{2R \cdot \frac{abc}{4R}}{S} = \frac{1}{S} \cdot \frac{abcS}{2rS} = \frac{1}{S} \cdot \frac{abcS}{2r^2s} = \frac{[I_a I_b I_c]}{[ABC]}
 \end{aligned}$$

**816. In  $\Delta ABC$  the following relationship holds:**

$$r_a \left( \frac{1}{b} + \frac{1}{c} \right) + r_b \left( \frac{1}{c} + \frac{1}{a} \right) + r_c \left( \frac{1}{a} + \frac{1}{b} \right) = \frac{s}{r}$$

*Proposed by Mehmet Sahin-Ankara-Turkey*

*Solution by Daniel Sitaru-Romania*

$$\begin{aligned}
 \sum r_a \left( \frac{1}{b} + \frac{1}{c} \right) &= \sum \left( \frac{S}{s-a} \cdot \frac{b+c}{bc} \right) = S \sum \frac{2s-a}{bc(s-a)} = \frac{S}{abc} \sum \frac{as+a(s-a)}{s-a} = \\
 &= \frac{sS}{abc} \sum \frac{a}{s-a} + \frac{S}{abc} \cdot 2s = \frac{sS}{abc} \left( \frac{4R-2r}{r} + 2 \right) = \frac{sS}{4RS} \cdot \frac{4R}{r} = \frac{s}{r}
 \end{aligned}$$



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**817. In  $\Delta ABC$  the following relationship holds:**

$$\frac{a^3}{r_b + r_c} + \frac{b^3}{r_c + r_a} + \frac{c^3}{r_a + r_b} = 2[I_a I_b I_c] - 4S,$$

*$\Delta I_a I_b I_c$  – excentral triangle*

*Proposed by Mehmet Sahin-Ankara-Turkey*

*Solution by Daniel Sitaru-Romania*

$$\begin{aligned} \sum \frac{a^3}{r_b + r_c} &= \sum \frac{a^3}{\frac{s}{s-b} + \frac{s}{s-c}} = \frac{1}{s} \sum \frac{a^3(s-b)(s-c)}{s-c+s-b} = \\ &= \frac{1}{s} \sum a^2(s-b)(s-c) = \frac{1}{rs} \cdot 4rs^2(R-r) = 4s(R-r) = 4sR - 4S = \\ &= \frac{4Rrs}{r} - 4S = \frac{4RS}{r} - 4S = \frac{abcS}{rS} - 4S = 2 \cdot \frac{abcS}{2r^2s} - 4S = 2[I_a I_b I_c] - 4S \end{aligned}$$

**818. In  $\Delta ABC$  the following relationship holds:**

$$\frac{R + r_a}{bc} + \frac{R + r_b}{ca} + \frac{R + r_c}{ab} = \frac{3}{2r} - \frac{1}{2R}$$

*Proposed by Mehmet Sahin-Ankara-Turkey*

*Solution by Daniel Sitaru-Romania*

$$\begin{aligned} \sum \frac{R + r_a}{bc} &= R \sum \frac{1}{bc} + \sum \frac{\frac{s}{s-a}}{bc} = R \cdot \frac{1}{abc} \cdot 2s + s \sum \frac{1}{bc(s-a)} = \\ &= \frac{2Rs}{4RS} + s \cdot \frac{1}{4RS} \cdot \sum \frac{a}{s-a} = \frac{s}{2S} + \frac{S}{4RS} \cdot \frac{2(2R-r)}{r} = \frac{s}{2rs} + \frac{2R-r}{2Rr} = \\ &= \frac{1}{2r} + \frac{2R-r}{2Rr} = \frac{R+2R-r}{2Rr} = \frac{3}{2r} - \frac{1}{2R} \end{aligned}$$

**819. In  $\Delta ABC$  the following relationship holds:**

$$\frac{r_a^2}{r_a^2 + s^2} + \frac{r_b^2}{r_b^2 + s^2} + \frac{r_c^2}{r_c^2 + s^2} = 1 - \frac{r}{2R}$$

*Proposed by Adil Abdullayev-Baku-Azerbaijan*



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*Solution by Daniel Sitaru-Romania*

$$\begin{aligned}
 \sum \frac{r_a^2}{r_a^2 + s^2} &= \sum \frac{\frac{s^2}{(s-a)^2}}{\frac{s^2}{(s-a)^2} + s^2} = \sum \frac{(s-b)(s-c)}{(s-b)(s-c) + s(s-a)} = \sum \frac{(s-b)(s-c)}{bc} = \\
 &= 3 + s^2 \sum \frac{1}{bc} - s \sum \frac{b+c}{bc} = 3 + \frac{2s^3}{4Rrs} - 2s \sum \frac{1}{a} = 3 + \frac{s^2}{2Rr} - \frac{2s}{4Rrs}(s^2 + r^2 + 4Rr) = \\
 &= 3 + \frac{s^2}{2Rr} - \frac{s^2}{2Rr} - \frac{r}{2R} - 2 = 1 - \frac{r}{2R}
 \end{aligned}$$

820. In  $\Delta ABC$ ,  $K$  – Lemoine's point,  $KD \perp BC$ ,  $KE \perp CA$ ,  $KF \perp AB$ ,

$KD = x$ ,  $KE = y$ ,  $KF = z$ . Prove that:

$$xh_a m_a^2 + yh_b m_b^2 + zh_c m_c^2 = 3S^2$$

*Proposed by Mehmet Sahin-Ankara-Turkey*

*Solution by Daniel Sitaru-Romania*

$$\begin{aligned}
 K - \text{Lemoine's point} \rightarrow \frac{x}{a} = \frac{y}{b} = \frac{z}{c} \stackrel{\text{denote}}{\equiv} q \\
 S = \frac{ax + by + cz}{2} = \frac{q}{2} \sum_{\text{cyc}(a,b,c)} a^2 \rightarrow \sum_{\text{cyc}(a,b,c)} a^2 = \frac{2S}{q} \\
 \sum_{\substack{\text{cyc}(a,b,c) \\ \text{cyc}(x,y,z)}} xh_a m_a^2 = \sum_{\substack{\text{cyc}(a,b,c) \\ \text{cyc}(x,y,z)}} qa \frac{2S}{a} m_a^2 = 2Sq \sum_{\text{cyc}(a,b,c)} m_a^2 = \\
 = 2Sq \cdot \frac{3}{4} \sum_{\text{cyc}(a,b,c)} a^2 = \frac{3Sq}{2} \cdot \frac{2S}{q} = 3S^2
 \end{aligned}$$

821. In  $\Delta ABC$  the following relationship holds:

$$\frac{2R - r_a}{h_a} + \frac{2R - r_b}{h_b} + \frac{2R - r_c}{h_c} = 1$$

*Proposed by Mehmet Sahin-Ankara-Turkey*



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*Solution by Daniel Sitaru-Romania*

$$\begin{aligned} \sum \frac{2R - r_a}{h_a} &= 2R \sum \frac{1}{h_a} - \sum \frac{r_a}{h_a} = 2R \sum \frac{a}{2s} - \sum \frac{\frac{s-a}{2s}}{a} = \\ &= \frac{R}{s} \sum a - \frac{1}{2} \sum \frac{a}{s-a} = \frac{2Rs}{rs} - \frac{1}{2} \cdot \frac{2(2R-r)}{r} = \frac{2R}{r} - \frac{2R}{r} + 1 = 1 \end{aligned}$$

**822.** In  $\Delta ABC$  the following relationship holds:

$$\sum \frac{1}{b \cos B + c \cos C - a \cos A} = \frac{R}{2s \cos A \cos B \cos C} + \frac{1}{a \cos A + b \cos B + c \cos C}$$

*Proposed by Daniel Sitaru – Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\begin{aligned} \sum a \cos A &= \sum (2R \sin A \cos A) = R(\sin 2A + \sin 2B + \sin 2C) = \\ &= R\{2 \sin C \cos(A-B) + 2 \sin C \cos C\} = 2R \sin C \{\cos(A-B) - \cos(A+B)\} \\ &= 2R \sin C \cdot 2 \sin A \sin B = 4R \frac{abc}{8R^3} \stackrel{(1)}{=} \frac{abc}{2R^2} \end{aligned}$$

$$\begin{aligned} \text{Now, } b \cos B + c \cos C - a \cos A &= R(\sin 2B + \sin 2C - \sin 2A) = \\ &= R\{2 \sin A \cos(B-C) + 2 \sin A \cos(B+C)\} = 2R \sin A \cdot 2 \cos B \cos C = \\ &= 4R \sin A \cos B \cos C = 2a \left( \frac{\prod \cos A}{\cos A} \right) \Rightarrow \frac{1}{b \cos B + c \cos C - a \cos A} = \\ &= \frac{1}{2 \cos A \cos B \cos C} \left( \frac{\cos A}{a} \right) = \frac{1}{2 \cos A \cos B \cos C} \left( \frac{b^2 + c^2 - a^2}{2abc} \right) \stackrel{(a)}{=} \frac{b^2 + c^2 - a^2}{4abcp} \quad (\text{where } p = \prod \cos A) \end{aligned}$$

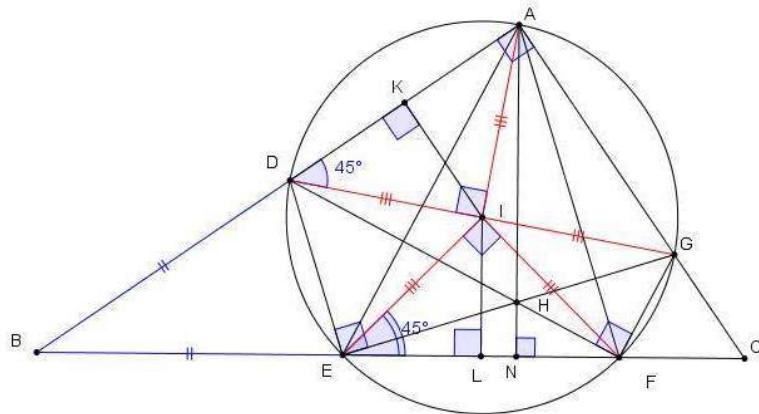
$$\begin{aligned} \text{Similarly, } \frac{1}{c \cos C + a \cos A - b \cos B} &\stackrel{(b)}{=} \frac{c^2 + a^2 - b^2}{4abcp} \& \frac{1}{a \cos A + b \cos B - c \cos C} &\stackrel{(c)}{=} \frac{a^2 + b^2 - c^2}{4abcp} \\ (a) + (b) + (c) \Rightarrow \sum \frac{1}{b \cos B + c \cos C - a \cos A} - \frac{1}{\sum a \cos A} &\stackrel{\text{by (1)}}{=} \frac{\sum a^2}{4abcp} - \frac{2R^2}{abc} = \frac{\sum a^2 - 8R^2 p}{4pabc} \\ = \frac{\sum a^2 - 8R^2 \left( \frac{s^2 - 4R^2 - 4Rr - r^2}{4R^2} \right)}{4pabc} &= \frac{2(s^2 - 4Rr - r^2) - 2(s^2 - 4R^2 - 4Rr - r^2)}{4pabc} = \\ = \frac{8R^2}{4p \cdot 4RS} = \frac{R}{2Sp} = \frac{R}{2S \cos A \cos B \cos C} &\Rightarrow \sum \frac{1}{b \cos B + c \cos C - a \cos A} = \end{aligned}$$

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$$= \frac{R}{2S(\prod \cos A)} + \frac{1}{\sum a \cos A} \quad (\text{Proved})$$

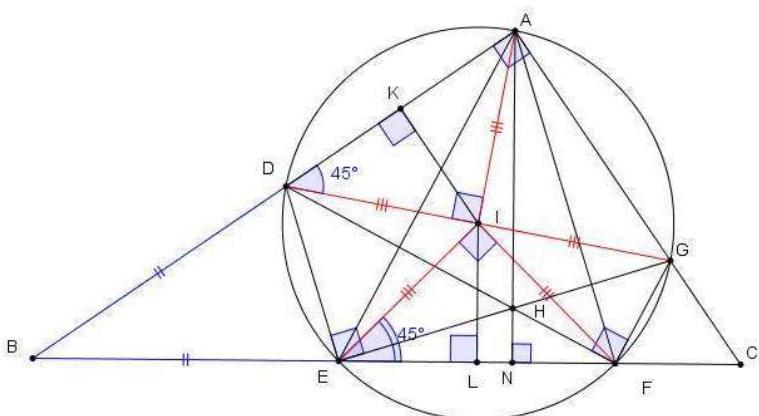
823.



$m(\angle A) = 90^\circ$ , the drawn circle has center in  $I$  – incenter of  $\triangle ABC$  and radii  
**A1.** Prove that  $H$  is orthocenter of  $\triangle AEF$ .

*Proposed by Rahul Sethi-India*

*Solution by Apostolis Manoloudis-Greece*



$\angle BAC = 90^\circ$ ,  $I$  = incentre, the circle  $(I, IA)$ ,  $H = DC \cap BG \Rightarrow$   
 $H$  = orthocenter of  $\triangle AEF$

*Is  $IK = IL \Rightarrow DA = EF$  and  $BK = BL$ . Is  $\angle DAI = 45^\circ = \angle IDA \Rightarrow$   
 $\angle AID = 90^\circ = \angle AIG =$*

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$= \angle EIF$ . But  $\angle AEF + \angle DFE = \frac{\angle AIF}{2} + \frac{\angle DIE}{2} = \frac{(\angle AIF + \angle DIE)}{2} = 90^\circ$ . So,  $DF \perp AE$ . Similar  
 $EG \perp AF \Rightarrow H = \text{orthocenter of } \triangle AEF$ .

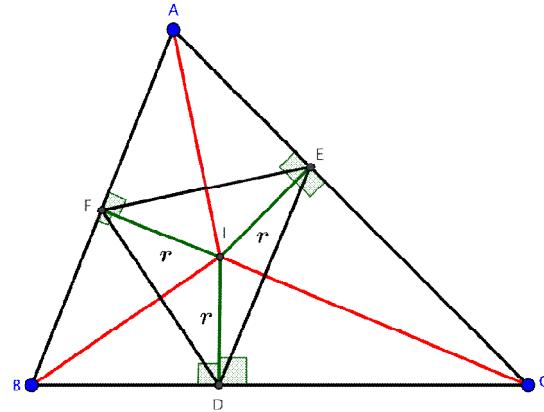
824. **Δ DEF – pedal triangle of I – incenter of acute Δ ABC,**

$h_1, h_2, h_3$  – altitudes in  $\triangle DEF$

$$\frac{\cos \frac{A}{2}}{h_1} + \frac{\cos \frac{B}{2}}{h_2} + \frac{\cos \frac{C}{2}}{h_3} = \frac{r_a + r_b + r_c}{s}$$

Proposed by Mehmet Sahin-Ankara-Turkey

Solution by Soumava Chakraborty-Kolkata-India



From  $\triangle FIE$ ,  $FE^2 = 2r^2 - 2r^2 \cos(\pi - A) = 4r^2 \cos^2 \frac{A}{2} \Rightarrow FE \stackrel{(1)}{=} 2r \cos \frac{A}{2}$

Similarly,  $DE \stackrel{(2)}{=} 2r \cos \frac{B}{2}$  &  $FB \stackrel{(3)}{=} 2r \cos \frac{C}{2}$ . Also,  $r$  is the circumradius of  $\triangle DEF$ .

$$\therefore \frac{1}{2} FE \cdot h_1 = \frac{FE \cdot DE \cdot FD}{4r} (= ar (\triangle DEF)) \Rightarrow \frac{1}{2} 2r \cos \frac{A}{2} \cdot h_1 = \frac{8r^3 \left( \frac{s}{4R} \right)}{4r} \quad (\text{using (1), (2), (3)})$$

$$\Rightarrow h_1 = \frac{rs}{2R \cos \frac{A}{2}} \Rightarrow \frac{\cos \frac{A}{2}}{h_1} = \frac{2R \cos^2 \frac{A}{2}}{s}. \text{ Similarly, } \frac{\cos \frac{B}{2}}{h_2} = \frac{2R \cos^2 \frac{B}{2}}{s} \& \frac{\cos \frac{C}{2}}{h_3} = \frac{2R \cos^2 \frac{C}{2}}{s}$$

$$(a) + (b) + (c) \Rightarrow LHS = \frac{R}{s} \sum (1 + \cos A) = \frac{R}{s} \left( 3 + 1 + \frac{r}{R} \right) = \frac{4R+r}{s} = \frac{\sum r_a}{s} \quad (\text{Proved})$$



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**825. In  $\Delta ABC$  the following relationship holds:**

$$\sum m_a^2 + \frac{a^2}{4} + \frac{b^2}{4} + \frac{c^2}{4} \geq ab + bc + ca$$

*Proposed by Seyran Ibrahimov-Maasili-Azerbaijan*

*Solution by Daniel Sitaru-Romania*

$$\begin{aligned} \sum \left( m_a^2 + \frac{a^2}{4} \right) &= \sum \left( \frac{1}{2}(b^2 + c^2) - \frac{a^2}{4} + \frac{a^2}{4} \right) = \sum \frac{1}{2}(b^2 + c^2) = \\ &= a^2 + b^2 + c^2 \geq ab + bc + ca \end{aligned}$$

**826. In  $\Delta ABC$  the following relationship holds:**

$$a^2 m_a + b^2 m_b + c^2 m_c \geq bch_a + cah_b + abh_c$$

*Proposed by Daniel Sitaru – Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\begin{aligned} \because m_a &\geq \frac{b^2+c^2}{4R}, \text{ etc (Tereshin), } \therefore \\ LHS &\geq \sum \frac{a^2(b^2+c^2)}{4R} = \frac{\sum a^2 b^2}{2R} = \sum ab \cdot \frac{ab}{2R} = \sum abh_c = bch_a + cah_b + abh_c \\ &\quad (\text{proved}) \end{aligned}$$

**827. In  $\Delta ABC$  the following relationship holds:**

$$r\sqrt{r} \left( \frac{h_a}{w_a} + \frac{h_b}{w_b} + \frac{h_c}{w_c} \right) \geq 3 \sqrt{\prod (h_a - 2r)}$$

*Proposed by Bogdan Fustei-Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\begin{aligned} \prod (a+b) &= 2abc + \sum ab(2s-c) \\ &= 2s(s^2 + 4Rr + r^2) - 4Rrs \stackrel{(1)}{=} 2s(s^2 + 2Rr + r^2) \end{aligned}$$



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$$\text{Also, } \frac{h_a}{w_a} = \frac{bc}{2R} \cdot \frac{b+c}{2bc \cos \frac{A}{2}} \stackrel{(2)}{=} \frac{1}{4R} \cdot \frac{b+c}{\cos \frac{A}{2}}$$

$$\begin{aligned} \text{Now, LHS} &\stackrel{A-G}{\geq} (3) 3r\sqrt{r^3 \sqrt{\prod \left( \frac{h_a}{w_a} \right)}} \stackrel{\text{by (1),(2)}}{=} 3r\sqrt{3^3 \sqrt{\frac{1}{(4R)^3} \cdot \frac{2s(s^2+2Rr+r^2) \cdot 4R}{s}}} \\ &= \frac{3r\sqrt{r}}{2R} \sqrt[3]{R(s^2 + 2Rr + r^2)} \end{aligned}$$

$$RHS = 3 \sqrt[3]{\prod \left( \frac{2rs}{a} - 2r \right)} = 3 \sqrt[3]{8r^3 \cdot \prod \left( \frac{s-a}{a} \right)} \stackrel{(4)}{=} 3r\sqrt{r} \sqrt{\frac{2r}{R}}$$

(3), (4)  $\Rightarrow$  it suffices to prove:

$$\frac{\sqrt[3]{R(s^2+2Rr+r^2)}}{2R} \geq \sqrt{\frac{2r}{R}} \Leftrightarrow (s^2 + 2Rr + r^2)^2 \geq 512Rr^3 \quad (5)$$

$$\begin{aligned} \text{LHS of (5)} &\stackrel{\text{Gerretsen}}{\geq} 4r^2(9R - 2r)^2 \stackrel{?}{\geq} 512Rr^3 \Leftrightarrow (R - 2r)(81R - 2r) \stackrel{?}{\geq} 0 \rightarrow \text{true} \\ &\because R \stackrel{\text{Euler}}{\geq} 2r \\ &\quad (\text{proved}) \end{aligned}$$

$$\begin{aligned} \text{LHS} &\stackrel{A-G}{\geq} 3r\sqrt{r^3 \sqrt{\frac{\prod h_a}{\prod w_a}}} \stackrel{w_a \leq \sqrt{s(s-a)}, \text{etc}}{\geq} (1) 3r\sqrt{r^3 \sqrt{\frac{2r^2 s^2}{R} \cdot \frac{1}{\sqrt{s(s-a)s(s-b)s(s-c)}}}} = \\ &= 3r\sqrt{r} \sqrt[3]{\frac{2r^2 s^2}{Rrs^2}} = 3r\sqrt{r} \sqrt[3]{\frac{2r}{R}} \end{aligned}$$

$$RHS = \sqrt[3]{\prod \left( \frac{2rs}{a} - 2r \right)} = \sqrt[3]{8r^3 \prod \left( \frac{s-a}{a} \right)} \stackrel{(2)}{=} 3r\sqrt{r} \sqrt{\frac{2r}{R}}$$

$$(1), (2) \Rightarrow \text{it suffices to show: } \left(\frac{2r}{R}\right)^2 \geq \left(\frac{2r}{R}\right)^3 \Leftrightarrow R \geq 2r \rightarrow \text{true (Euler)}$$

(Proved)

**828. In  $\Delta ABC$  the following relationship holds:**

$$\sqrt{R(r_b + r_c)} \geq \frac{b+c}{2} \geq \sqrt{2r(r_b + r_c)}$$

*Proposed by Bogdan Fustei-Romania*



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*Solution by Soumava Chakraborty-Kolkata-India*

$$\begin{aligned}
 \sqrt{R(r_b + r_c)} &\stackrel{(a)}{\geq} \frac{b+c}{2} \stackrel{(b)}{\geq} \sqrt{2r(r_b + r_c)} \\
 r_b + r_c = s \left( \frac{\sin \frac{B}{2}}{\cos^2 \frac{B}{2}} + \frac{\sin \frac{C}{2}}{\cos^2 \frac{C}{2}} \right) &= s \left( \frac{\sin \left( \frac{B+C}{2} \right)}{\cos \frac{B}{2} \cos \frac{C}{2}} \right) = \frac{s \cos^2 \frac{A}{2}}{\prod \cos \frac{A}{2}} = \frac{s \cos^2 \frac{A}{2}}{\frac{s}{4R}} = 4R \cos^2 \frac{A}{2} \rightarrow (1) \\
 \therefore (a) \stackrel{by (1)}{\Leftrightarrow} \sqrt{4R^2 \cos^2 \frac{A}{2}} &\geq \frac{b+c}{2} = 2R \sin \left( \frac{B+C}{2} \right) \cos \frac{B-C}{2} \Leftrightarrow \\
 \Leftrightarrow 2R \cos \frac{A}{2} &\geq 2R \cos \frac{A}{2} \cos \frac{B-C}{2} \Leftrightarrow \cos \frac{B-C}{2} \leq 1 \rightarrow (a_1) \\
 \because -\frac{\pi}{2} < \frac{B-C}{2} < \frac{\pi}{2} \therefore 0 < \cos \frac{B-C}{2} &\leq 1 \Rightarrow (a_1) \Rightarrow (a) \text{ is true} \\
 2r(r_b + r_c) \stackrel{by (1)}{\Leftrightarrow} 8Rr \cos^2 \frac{A}{2} &= 8 \left( \frac{abc}{4S} \right) \left( \frac{S}{s} \right) \left( \frac{s(s-a)}{bc} \right) \stackrel{(2)}{=} a(b+c-a) \\
 \therefore (b) \Leftrightarrow \frac{(b+c)^2}{4} &\geq a(b+c) - a^2 \Leftrightarrow (b+c)^2 + a^2 - 4a(b+c) \geq 0 \\
 \Leftrightarrow (b+c-2a)^2 &\geq 0 \rightarrow \text{true} \Rightarrow (b) \text{ is true (Done)}
 \end{aligned}$$

**829.** In  $\Delta ABC$  the following relationship holds:

$$\sqrt{\frac{a}{s-a}} + \sqrt{\frac{b}{s-b}} + \sqrt{\frac{c}{s-c}} \geq 3\sqrt{2}$$

*Proposed by Bogdan Fustei-Romania*

*Solution by Daniel Sitaru-Romania*

$$\begin{aligned}
 \mathbf{a} &= \mathbf{y} + \mathbf{z}, \mathbf{b} = \mathbf{z} + \mathbf{x}, \mathbf{c} = \mathbf{x} + \mathbf{y} \\
 3\sqrt{2} &= \frac{3}{\sqrt{xyz}} \cdot \sqrt{2xyz} = \frac{3}{\sqrt{xyz}} \cdot \sqrt[6]{8x^3y^3z^3} = \frac{3}{\sqrt{xyz}} \cdot \sqrt[6]{x^2y^2z^2 \cdot 8xyz} \leq \\
 &\stackrel{CESARO}{\leq} \frac{3}{\sqrt{xyz}} \cdot \sqrt[6]{x^2y^2z^2 \cdot (y+z)(z+x)(x+y)} = \\
 &= \frac{1}{\sqrt{xyz}} \cdot 3 \sqrt[3]{\sqrt{yz(y+z)} \cdot \sqrt{zx(z+x)} \cdot \sqrt{xy(x+y)}} \stackrel{GM-AM}{\leq} \frac{1}{\sqrt{xyz}} \cdot \sum \sqrt{yz(y+z)} =
 \end{aligned}$$



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$$= \sum \sqrt{\frac{y+z}{x}} = \sum \sqrt{\frac{a}{s-a}} = \sqrt{\frac{a}{s-a}} + \sqrt{\frac{b}{s-b}} + \sqrt{\frac{c}{s-c}}$$

**830. In  $\Delta ABC$  the following relationship holds:**

$$(m_a + m_b + m_c)^2 \leq 4s^2 - 16Rr + 5r^2$$

*Proposed by X.G. Chu, X.Z Yang – China*

*Solution by Bogdan Fustei-Romania*

**Jack Garfunkel inequality:** any  $\Delta ABC$ :  $m_a + l_b + h_c \leq \frac{\sqrt{3}}{2}(a + b + c) = \frac{\sqrt{3}}{2} \cdot 2p = p\sqrt{3}$ ;

$$x = p - a > 0; y = p - b > 0; z = p - c > 0 \Rightarrow a = y + z; b = z + x; c = x + y;$$

$$\begin{aligned} m_a &= \frac{1}{2}\sqrt{2(b^2 + c^2) - a^2} = \frac{1}{2}\sqrt{2(x+z)^2 + 2(x+y)^2 + (y+z)^2} \\ &= \frac{1}{2}\sqrt{4x^2 + 4x(y+z) + y^2 - 2yz + z^2} = \sqrt{\left(x + \frac{y+z}{2}\right)^2 - yz} \\ &= \frac{1}{\sqrt{3}}\sqrt{3\left(x + \frac{y+z}{2} - \sqrt{yz}\right)\left(x + \frac{y+z}{2} + \sqrt{yz}\right)} \leq \frac{1}{\sqrt{3}} \cdot \frac{3\left(x + \frac{y+z}{2} - \sqrt{yz}\right)\left(x + \frac{y+z}{2} + \sqrt{yz}\right)}{2} \\ &= \frac{2x + y + z - \sqrt{yz}}{\sqrt{3}} \\ l_b &= \frac{2\sqrt{ac}}{a+c}\sqrt{p(p-b)} \leq \sqrt{p(p-b)} = \sqrt{y(x+y+z)} \\ l_c &= \frac{2\sqrt{ab}}{a+b}\sqrt{p-c} \leq \sqrt{p(p-c)} = \sqrt{z(x+y+z)} \\ \Rightarrow m_a + l_b + l_c &\leq \frac{2x + y + z - \sqrt{yz}}{\sqrt{3}} + \sqrt{x+y+z}(\sqrt{y} + \sqrt{z}) \leq \frac{2x + y + z - \sqrt{yz}}{\sqrt{3}} + \\ &+ \frac{2}{3}\sqrt{x+y+z} \cdot \frac{\sqrt{3}}{2}(\sqrt{y} + \sqrt{z}) \leq \frac{1}{\sqrt{3}}2x + y + z - \sqrt{yz} + x + y + z + \frac{3}{4}(\sqrt{y} + \sqrt{z})^2 \leq \\ &\leq \frac{1}{\sqrt{3}}[3(x+y+z) - (\sqrt{y} - \sqrt{z})^2] \leq \frac{1}{\sqrt{3}} \cdot 3(x+y+z) \leq \frac{\sqrt{3}}{2}(a+b+c) \\ \Rightarrow m_a + l_b + l_c &\leq \frac{\sqrt{3}}{2}(a+b+c) \quad (1) \end{aligned}$$



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$$\Rightarrow m_a + l_b + h_c \leq \frac{\sqrt{3}}{2}(a + b + c)$$

*Take into account:  $m_a + l_b + l_c \leq \frac{\sqrt{3}}{2}(a + b + c) = p\sqrt{3}$  (and the analogs)*

$$\sum m_a + \sum l_a + \sum l_a = \sum m_a + 2 \sum l_a \leq 3\sqrt{3}p \leq \sqrt{4p^2 - 16Rr + 5r^2} + \\ + 2\sqrt{4p^2 - 16Rr + 5r^2}$$

$$l_a + l_b + l_c \leq p\sqrt{3} \Rightarrow 2(l_a + l_b + l_c) \leq 2\sqrt{3}p$$

$$p\sqrt{3} \leq \sqrt{4p^2 - 16Rr + 5r^2} \Rightarrow 3p^2 \leq 4p^2 - 16Rr + 5r^2 \Rightarrow 16Rr - 5r^2 \leq p^2 \Rightarrow \\ \Rightarrow Gerretsen's\ inequality$$

$$\Rightarrow \sum m_a \leq \sqrt{4p^2 - 16Rr + 5r^2} \Rightarrow (m_a + m_b + m_c)^2 \leq 4p^2 - 16Rr + 5r^2$$

*Q.E.D.*

**831. In  $\Delta ABC$  the following relationship holds:**

$$\frac{w_a^2}{r_b + r_c} + \frac{w_b^2}{r_c + r_a} + \frac{w_c^2}{r_a + r_b} \leq \frac{1}{2}(h_a + h_b + h_c)$$

*Proposed by Bogdan Fustei-Romania*

**Solution by Daniel Sitaru-Romania**

$$\begin{aligned} \sum \frac{w_a^2}{r_b + r_c} &= \sum \frac{w_a^2}{\frac{s}{s-b} + \frac{s}{s-c}} = \frac{1}{s} \sum \frac{w_a^2(s-b)(s-c)}{a} \leq \\ &\leq \frac{1}{s} \sum \frac{s(s-a)(s-b)(s-c)}{a} = \frac{S^2}{S} \sum \frac{1}{a} = S \cdot \frac{ab + bc + ca}{abc} = \\ &= S \cdot \frac{s^2 + r^2 + 4Rr}{4RS} = \frac{1}{2} \cdot \frac{s^2 + r^2 + 4Rr}{2R} = \frac{1}{2}(h_a + h_b + h_c) \end{aligned}$$

**832. In  $\Delta ABC$ :**

$$\prod \cos \frac{3A}{2} = 0 \rightarrow \sum a \sin^2 A \geq \frac{9\sqrt{3}}{2}r$$

*Proposed by Daniel Sitaru – Romania*



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*Solution by Rajsekhar Azaad-India*

$$\prod \cos \frac{3A}{2} = 0 \Rightarrow \text{Any one angle of triangle is } \frac{\pi}{3}$$

$$\text{Let } A = \frac{\pi}{3} \Rightarrow \cos \frac{\pi}{3} = \frac{b^2 + c^2 - a^2}{2bc} \Rightarrow b^2 + c^2 - bc = a^2 \quad (i)$$

$$\begin{aligned} \text{Now, } \sum a \sin^2 A &= \frac{\sum a^3}{4R^2} = \frac{a^3 + b^3 + c^3}{4R^2} = \frac{a^3 + (b+c)(b^2 - bc + c^2)}{4R^2} = \frac{a^3 + a^2(b+c)}{4R^2} \{ \text{from (i)} \} \\ &= \frac{a^2}{4R^2} \cdot 2S = 2S \cdot \sin^2 A = 2S \cdot \sin^2 A = 2S \left( \frac{\sqrt{3}}{2} \right)^2 \\ &= \frac{3S}{2} \geq \frac{3}{2} \cdot 3\sqrt{3}r = \frac{9\sqrt{3}r}{2} \quad (\text{proved}) \end{aligned}$$

**833.** In  $\Delta ABC$  the following relationship holds:

$$\frac{4s^2}{9R} \sqrt{\frac{2}{R}} \leq \frac{a}{\sqrt{r_a}} + \frac{b}{\sqrt{r_b}} + \frac{c}{\sqrt{r_c}} \leq \frac{3R}{\sqrt{r}}$$

*Proposed by Mehmet Sahin-Ankara-Turkey*

*Solution 1 by Soumava Chakraborty-Kolkata-India*

$$\frac{4s^2}{9R} \sqrt{\frac{2}{R}} \stackrel{(a)}{\leq} \frac{a}{\sqrt{r_a}} + \frac{b}{\sqrt{r_b}} + \frac{c}{\sqrt{r_c}} \stackrel{(b)}{\leq} \frac{3R}{\sqrt{r}}$$

$$\sum \frac{a}{\sqrt{r_a}} = \sum \sqrt{a} \sqrt{\frac{a}{r_a}} \stackrel{CBS}{\leq} \sqrt{2s} \sqrt{\sum \frac{a}{r_a}}$$

$$\text{Now, } \sum \frac{a}{r_a} = \frac{1}{\Delta} \sum a(s-a) = \frac{1}{\Delta} \{ s(2s) - \sum a^2 \} = \frac{1}{\Delta} \{ 2s^2 - 2(s^2 - 4Rr - r^2) \} \stackrel{(2)}{=} \frac{2(4R+r)}{s}$$

$$\therefore (1), (2) \Rightarrow \sum \frac{a}{\sqrt{r_a}} \leq \sqrt{2(4R+r) \cdot 2} = 2\sqrt{\frac{(4R+r)r}{r}} \stackrel{\text{Euler}}{\leq} 2\sqrt{\frac{(4R+\frac{r}{2})(\frac{r}{2})}{r}} = \frac{3R}{\sqrt{r}} \Rightarrow (b) \text{ is true}$$

$$\text{Now, } \sum \frac{a}{\sqrt{r_a}} = \sum \frac{a^2}{a\sqrt{r_a}} \stackrel{\text{Bergström}}{\geq} \frac{4s^2}{\sum \sqrt{a}\sqrt{ar_a}} \stackrel{(3)}{\leq} \frac{4s^2}{\sqrt{2s}\sqrt{\sum ar_a}}$$

$$\begin{aligned} \sum ar_a &= \sum \left( a \frac{\Delta}{s-a} \right) = \Delta \sum \frac{a-s+s}{s-a} = \Delta(-3) + \frac{\Delta s}{(s-a)(s-b)(s-c)} \sum (s-b)(s-c) \\ &= \Delta \left[ -3 + \frac{s}{r^2 s} (3s^2 - 4s^2 + s^2 + 4Rr + r^2) \right] = \Delta \left( -3 + \frac{4R+r}{r} \right) = \left( \frac{4R-2r}{r} \right) \cdot rs \stackrel{(4)}{=} s(4R-2r) \end{aligned}$$



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$$(3), (4) \Rightarrow \sum \frac{a}{\sqrt{r_a}} \geq \frac{4s^2}{2s\sqrt{2R-r}} \stackrel{?}{\geq} \frac{4s^2}{9R} \sqrt{\frac{2}{R}} \Leftrightarrow 9R\sqrt{R} \stackrel{?}{\geq} 2s\sqrt{4R-2r} \Leftrightarrow 81R^3 \stackrel{(5)}{\stackrel{?}{\geq}} 8(2R-r)s^2$$

$$\text{Now, RHS of (5)} \stackrel{\text{Gerretsen}}{\leq} 8(2R-r)(4R^2 + 4Rr + 3r^2) \stackrel{?}{\leq} 81R^3$$

$$\Leftrightarrow 17t^3 - 32t^2 - 16t + 24 \geq 0 \left( t = \frac{R}{r} \right) \Leftrightarrow$$

$$\Leftrightarrow (t-2)\{(17t+36)(t-2) + 60\} \geq 0 \rightarrow \text{true} \therefore t \stackrel{\text{Euler}}{\geq} 2 \Rightarrow (a) \text{ is true}$$

(Done)

*Solution by Soumitra Mandal-Chandar Nagore-India*

$$\sum_{cyc} r_a = 4R + r, \sum_{cyc} r_a r_b = s^2 \text{ and } \prod_{cyc} r_a = s^2 r$$

$$\text{Let } a \geq b \geq c \text{ then } \frac{1}{r_a} \leq \frac{1}{r_b} \leq \frac{1}{r_c}$$

$$\sum_{cyc} \frac{a}{\sqrt{r_a}} \stackrel{\text{CHEBYSHEV'S INEQUALITY}}{\leq} \frac{1}{3} \left( \sum_{cyc} a \right) \left( \sum_{cyc} \frac{1}{\sqrt{r_a}} \right) = 2s \cdot \sqrt{\frac{1}{r} \sum_{cyc} \frac{1}{r_a}} = \frac{2s}{\sqrt{3r}} \leq \frac{3\sqrt{3}R}{\sqrt{3r}} = \frac{3R}{\sqrt{r}}$$

$$\begin{aligned} \sum_{cyc} \frac{a}{\sqrt{r_a}} &= \sum_{cyc} \frac{a^2}{a\sqrt{r_a}} \stackrel{\text{CAUCHY SCHWARZ}}{\geq} \frac{(a+b+c)^2}{\sum_{cyc} a\sqrt{r_a}} \geq \frac{4s^2}{\sqrt{(\sum_{cyc} a^2)(\sum_{cyc} r_a)}} \\ &\geq \frac{4s^2}{\sqrt{9R^2(4R+r)}} = \frac{4s^2}{3R} \cdot \frac{1}{\sqrt{4R+r}} \geq \frac{4s^2}{3R} \cdot \frac{1}{\sqrt{4R+\frac{R}{2}}} = \frac{4s^2}{9R} \sqrt{\frac{2}{R}} \end{aligned}$$

(proved)

**834.** In  $\Delta ABC$  the following relationship holds:

$$\left( \frac{h_b + h_c}{m_a} \right)^2 + \left( \frac{h_c + h_a}{m_b} \right)^2 + \left( \frac{h_a + h_b}{m_c} \right)^2 \leq \frac{8(2R-r)}{R}$$

*Proposed by Bogdan Fustei – Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\therefore m_a \geq \frac{b+c}{2} \cos \frac{A}{2} \text{ etc,}$$



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$$\begin{aligned}
 \therefore LHS &\leq \sum \left\{ \left( \frac{2\Delta}{b} + \frac{2\Delta}{c} \right) \frac{2}{(b+c) \cos \frac{A}{2}} \right\}^2 = \sum 16\Delta^2 \left[ \frac{b+c}{bc(b+c) \cos \frac{A}{2}} \right]^2 = \\
 &= \sum \left[ 16\Delta^2 \frac{1}{b^2 c^2} \cdot \frac{bc}{s(s-a)} \right] = \frac{16\Delta^2}{s} \sum \frac{1}{bc(s-a)} = \frac{16\Delta^2 \{\sum a(s-b)(s-c)\}}{abc s(s-a)(s-b)(s-c)} \\
 &= \frac{16\Delta^2}{4Rrs\Delta^2} \sum \{a(s^2 - s(b+c) + bc)\} \\
 &= \frac{4}{Rrs} \{s^2(2s) - 2s(s^2 + 4Rr + r^2) + 12Rrs\} = \frac{8(2Rr - r^2)}{Rr} = \frac{8(2R - r)}{R} \\
 &\quad (\text{proved})
 \end{aligned}$$

**835. In  $\Delta ABC$  the following relationship holds:**

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{4}{9R^2} (w_a h_b + w_b h_c + w_c h_a)$$

*Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan*

*Solution by Daniel Sitaru-Romania*

$$\begin{aligned}
 LHS &= \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \stackrel{AM-GM}{\geq} 3 \sqrt[3]{\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{a}} = 3, \quad RHS = \frac{4}{9R^2} \sum w_a h_b \leq \frac{4}{9R^2} \sum w_a w_b \leq \\
 &\leq \frac{4}{9R^2} \sum \sqrt{s(s-a) \cdot s(s-b)} \stackrel{AM-GM}{\leq} \frac{4s}{9R^2} \sum \frac{s-a+s-b}{2} = \frac{4s^2}{9R^2} \leq \\
 &\stackrel{MITRINOVIC}{\leq} \frac{4}{9R^2} \cdot \frac{27R^2}{4} = 3, \quad LHS \geq 3 \geq RHS
 \end{aligned}$$

**836. In  $\Delta ABC$  the following relationship holds:**

$$(s_a + m_b + h_c)(m_a + h_b + s_c)(h_a + s_b + m_c) \geq 729r^3$$

*Proposed by Daniel Sitaru – Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\because m_a \geq s_a \geq h_a \text{ etc, } \therefore LHS \geq (\sum h_a)^3 \stackrel{?}{\geq} 729r^3 \Leftrightarrow \sum h_a \stackrel{?}{\geq} 9r$$



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$$\Leftrightarrow \sum ab \stackrel{?}{\geq} 18Rr \Leftrightarrow s^2 \stackrel{?}{\geq} 14Rr - r^2$$

Now,  $s^2 \stackrel{Gerretsen}{\geq} 16Rr - 5r^2 \stackrel{?}{\geq} 14Rr - r^2 \Leftrightarrow 2Rr \stackrel{?}{\geq} 4r^2 \Leftrightarrow R \stackrel{?}{\geq} 2r \rightarrow \text{true}$   
*(Euler) (Proved)*

**837. In  $\Delta ABC$  the following relationship holds:**

$$3\sqrt[3]{w_a^2 w_b^2 w_c^2} \leq s^2 \leq \sqrt{3(m_a^4 + m_b^4 + m_c^4)}$$

*Proposed by Daniel Sitaru – Romania*

**Solution 1 by Alexandru Capmare-Romania**

1)  $3\sqrt[3]{w_a^2 w_b^2 w_c^2} \leq s^2$  but  $3\sqrt[3]{w_a^2 w_b^2 w_c^2} \leq w_a^2 + w_b^2 + w_c^2$  so we have:

$$w_a^2 + w_b^2 + w_c^2 \leq s^2$$

$$w_a = \frac{2bc}{b+c} \sqrt{\frac{s(s-a)}{bc}} |^2$$

$$w_a^2 \leq \frac{4(bc)^2}{(b+c)^2} \cdot \frac{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b+c-2a}{2}\right)}{bc}$$

$$w_a^2 = \frac{bc[(b+c)^2 - a^2]}{(b+c)^2} \Rightarrow \sum \frac{bc[(b+c)^2 - a^2]}{(b+c)^2} \leq \frac{(a+b+c)^2}{4}$$

$$\text{but } bc \leq \frac{(b+c)^2}{4} \Rightarrow \sum \frac{(b+c)[(b+c)^2 - a^2]}{4(b+c)^2} \leq \frac{(a+b+c)^2}{4} | \cdot 4$$

$$\Rightarrow \sum (b+c)^2 - a^2 \leq (a+b+c)^2$$

$$\sum a^2 + 2 \sum ab \leq (a+b+c)^2$$

$$(a+b+c)^2 \leq (a+b+c)^2 \text{ true (1)}$$

2)  $\frac{(a+b+c)^2}{4} \leq \sqrt{3(m_a^4 + m_b^4 + m_c^4)}$  but  $\frac{(m_a^2)^2}{1} + \frac{(m_b^2)^2}{1} + \frac{(m_c^2)^2}{1} \geq \frac{(m_a^2 + m_b^2 + m_c^2)^2}{3}$  so we have  $\Rightarrow$



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$$\begin{aligned}
 \Rightarrow \frac{(a+b+c)^2}{4} &\leq \sqrt{3 \cdot \frac{(m_a^2 + m_b^2 + m_c^2)^2}{3}} \Rightarrow \frac{(a+b+c)^2}{4} \leq m_a^2 + m_b^2 + m_c^2 \\
 m_a^2 + m_b^2 + m_c^2 &\geq \frac{(a+b+c)^2}{4}; \sum \frac{2(b^2 + c^2) - a^2}{4} \geq \frac{(a+b+c)^2}{4} \cdot 4 \\
 \sum 2(b^2 + c^2) - a^2 &\geq (a+b+c)^2; 3 \sum a^2 \geq \sum a^2 + 2 \sum ab; 2 \sum a^2 \geq 2 \sum ab \\
 \sum a^2 &\geq \sum ab \text{ true (2)} \\
 (1), (2) \Rightarrow 3\sqrt[3]{w_a^2 w_b^2 w_c^2} &\leq s^2 \leq \sqrt{3(m_a^4 + m_b^4 + m_c^4)}
 \end{aligned}$$

*Solution 2 by Rajsekhar Azaad-India*

$$w_a^2 = \frac{4bc}{(b+c)^2} \cdot s(s-a) \leq s(s-a) \quad (1)$$

$$\begin{aligned}
 \text{Now, } 3 \cdot \sqrt[3]{w_a^2 w_b^2 w_c^2} &\leq w_a^2 + w_b^2 + w_c^2 \quad (GM \leq AM) \\
 &\leq \sum s(s-a) \text{ from (1)} \\
 &= s(3s - 2s) = s^2 \quad (2)
 \end{aligned}$$

$$\text{Again, } m_a^2 = \frac{2(b^2 + c^2) - a^2}{4} \geq s(s-a)$$

$$\begin{aligned}
 \text{Now, } \sqrt{3(m_a^4 + m_b^4 + m_c^4)} &\geq \sum m_a^2 \quad [\because (\sum a)^2 \leq 3 \sum a^2] \\
 \geq \sum s(s-a) &= s^2 \quad (3). \text{ From (2) and (3) our inequality proved.}
 \end{aligned}$$

**838. In  $\Delta ABC$  the following relationship holds:**

$$\sum \sqrt{\frac{r_a}{h_a}} \geq \frac{1}{6} \left( \sum \sqrt{\frac{m_a}{h_a}} \right) \left( \sum \frac{h_b + h_c}{m_a} \right)$$

*Proposed by Bogdan Fustei – Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\sum \sqrt{\frac{r_a}{h_a}} = \sum \sqrt{\frac{\Delta}{s-a} \cdot \frac{a}{2\Delta}} \stackrel{(1)}{=} \frac{1}{\sqrt{2}} \sum \sqrt{\frac{a}{s-a}}$$



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$$\begin{aligned}
 \sum \frac{h_b + h_c}{m_a} &\stackrel{m_a \geq \frac{b+c}{2} \cos \frac{A}{2}}{\leq} \sum \frac{\frac{2\Delta}{b} + \frac{2\Delta}{c}}{\frac{b+c}{2} \cos \frac{A}{2}} = 2\Delta \sum \left[ \left( \frac{b+c}{bc} \right) \cdot \frac{2}{(b+c)} \sqrt{\frac{bc}{s(s-a)}} \right] \\
 &= \frac{4\Delta}{\sqrt{s}} \sum \frac{1}{\sqrt{bc(s-a)}} = \frac{4\Delta}{\sqrt{sabc}} \sum \sqrt{\frac{a}{s-a}} \Rightarrow \sum \frac{h_b + h_c}{m_a} \stackrel{(2)}{=} \left( \sum \sqrt{\frac{a}{s-a}} \right) \left( \frac{4\Delta}{\sqrt{sabc}} \right) \\
 &\text{Also, } \sum \sqrt{\frac{m_a}{h_a}} \stackrel{CBS}{\leq} \sqrt{\sum m_a} \sqrt{\sum \frac{1}{h_a}} \stackrel{\sum m_a \leq 4R+r}{\underset{(3)}{\leq}} \sqrt{\frac{4R+r}{r}}
 \end{aligned}$$

$$\begin{aligned}
 (1), (2), (3) \Rightarrow \text{it suffices to prove: } \frac{1}{\sqrt{2}} &\geq \frac{1}{6} \sqrt{\frac{4R+r}{r}} \frac{4rs}{\sqrt{4Rrs^2}} \Leftrightarrow \frac{1}{2} \geq \frac{1}{36} \left( \frac{4R+r}{r} \right) \left( \frac{16r^2s^2}{4Rrs^2} \right) \\
 \Leftrightarrow 9R &\geq 2(4R+r) \Leftrightarrow R \geq 2r \rightarrow \text{true (Euler) (Proved)}
 \end{aligned}$$

**839. In  $\Delta ABC$  the following relationship holds:**

$$\sqrt[3]{r_a r_b} + \sqrt[3]{r_b r_c} + \sqrt[3]{r_c r_a} \geq 3 \sqrt[3]{9r^2}$$

*Proposed by Mehmet Sahin-Ankara-Turkey*

**Solution 1 by Daniel Sitaru-Romania**

$$\begin{aligned}
 \sum \sqrt[3]{r_a r_b} &= \sum \sqrt[3]{\frac{s^2}{(s-a)(s-b)}} = \sum \sqrt[3]{\frac{s(s-a)(s-b)(s-c)}{(s-a)(s-b)}} = \\
 &= \sqrt[3]{s} \sum \sqrt[3]{s-c} \stackrel{AM-GM}{\geq} \sqrt[3]{s} \cdot \sqrt[3]{\prod \sqrt[3]{s-c}} = 3 \sqrt[3]{s} \cdot \sqrt[3]{(s-a)(s-b)(s-c)} = \\
 &= 3 \sqrt[9]{s^2 S^2} = 3 \sqrt[9]{s^4 r^2} \stackrel{MITRINOVIC}{\geq} 3 \sqrt[9]{(3\sqrt{3}r)^4 r^2} = 3 \sqrt[9]{(9r^2)^3} = 3 \sqrt[3]{9r^2}
 \end{aligned}$$

**Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia**

$$(s-a)(s-b)(s-c) \leq \frac{s^3}{27} \quad (1) \text{ (True)}$$

$$(1) \Rightarrow \frac{\Delta^3}{(s-a)(s-b)(s-c)} \geq \frac{27\Delta^3}{s^3} \Rightarrow r_a r_b r_c \geq 27r^3$$

$$\frac{\sqrt[3]{r_a r_b} + \sqrt[3]{r_b r_c} + \sqrt[3]{r_c r_a}}{3} \stackrel{AM \geq GM}{\geq} \sqrt[9]{(r_a r_b)(r_b r_c)(r_c r_a)} \geq \sqrt[3]{9r^2}$$



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**840. If  $x, y, z$  are the distances from  $I$  – incenter to  $BC, CA$  respectively  $AB$  in  $\Delta ABC$  then:**

$$\left( \frac{r}{2x \sin \frac{A}{2}} \right)^{\frac{r}{\sin \frac{A}{2}}} + \left( \frac{r}{2y \sin \frac{B}{2}} \right)^{\frac{r}{\sin \frac{B}{2}}} + \left( \frac{r}{2z \sin \frac{C}{2}} \right)^{\frac{r}{\sin \frac{C}{2}}} \geq 3$$

*Proposed by Daniel Sitaru – Romania*

**Solution by Soumava Chakraborty-Kolkata-India**

$$\because x = y = z = r, \therefore LHS \stackrel{A-G}{\geq_{(1)}} 3 \sqrt[3]{\prod \left( \frac{1}{2 \sin \frac{A}{2}} \right)^{\frac{r}{\sin \frac{A}{2}}}}.$$

By weighted GM – weighted HM inequality:  $S = \sqrt{\prod \left( \frac{1}{2 \sin \frac{A}{2}} \right)^{\frac{r}{\sin \frac{A}{2}}}} \geq \frac{\sum r \csc \frac{A}{2}}{\sum \frac{r}{\sin \frac{A}{2}} \times 2 \sin \frac{A}{2}}$

$$= \frac{\sum \csc \frac{A}{2}}{6} \stackrel{\text{Jensen}}{\geq} \frac{3 \csc \left( \frac{A+B+C}{6} \right)}{6} \quad (\because f(x) = \csc \frac{x}{2}, \forall x \in (0, \pi) \text{ is convex}) = 1$$

$$\Rightarrow \ln S \geq 0 \Rightarrow \left( \sum r \csc \frac{A}{2} \right) \ln S \geq 0 \Rightarrow \ln S^{\sum r \csc \frac{A}{2}} \geq 0 \Rightarrow \prod \left( \frac{1}{2 \sin \frac{A}{2}} \right)^{\frac{r}{\sin \frac{A}{2}}} \stackrel{(2)}{\geq} 1$$

**(1),(2)  $\Rightarrow LHS \geq 3$  (proved)**

**841. In  $\Delta ABC$  the following relationship holds:**

$$\frac{m_a}{w_a} + \frac{m_b}{w_b} + \frac{m_c}{w_c} \geq \sqrt{\frac{r}{2R}} \left( \frac{r_a + r}{r_a - r} + \frac{r_b + r}{r_b - r} + \frac{r_c + r}{r_c - r} \right)$$

*Proposed by Bogdan Fustei – Romania*

**Solution by Soumava Chakraborty-Kolkata-India**

$$\sum \frac{r_a + r}{r_a - r} = \sum \frac{\frac{\Delta}{s-a} + \frac{\Delta}{s}}{\frac{\Delta}{s-a} - \frac{\Delta}{s}} = \sum \frac{b+c}{a} = \frac{\sum ab(2s-c)}{abc} =$$



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$$\begin{aligned}
 &= \frac{2s(s^2 + 4Rr + r^2) - 12Rrs}{4Rrs} \stackrel{(1)}{=} \frac{s^2 - 2Rr + r^2}{2Rr} \\
 &\because m_a \geq \frac{b+c}{2} \cos \frac{A}{2} \text{ etc, } \therefore \sum \frac{m_a}{w_a} \stackrel{(2)}{\geq} \sum \frac{(b+c) \cos \frac{A}{2}}{2} \cdot \frac{(b+c)}{2bc \cos \frac{A}{2}} = \sum \frac{(b+c)^2}{4bc} = \frac{\sum a(b^2 + c^2 + 2bc)}{4abc} \\
 &= \frac{\sum ab(2s - c) + 6abc}{4abc} = \frac{2s(s^2 + 4Rr + r^2) + 12Rrs}{16Rrs} = \frac{s^2 + 10Rr + r^2}{8Rr} \\
 &(1), (2) \Rightarrow \text{it suffices to prove: } R(s^2 + 10Rr + r^2)^2 \geq 8r(s^2 - 2Rr + r^2)^2 \\
 &\Leftrightarrow R[s^4 + r^2(10R + r)^2 + 2s^2(10Rr + r^2)] \geq 8r[s^4 + r^2(2R - r)^2 - 2s^2(2Rr - r^2)] \\
 &\Leftrightarrow (R - 2r)s^4 + 2s^2R(10Rr + r^2) + 16s^2r(2Rr - r^2) + Rr^2(10R + r)^2 - 8r^3(2R - r)^2 \stackrel{(3)}{\geq} 6rs^4 \\
 &\text{LHS of (3)} \stackrel{\substack{\text{Gerretsen} \\ (4)}}{\geq} s^2(R - 2r)(16Rr - 5r^2) + 2s^2R(10Rr + r^2) + \\
 &\quad + 16s^2r(2Rr - r^2) + Rr^2(10R + r)^2 - 8r^3(2R - r)^2 \\
 &\text{RHS of (3)} \stackrel{\substack{\text{Gerretsen} \\ (5)}}{\geq} 6rs^2(4R^2 + 4Rr + 3r^2) \\
 &(4), (5) \Rightarrow \text{in order to prove (3), it suffices to show} \\
 &s^2\{(R - 2r)(16R - 5r) + 2R(10R + r) + 16(2Rr - r^2) - 6(4R^2 + 4Rr + 3r^2)\} + \\
 &+ Rr(10R + r)^2 - 8r^2(2R + r)^2 \geq 0 \Leftrightarrow s^2(12R^2 - 27Rr - 24r^2) + Rr(10R + r)^2 - \\
 &- 8r^2(2R - r)^2 \geq 0 \Leftrightarrow \underbrace{s^2(12R^2 - 27Rr + 6r^2)}_{=(12R-3r)(R-2r) \geq 0} + Rr(10R + r)^2 - 8r^2(2R - r)^2 \stackrel{(6)}{\geq} 30r^2s^2 \\
 &\text{LHS of (6)} \stackrel{\substack{\text{Gerretsen} \\ (7)}}{\geq} (16Rr - 5r^2)(12R^2 - 27Rr + 6r^2) + Rr(10R + r)^2 - 8r^2(2R - r)^2 \\
 &\quad \& \text{RHS of (6)} \stackrel{\substack{\text{Gerretsen} \\ (8)}}{\geq} 30r^2(4R^2 + 4Rr + 3r^2)
 \end{aligned}$$

$$\begin{aligned}
 &(7), (8) \Rightarrow \text{in order to prove (6), it suffices to show:} \\
 &(16R - 5r)(12R^2 - 27Rr + 6r^2) + R(10R + r)^2 - 8r(2R - r)^2 - \\
 &- 30r(4R^2 + 4Rr + 3r^2) \geq 0 \Leftrightarrow (t - 2)\{(t - 2)(73t + 136) + 288\} \geq 0 \rightarrow \text{true} \\
 &\left(t = \frac{R}{r}\right) \because t \stackrel{\text{Euler}}{\geq} 2 \quad (\text{proved})
 \end{aligned}$$



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**842. In  $\Delta ABC$  the following relationship holds:**

$$\sqrt{\frac{r_a}{h_a}} + \sqrt{\frac{r_b}{h_b}} + \sqrt{\frac{r_c}{h_c}} \geq 2 \left( \frac{w_a}{r_b + r_c} + \frac{w_b}{r_c + r_a} + \frac{w_c}{r_a + r_b} \right)$$

*Proposed by Bogdan Fustei – Romania*

**Solution 1 by Soumitra Mandal-Chandar Nagore-India**

We know,  $abc \geq 8(s-a)(s-b)(s-c)$  now,

$$\begin{aligned} 2 \sum_{cyc} \frac{w_a}{r_b + r_c} &\stackrel{A.M. \geq G.M.}{\leq} \sum_{cyc} \frac{w_a}{\sqrt{r_b r_c}} \leq \sum_{cyc} \frac{\sqrt{s(s-a)}}{\sqrt{\frac{\Delta}{s-b} \cdot \frac{\Delta}{s-c}}} = \frac{3\sqrt{s(s-a)(s-b)(s-c)}}{\Delta} = 3 \\ \sum_{cyc} \sqrt{\frac{r_a}{h_a}} &= \sum_{cyc} \sqrt{\frac{\frac{\Delta}{s-a}}{\frac{2\Delta}{a}}} = \sum_{cyc} \sqrt{\frac{a}{2(s-a)}} \stackrel{A.M. \geq G.M.}{\geq} 3 \sqrt[3]{\sqrt{\frac{abc}{8 \prod_{cyc} (s-a)}}} \geq 3 \\ \therefore \sum_{cyc} \sqrt{\frac{r_a}{h_a}} &\geq 2 \sum_{cyc} \frac{w_a}{r_b + r_c} \end{aligned}$$

**Solution 2 by Soumava Chakraborty-Kolkata-India**

$$\frac{r_a}{h_a} = \frac{s \tan \frac{A}{2}}{\frac{2rs}{a}} = \frac{4R \sin \frac{A}{2} \cos \frac{A}{2} \tan \frac{A}{2}}{2r} = \frac{2R}{r} \sin^2 \frac{A}{2} \Rightarrow \sqrt{\frac{r_a}{h_a}} \stackrel{(1)}{=} \sqrt{\frac{2R}{r}} \sin \frac{A}{2}$$

$$\text{Similarly, } \sqrt{\frac{r_b}{h_b}} \stackrel{(2)}{=} \sqrt{\frac{2R}{r}} \sin \frac{B}{2} \text{ & } \sqrt{\frac{r_c}{h_c}} \stackrel{(3)}{=} \sqrt{\frac{2R}{r}} \sin \frac{C}{2}$$

$$\begin{aligned} (1) + (2) + (3) \Rightarrow LHS &= \sqrt{\frac{2R}{r}} \sum \sin \frac{A}{2} \stackrel{A-G}{\geq} 3 \sqrt{\frac{2R}{r}} \sqrt[3]{\prod \sin \frac{A}{2}} = 3 \sqrt{\frac{2R}{r}} \sqrt[3]{\frac{r}{4R}} \stackrel{?}{\geq} 3 \Leftrightarrow \\ &\Leftrightarrow \left(\frac{2R}{r}\right)^3 \left(\frac{r}{4R}\right)^2 \stackrel{?}{\geq} 1 \Leftrightarrow R \stackrel{?}{\geq} 2r \rightarrow \text{true} \end{aligned}$$

$$\therefore LHS \geq 3 \quad (3)$$

$$\begin{aligned} \text{Now, } \frac{w_a}{r_b + r_c} &= \frac{w_a}{\frac{\Delta}{s-b} + \frac{\Delta}{s-c}} = \frac{w_a(s-b)(s-c)}{\Delta(2s-b-c)} = \frac{w_a((s-b)(s-c))^2}{a\sqrt{s(s-a)(s-b)(s-c)}} \stackrel{G-A}{\leq} \frac{w_a \left( \frac{s-b+s-c}{2} \right)}{a\sqrt{s(s-a)}} = \\ &= \frac{1}{2} \cdot \frac{w_a}{\sqrt{s(s-a)}} \stackrel{w_a \leq \sqrt{s(s-a)}}{\stackrel{(4)}{\leq}} \frac{1}{2}. \text{ Similarly, } \frac{w_b}{r_c + r_a} \stackrel{(5)}{\leq} \frac{1}{2} \text{ & } \frac{w_c}{r_a + r_b} \stackrel{(6)}{\leq} \frac{1}{2} \end{aligned}$$

$$(4)+(5)+(6) \Rightarrow RHS \stackrel{(7)}{\leq} 2 \cdot \frac{3}{2} = 3$$

(3), (7)  $\Rightarrow LHS \geq RHS$  (Proved)

**843.** In  $\triangle ABC$  the following relationship holds:

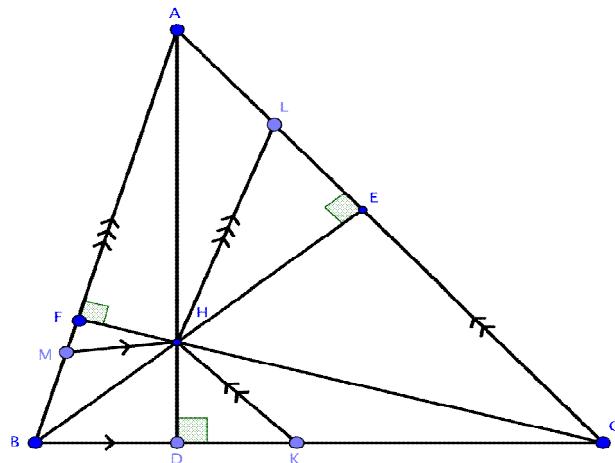
$$\frac{4\cot^2 \frac{A}{2}\cot^2 \frac{B}{2}\cot^2 \frac{C}{2}}{9\left(\cot^2 \frac{A}{2} + \cot^2 \frac{B}{2} + \cot^2 \frac{C}{2}\right) + 27} \leq 1$$

*Proposed by Nguyen Van Nho-Nghe An-Vietnam*

*Solution by Daniel Sitaru-Romania*

$$\begin{aligned} \frac{4\left(\prod \cot^2 \frac{A}{2}\right)^2}{9 \sum \cot^2 \frac{A}{2} + 27} &= \frac{4\left(\frac{s}{r}\right)^2}{9 \cdot \frac{s^2 - 2r^2 - 8Rr}{r^2} + 27} = \frac{\frac{4s^2}{r^2}}{\frac{9s^2 + 9r^2 - 72Rr}{r^2}} = \\ &= \frac{4s^2}{9s^2 + 9r^2 - 72Rr} \leq 1 \Leftrightarrow 5s^2 \geq 72Rr - 9r^2 \\ 5s^2 &\stackrel{GERRETSEN}{\geq} 5(16Rr - 5r^2) \geq 72Rr - 9r^2 \Leftrightarrow 8Rr \geq 16r^2 \Leftrightarrow R \stackrel{EULER}{\geq} 2r \end{aligned}$$

**844.**



If  $HM = x$ ,  $HK = y$ ,  $HL = z$  then:

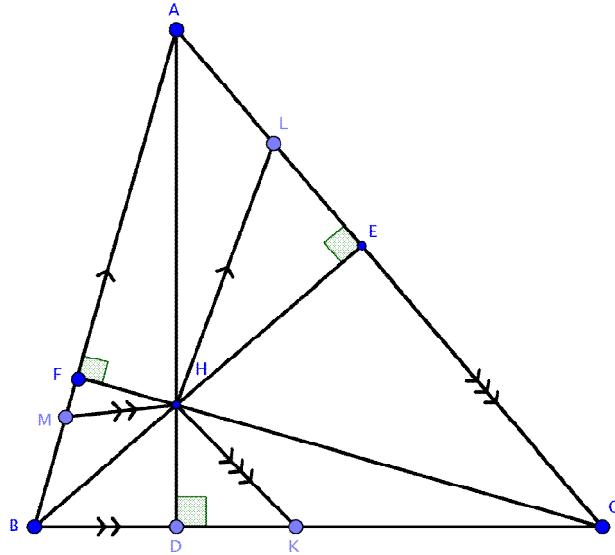
# R M M

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$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} \geq \frac{4\sqrt{3}S}{R^2} \text{ and } xyz \leq \frac{R^5}{4S}$$

*Proposed by Mehmet Sahin-Ankara-Turkey*

*Solution by Soumava Cakraborty-Kolkata-India*



$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} \stackrel{(a)}{\geq} \frac{4\sqrt{3}S}{R^2} \quad \& \quad xyz \stackrel{(b)}{\leq} \frac{R^5}{4S}$$

$$\frac{HM}{BD} = \frac{AH}{AD} \quad (\because AHD \sim \Delta ADB) \Rightarrow \frac{x}{c|\cos B|} = \frac{2R|\cos A|}{\frac{bc}{2R}} \Rightarrow x = \frac{4R^2 c |\cos A| |\cos B|}{bc} =$$

$$= \frac{4R^2 |\cos A \cos B|}{2R \sin B} \Rightarrow \frac{x}{a} = \frac{2R |\cos A \cos B|}{3R \sin A \sin B} \Rightarrow \frac{a}{x} \stackrel{(1)}{=} \frac{\sin A \sin B}{|\cos A \cos B|}. \text{ Similarly, } \frac{b}{y} \stackrel{(2)}{=} \frac{\sin B \sin C}{|\cos B \cos C|} \quad \&$$

$$\frac{c}{z} \stackrel{(3)}{=} \frac{\sin C \sin A}{|\cos C \cos A|}$$

$$(1) + (2) + (3) \Rightarrow \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \sum \frac{\sin A \sin B}{|\cos A \cos B|} \stackrel{A-G}{\geq} 3^3 \sqrt[3]{\frac{\sin^2 A \sin^2 B \sin^2 C}{\cos^2 A \cos^2 B \cos^2 C}} =$$

$$= 3 \sqrt[3]{\frac{S^2}{4R^4(\cos A \cos B \cos C)^2}} \quad (\because S = 2R^2 \sin A \sin B \sin C)$$



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$$\begin{aligned}
 & \prod \cos A \leq \frac{1}{8} \quad 3 \sqrt[3]{\frac{S^2}{4R^4} \cdot 64} \stackrel{\text{Mitrinovic}}{\geq} 3^3 \sqrt[3]{\frac{16S^3}{R^4 r \left(\frac{3\sqrt{3}R}{2}\right)}} \stackrel{\text{Euler}}{\geq} 3S^3 \sqrt[3]{\frac{16}{R^4 \left(\frac{R}{2}\right) \left(\frac{3\sqrt{3}R}{2}\right)}} = 3S \sqrt[3]{\frac{64}{3\sqrt{3}R^6}} \\
 & = \frac{3 \cdot 4S}{\sqrt{3}R^2} = \frac{4\sqrt{3}S}{R^2} \Rightarrow (\text{a}) \text{ is proved. Also, } xyz = \prod \left( \frac{2R|\cos A \cos B|}{\sin B} \right) = \frac{8R^3(\cos A \cos B \cos C)^2}{\frac{abc}{8R^3}} \leq \\
 & \leq \frac{\prod \cos A \leq \frac{1}{8} \cdot 64R^6 \cdot \frac{1}{64}}{4Rr8} = \frac{R^5}{4S} \Rightarrow (\text{b}) \text{ is proved (Done)}
 \end{aligned}$$

**845.** In  $\Delta ABC$  the following relationship holds:

$$\frac{1}{h_a w_a m_a} + \frac{1}{h_b w_b m_b} + \frac{1}{h_c w_c m_c} \leq \frac{2R - r}{S^2}$$

*Proposed by Bogdan Fustei – Romania*

**Solution 1 by Marian Ursărescu – Romania**

We know  $m_a w_a \geq s(s-a) \Rightarrow \frac{1}{m_a w_a} \leq \frac{1}{s(s-a)}$ . Inequality becomes:

$$\sum \frac{1}{h_a w_a m_a} \leq \sum \frac{1}{s(s-a) h_a} = \sum \frac{a}{2Ss(s-a)} \Rightarrow$$

We must show this:  $\frac{1}{2Ss} \sum \frac{a}{s-a} \leq \frac{2R-r}{S^2} \Leftrightarrow$

$$\frac{1}{s} \sum \frac{a}{s-a} \leq \frac{2(2R-r)}{S} \quad (1)$$

$$\text{But } \sum \frac{a}{s-a} = \frac{2(2R-r)}{r} \quad (2)$$

$$\text{From (1)+(2)} \Rightarrow \frac{1}{s} \cdot \sum \frac{a}{s-a} = \frac{1}{s} \cdot \frac{2(2R-r)}{r} = \frac{2(2R-r)}{S} \Rightarrow (1) \text{ its true.}$$

**Solution 2 by Soumava Chakraborty-Kolkata-India**

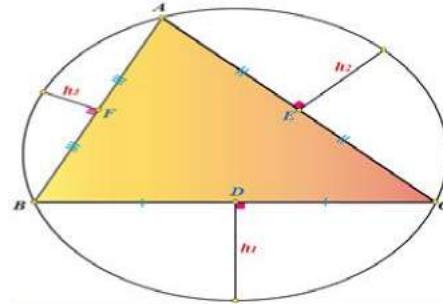
$$\therefore m_a \geq \frac{b+c}{2} \cos \frac{A}{2} \text{ etc,}$$

$$\therefore m_a w_a h_a \geq \frac{b+c}{2} \cos \frac{A}{2} \cdot \frac{2bc}{b+c} \cos \frac{A}{2} \cdot \frac{2S}{a} = bc \cdot \frac{s(s-a)}{bc} \cdot \frac{2rs}{a} = \frac{2rs^2(s-a)}{a}$$

$$\Rightarrow \sum \frac{1}{m_a w_a h_a} \leq \frac{1}{2rs^2} \sum \frac{a}{s-a} = \frac{1}{2rs^2} \sum \left( \frac{a-s+s}{s-a} \right) = \frac{1}{2rs^2} \left( -3 + s \sum \frac{1}{s-a} \right) =$$

$$\begin{aligned}
 &= \frac{1}{2rs^2} \left\{ -3 + \frac{s}{r^2 S} \sum (s-b)(s-c) \right\} = \\
 &= \frac{1}{2rs^2} \left\{ -3 + \frac{1}{r^2} (3s^2 - 4s^2 + s^2 + 4Rr + r^2) \right\} = \\
 &= \frac{1}{2rs^2} \left( -3 + \frac{4R+r}{r} \right) = \frac{4R-2r}{2r^2 s^2} = \frac{2R-r}{s^2} \quad (\text{proved})
 \end{aligned}$$

846.



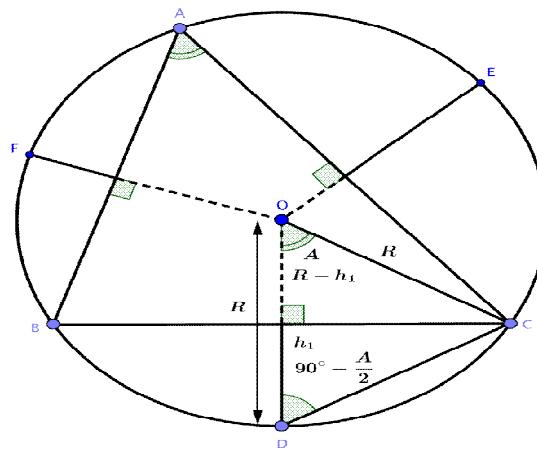
$R$  – circumradius of  $ABC$ ,  $r$  – inradius of  $ABC$ ,  $D, E, F$  midpoints of sides

$h_1, h_2, h_3$  perpendiculars from midpoints to circumcircle

$$\text{Prove: } h_1^2 + h_2^2 + h_3^2 \geq 3r^2$$

*Proposed by Abdilkadir Altintas-Afyonkarashisar-Turkey*

*Solution by Mehmet Sahin-Ankara-Turkey*



$$\sin(90^\circ - A) = \frac{R-h_1}{R} \Rightarrow \cos A = 1 - \frac{h_1}{R}, \cos B = 1 - \frac{h_2}{R}, \cos C = 1 - \frac{h_3}{R}$$



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$$1 + \frac{r}{R} = 3 - (h_1 + h_2 + h_3) \cdot \frac{1}{R}, \quad h_1 + h_2 + h_3 = 2R - r$$

*Using Cauchy-Schwarz's inequality:*  $(h_1 + h_2 + h_3)^2 \leq 3(h_1^2 + h_2^2 + h_3^2)$

$$(2R - r)^2 \leq 3(h_1^2 + h_2^2 + h_3^2) \text{ and}$$

$$R \geq 2r \Rightarrow 3(h_1^2 + h_2^2 + h_3^2) \geq (2R - r)^2 \geq (3r)^2 \Rightarrow h_1^2 + h_2^2 + h_3^2 \geq 3r^2 \therefore$$

**847. In  $\Delta ABC$  the following relationship holds:**

$$4(m_a + m_b + m_c) \geq \sum \frac{(h_b + h_c)(r_a + r)}{r_a - r}$$

*Proposed by Bogdan Fustei – Romania*

**Solution 1 by Soumava Chakraborty-Kolkata-India**

$$\begin{aligned} \frac{(h_b + h_c)(r_a + r)}{r_a - r} &= \frac{\left(\frac{2\Delta}{b} + \frac{2\Delta}{c}\right)\left(\frac{\Delta}{s-a} + \frac{\Delta}{s}\right)}{\frac{\Delta}{s-a} - \frac{\Delta}{s}} = \frac{2\Delta}{bc}(b+c)(b+c) \cdot \frac{1}{a} = \frac{2\Delta(b+c)^2}{4R\Delta} = \\ &= \frac{(b+c)^2}{2R} \stackrel{\substack{\text{Reverse} \\ \text{Chebyshev}}}{\leq} \frac{2(b^2 + c^2)}{2R} = \frac{4(b^2 + c^2)}{4R} \stackrel{\substack{\text{Tereshin} \\ (1)}}{\leq} 4m_a \\ \text{Similarly, } \frac{(h_c + h_a)(r_b + r)}{r_b - r} &\stackrel{(2)}{\leq} 4m_b \text{ & } \frac{(h_a + h_b)(r_c + r)}{r_c - r} \stackrel{(3)}{\leq} 4m_c \end{aligned}$$

$$(1) + (2) + (3) \Rightarrow 4(\sum m_a) \geq \sum \frac{(h_b + h_c)(r_a + r)}{r_a - r} \text{ (proved)}$$

**Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia**

$$\Delta ABC: 4 \sum m_a \geq \sum \frac{(h_b + h_c)(r_a + r)}{r_a - r} \quad (1)$$

$$a) h_b + h_c = 2S\left(\frac{1}{b} + \frac{1}{c}\right) = \frac{2S}{bc} \cdot (b+c) = \frac{2bc \sin A}{2bc} (b+c) = \sin A (b+c) \quad (*)$$

$$r_a + r = \frac{S}{s-a} + \frac{S}{s} = S\left(\frac{b+c}{s(s-a)}\right) = (b+c) \cdot \tan \frac{A}{2} \quad (**)$$

$$b) r_a - r = \frac{S}{p-a} - \frac{S}{p} = a \cdot \tan \frac{A}{2} \quad (**)$$



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$$\begin{aligned}
 (1) \Rightarrow & \left( \underbrace{\mathbf{m}_a \geq \frac{b^2+c^2}{4R}}_{(\text{True})} \Rightarrow 4\mathbf{m}_a \geq \frac{b^2+c^2}{R} \geq \frac{(b+c)^2}{2R} \right) \\
 \begin{cases} 4\mathbf{m}_a \geq \frac{(b+c)^2}{2R} \\ 4\mathbf{m}_b \geq \frac{(a+c)^2}{2R} \Rightarrow 4 \sum \mathbf{m}_a \geq \sum \frac{(b+c)^2}{2R} = \sum \frac{\sin A (b+c)^2}{a} = \\ 4\mathbf{m}_c \geq \frac{(a+b)^2}{2R} \end{cases} \\
 & = \sum \frac{\sin A (b+c)(b+c) \tan \frac{A}{2}}{a \tan \frac{A}{2}} \stackrel{(*) ; (**)}{=} \sum \frac{(h_b + h_c)(r_a + r)}{r_a - r}
 \end{aligned}$$

**848.** In  $\Delta ABC$  the following relationship holds:

$$\sum \mathbf{m}_a \sqrt{\frac{r_a}{h_a}} \geq \left( 6 + \frac{h_a}{r_a} + \frac{h_b}{r_b} + \frac{h_c}{r_c} \right) \sqrt{\frac{Rr}{2}}$$

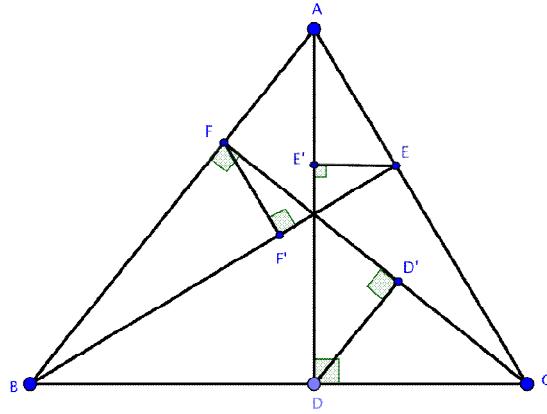
*Proposed by Bogdan Fustei – Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\begin{aligned}
 \sum \mathbf{m}_a \sqrt{\frac{r_a}{h_a}} &= \sum \mathbf{m}_a \sqrt{\frac{s \tan \frac{A}{2} \cdot 4R \sin \frac{A}{2} \cos \frac{A}{2}}{2rs}} \geq \\
 &\stackrel{(1)}{\geq} \sum \frac{b+c}{2} \cos \frac{A}{2} \sqrt{\frac{2R}{r}} \sin \frac{A}{2} = \frac{1}{4} \sum (b+c) \left( \frac{a}{2R} \right) \sqrt{\frac{2R}{r}} = \sqrt{\frac{2}{Rr}} \left( \frac{\sum ab}{4} \right) \\
 RHS &= \left( 6 + \sum \frac{2\Delta}{a} \cdot \frac{s-a}{\Delta} \right) \sqrt{\frac{Rr}{2}} = \left( 6 + 2 \sum \frac{s-a}{a} \right) \sqrt{\frac{Rr}{2}} = \left( 6 + 2s \frac{\sum ab}{4Rrs} - 6 \right) \sqrt{\frac{Rr}{2}} = \\
 &\stackrel{(2)}{=} \frac{\sum ab}{2Rr} \sqrt{\frac{Rr}{2}}
 \end{aligned}$$

(1), (2)  $\Rightarrow$  it suffices to prove:  $\frac{\sum ab}{4} \geq \frac{\sum ab}{2Rr} \cdot \frac{Rr}{2} \Leftrightarrow 1 \geq 1 \rightarrow \text{true (Proved)}$

849.

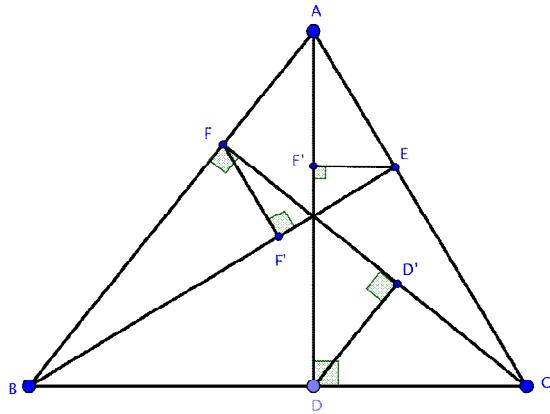


In acute  $\Delta ABC$ ,  $DD' = x$ ,  $EE' = y$ ,  $FF' = z$ . Prove that:

$$xyz \leq \frac{3\sqrt{3}R^2}{64} \text{ and } \frac{a}{z} + \frac{b}{x} + \frac{c}{y} \geq 8 \left(1 + \frac{r}{R}\right)$$

*Proposed by Mehmet Sahin-Ankara-Turkey*

*Solution by Marian Ursarescu-Romania*



$$m(\widehat{DCH}) = \frac{\pi}{2} - B \Rightarrow$$

$$\left. \begin{array}{l} \sin\left(\frac{\pi}{2} - B\right) = \frac{x}{DC} \\ \cos C = \frac{CD}{b} \Rightarrow CD = b \cos C \end{array} \right\} \Rightarrow x = b \cos B \cos C \text{ and similarly: } y = c \cos C \cos A,$$

$$z = a \cos A \cos B$$



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$$xyz \leq \frac{3\sqrt{3}}{64} R^3 \Leftrightarrow abc(\cos A \cos B \cos C)^2 \leq \frac{3\sqrt{3}}{64} R^3 \quad (1)$$

$$\text{But } \cos A \cos B \cos C \leq \frac{1}{8} \text{ (in acute } \Delta) \quad (2)$$

(1)+(2)  $\Rightarrow$  we must show:  $abc \leq 3\sqrt{3}R^3 \Leftrightarrow 8R^3 \sin A \sin B \sin C \leq 3\sqrt{3}R^3 \Leftrightarrow$

$$\Leftrightarrow \sin A \sin B \sin C \leq \frac{3\sqrt{3}}{8} \text{ (true)}$$

$$\frac{a}{z} + \frac{b}{x} + \frac{c}{y} \geq \frac{8(R+r)}{R} \Leftrightarrow \frac{1}{\cos A \cos B} + \frac{1}{\cos B \cos C} + \frac{1}{\cos C \cos A} \geq \frac{8(R+r)}{R} \quad (3)$$

$$\text{But } \sum \frac{1}{\cos A \cos B} = \frac{4R(R+r)}{s^2 - (2R+r)^2} \quad (4). \text{ We must show: } \frac{4R(R+r)}{s^2 - (2R+r)^2} \geq \frac{8(R+r)}{R} \Leftrightarrow$$

$\Leftrightarrow R^2 \geq 2s^2 - 2(2R+r)^2 \Leftrightarrow 2s^2 \leq 2(2R+r)^2$  true, because is Carlitz inequality.

**850. If in  $\Delta ABC$ , I – incenter then:**

$$AI + BI + CI \leq 4r \left( \frac{m_a}{h_b + h_c} + \frac{m_b}{h_c + h_a} + \frac{m_c}{h_a + h_b} \right)$$

*Proposed by Bogdan Fustei – Romania*

**Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia**

$$AI = \frac{r}{\sin \frac{A}{2}}; BI; CI \dots$$

$$\begin{aligned} \sum AI &= \sum \frac{r}{\sin \frac{A}{2}} = r \cdot \sum \frac{2 \cos \frac{A}{2}}{2 \sin \frac{A}{2} \cos \frac{A}{2}} = 2r \sum \frac{\cos \frac{A}{2}}{\sin A} = 2r \sum \frac{2 \frac{b+c}{2} \cos \frac{A}{2}}{\sin A (b+c)} = \\ &= 4r \cdot \sum \frac{\frac{b+c}{2} \cos \frac{A}{2}}{\frac{a}{2R} (b+c)} \leq 4r \sum \frac{m_a}{h_b + h_c} \end{aligned}$$

**851. In  $\Delta ABC$  the following relationship holds:**

$$((s-a)(s-b))^3 + ((s-b)(s-c))^3 + ((s-c)(s-a))^3 < (rr_a + rr_b + rr_c)^3$$

*Proposed by Rovsen Pirguliyev-Sumgait-Azerbaijan*



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**Solution 1 by Soumava Chakraborty-Kolkata-India**

$$RHS = \left( \sum \frac{\Delta}{s} \cdot \frac{\Delta}{s-a} \right)^3 = \left\{ \sum \frac{s(s-a)(s-b)(s-c)}{s(s-a)} \right\}^3 \stackrel{(1)}{=} \left\{ \sum (s-b)(s-c) \right\}^3$$

$$\text{Let } (s-a)(s-b) = x, (s-b)(s-c) = y, (s-c)(s-a) = z$$

*Then, using (1), given inequality becomes:  $\sum x^3 < (\sum x)^3, \forall x, y, z > 0$*

*which is true (Hence proved)*

**Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia**

$$\sum \left( \frac{(s-a)}{\Delta} \cdot \frac{(s-b)}{\Delta} \cdot \Delta^2 \right)^3 \stackrel{\Delta=\sqrt{r \cdot r_a \cdot r_b \cdot r_c}}{=} \sum \left( \frac{r \cdot r_a \cdot r_b \cdot r_c}{r_a \cdot r_b} \right)^3 = \sum (r \cdot r_c)^3$$

$$x^3 + y^3 + z^3 < (x+y+z)^3 \Leftrightarrow \sum_{\Delta} (rr_c)^3 < \left( \sum_{\Delta} rr_c \right)^3$$

**852. In  $\Delta ABC$  the following relationship holds:**

$$\sum m_a \sqrt{\frac{a}{s-a}} \geq \sqrt{2} \cdot \frac{(w_a + w_b + w_c)^2}{m_a + m_b + m_c}$$

*Proposed by Bogdan Fustei – Romania*

**Solution by Soumava Chakraborty-Kolkata-India**

$$w_a \leq m_a, \text{etc} \therefore \sum w_a \leq \sum m_a \Rightarrow RHS \leq \sqrt{2} \sum w_a \stackrel{w_a \leq \sqrt{s(s-a)}, \text{etc}}{\leq} \sqrt{2s} \sum \sqrt{s-a} \leq$$

$$\stackrel{CBS}{\leq} \sqrt{6s} \sqrt{\sum (s-a)} = \sqrt{6s} \therefore RHS \stackrel{(1)}{\leq} \sqrt{6s} \because m_a \leq \frac{b+c}{2} \cos \frac{A}{2}, \text{etc},$$

$$\therefore LHS \geq \sum \left( \frac{b+c}{2} \cos \frac{A}{2} \right) \sqrt{\frac{a}{s-a}} = \sum \left\{ \left( \frac{b+c}{2} \right) \sqrt{\frac{s(s-a)}{bc}} \sqrt{\frac{a}{s-a}} \right\} =$$

$$= \sum \left\{ \left( \frac{b+c}{2} \right) \sqrt{\frac{sa^2}{abc}} \right\} = \sqrt{\frac{s}{abc}} \left( \frac{2 \sum ab}{2} \right) = \sqrt{\frac{1}{4Rr}} (s^2 + 4Rr + r^2)$$



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$$\therefore LHS \stackrel{(2)}{\geq} \sqrt{\frac{1}{4Rr}(s^2 + 4Rr + r^2)}$$

(1), (2)  $\Rightarrow$  it suffices to prove

$$: \sqrt{\frac{1}{4Rr}(s^2 + 4Rr + r^2)} \geq \sqrt{6}s \Leftrightarrow (s^2 + 4Rr + r^2)^2 \geq 24Rrs^2$$

$$\Leftrightarrow s^4 + r^2 + (4R + r)^2 + 2s^2(4Rr + r^2) - 24Rrs^2 \stackrel{(3)}{\geq} 0. Now,$$

$$\begin{aligned} LHS \text{ of (1)} &\stackrel{Gerretsen}{\geq} s^2(16Rr - 5r^2) + r^2(4R + r)^2 + 2s^2(4Rr + r^2) - 24Rrs^2 = \\ &= r^2(4R + r)^2 - 3r^2s^2 = r^2((4R + r)^2 - 3s^2) \stackrel{Trucht}{\geq} 0 \Rightarrow (3) \text{ is true (Proved)} \end{aligned}$$

**853. In  $\Delta ABC$  the following relationship holds:**

$$\frac{a}{R + r_a} + \frac{b}{R + r_b} + \frac{c}{R + r_c} \geq \frac{2s}{3R - r}$$

*Proposed by Mehmet Sahin-Ankara-Turkey*

**Solution 1 by Marian Ursarescu-Romania**

$$\begin{aligned} \frac{a}{R + r_a} + \frac{b}{R + r_b} + \frac{c}{R + r_c} &= \frac{a^2}{aR + ar_a} + \frac{b^2}{bR + br_b} + \frac{c^2}{cR + cr_c} \geq \\ &\geq \frac{(a+b+c)^2}{R(a+b+c)+ar_a+br_b+cr_c} \quad (\text{from Cauchy's inequality}) \Rightarrow \end{aligned}$$

$$\Rightarrow \sum \frac{a}{R+r_a} \geq \frac{4s^2}{2Rs+ar_a+br_b+cr_c} \quad (1)$$

$$\text{But } \sum ar_a = 2s(2R - r) \quad (2)$$

$$\begin{aligned} \text{From (1)+(2)} \Rightarrow \sum \frac{a}{R+r_a} &\geq \frac{4s^2}{2Rs+2s(2R-r)} \Leftrightarrow \sum \frac{a}{R+r_a} \geq \frac{4s^2}{2s(R+2R-r)} \Leftrightarrow \\ &\Leftrightarrow \sum \frac{a}{R+r_a} \geq \frac{2s}{3R-r} \quad (\text{done}) \end{aligned}$$

**Solution 2 by Soumitra Mandal-Chandar Nagore-India**

$$\sum_{cyc} (s-a)(s-b) = r(4R + r)$$



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$$\sum_{cyc} \frac{a}{R + r_a} = \sum_{cyc} \frac{a^2}{aR + ar_a} \geq \frac{(a + b + c)^2}{R(a + b + c) + \sum_{cyc} ar_a} = \frac{4s^2}{2sR + \sum_{cyc} ar_a}$$

we need to prove,  $\frac{4s^2}{2sR + \sum_{cyc} ar_a} \geq \frac{2s}{3R - r} \Leftrightarrow 2s(3R - r) \geq 2sR + \sum_{cyc} ar_a \Leftrightarrow$

$$\Leftrightarrow 2s(2R - r) \geq \Delta \sum_{cyc} \frac{s - (s - a)}{s - a} \Leftrightarrow 2s(2R - r) + 3\Delta \geq \Delta s \sum_{cyc} \frac{1}{s - a} =$$

$$= s^2 r \frac{\sum_{cyc} (s - a)(s - b)}{\prod_{cyc} (s - a)} \Leftrightarrow s(4R + r) \geq \frac{s^2 r}{sr^2} \sum_{cyc} (s - a)(s - b) \Leftrightarrow$$

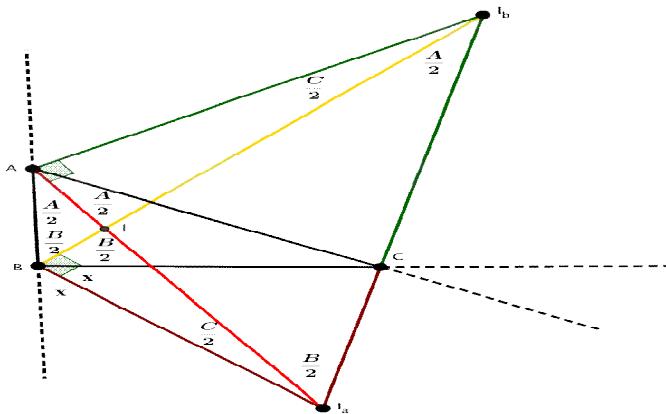
$\Leftrightarrow r(4R + r) \geq \sum_{cyc} (s - a)(s - b)$ . Which is true Hence Proved

**854. If in  $\triangle ABC$ ,  $N$  – ninepoint center,  $I$  – incenter,  $I_a, I_b, I_c$  – excenters then:**

$$\frac{NI_a + NI_b + NI_c}{AI_a \cdot II_a + BI_b \cdot II_b + CI_c \cdot II_c} \geq \frac{6r}{R^2}$$

Proposed by Daniel Sitaru – Romania

*Solution by Soumava Chakraborty-Kolkata-India*



$$\sin \frac{C}{2} = \frac{BI}{II_a} = \frac{r}{\sin \frac{B}{2} \cdot II_a} \Rightarrow II_a \stackrel{(1)}{=} \frac{r}{\sin \frac{B}{2} \sin \frac{C}{2}} \therefore AI_a = \frac{r}{\sin \frac{A}{2}} + \frac{r \sin \frac{A}{2}}{\pi \sin \frac{A}{2}} =$$



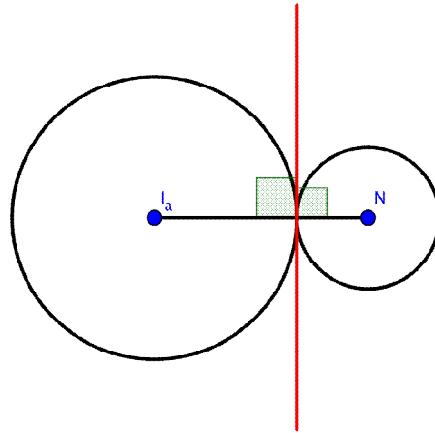
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$$\stackrel{(2)}{=} \frac{r \left( \sin \frac{B}{2} \sin \frac{C}{2} + \sin \frac{A}{2} \right)}{\pi \sin \frac{A}{2}} \therefore AI_a \cdot II_a \stackrel{\text{by (1),(2)}}{=} \frac{r^2 \left( \pi \sin \frac{A}{2} + \sin^2 \frac{A}{2} \right)}{\left( \pi \sin \frac{A}{2} \right)^2}$$

$$= \frac{r^2 \frac{r}{4R} + r^2 \sin^2 \frac{A}{2}}{\frac{r^2}{16R^2}} \stackrel{(a)}{=} \frac{4Rr + 16R^2 \sin^2 \frac{A}{2}}{1}$$

Similarly,  $BI_b \cdot II_b \stackrel{(b)}{=} 4Rr + 16R^2 \sin^2 \frac{B}{2}$  &  $CI_c \cdot II_c \stackrel{(c)}{=} 4Rr + 16R^2 \sin^2 \frac{C}{2}$

$$(a) + (b) + (c) \Rightarrow \sum AI_a \cdot II_a = 12Rr + 8R^2 \sum (1 - \cos A) = 12Rr + 8R^2 \left( 3 - 1 - \frac{r}{R} \right) = \\ = 12Rr + 8R(2R - r) \stackrel{(3)}{=} 16R^2 + 4Rr$$



$\because$  the three excircles touch the nine-point circle,  $\therefore NI_a = r_a + \frac{R}{2}$ ,  $NI_b = r_b + \frac{R}{2}$ ,

$$NI_c = r_c + \frac{R}{2} \therefore \sum NI_a = \sum r_a + \frac{3R}{2} = 4R + r + \frac{3R}{2} \stackrel{(4)}{=} \frac{11R + 2r}{2}$$

$$(3), (4) \Rightarrow LHS = \frac{11R + 2r}{2(16R^2 + 4Rr)} \stackrel{\text{Euler}}{\geq} \frac{22r + 2r}{32R^2 + 4R^2} = \frac{24r}{36R^2} = \frac{2r}{3R^2} \quad (\text{Proved})$$

855. In  $\Delta ABC$  the following relationship holds:

$$\frac{m_a \cdot h_a}{(b+c)^2} + \frac{m_b \cdot h_b}{(c+a)^2} + \frac{m_c \cdot h_c}{(a+b)^2} \geq \frac{9r^2}{4R^2}$$

*Proposed by Mehmet Sahin-Ankara-Turkey*



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**Solution 1 by Myagmarsuren Yadamsuren-Darkhan-Mongolia**

$$\begin{aligned}
 \frac{w_a^2}{4bc(s-a)} &= \frac{1}{(b+c)^2} \quad (*) \\
 (*) \Rightarrow \sum \frac{m_a h_a w_a^2}{4bc(s-a)} &= \sum \frac{m_a \cdot \frac{bc}{2R} w_a^2}{4bc(s-a)} = \frac{1}{8sR} \cdot \sum \frac{m_a \cdot w_a^2}{s-a} \geq \left[ \begin{array}{l} m_a \geq \frac{b+c}{2} \cdot \cos \frac{A}{2} \\ w_a = \frac{2\sqrt{bc} \cdot \sqrt{s(s-a)}}{b+c} = \frac{2bc}{b+c} \cos \frac{A}{2} \end{array} \right] \geq \\
 &\geq \frac{1}{8sR} \cdot \sum \frac{\frac{b+c}{2} \cdot \cos \frac{A}{2} \cdot \frac{2bc}{b+c} \cdot \cos \frac{A}{2} \cdot w_a}{s-a} = \frac{1}{8sR} \cdot \sum \frac{bc \cdot \cos^2 \frac{A}{2} \cdot w_a}{s-a} = \\
 &= \frac{1}{8R} \cdot \sum \frac{bc}{s(s-a)} \cdot \frac{s(s-a)}{bc} = \frac{1}{8R} \cdot \sum w_a \geq \frac{1}{8R} \sum h_a = \frac{1}{16R^2} \cdot \sum ab = \\
 &= \frac{1}{16R^2} \cdot (s^2 + 4Rr + r^2) \stackrel{R \geq 2r}{\stackrel{s \geq 3\sqrt{3}r}{\geq}} \frac{1}{16R^2} \cdot (27r^2 + 8r^2 + r^2) = \frac{1}{16R^2} \cdot 36r^2 = \frac{9r^2}{4R^2}
 \end{aligned}$$

**Solution 2 by Soumava Chakraborty-Kolkata-India**

$$\begin{aligned}
 m_a &\geq \frac{b+c}{2} \cos \frac{A}{2}, \text{ etc.} \therefore \sum \frac{m_a h_a}{(b+c)^2} \geq \frac{2S}{2} \sum \left( \frac{\cos \frac{A}{2}}{a(b+c)} \right) = S \sum \left( \frac{\cos \frac{A}{2}}{a2R(\sin B + \sin C)} \right) = \\
 &= S \sum \left( \frac{\cos \frac{A}{2}}{4Ra \sin \frac{B+C}{2} \cos \frac{B-C}{2}} \right) = \frac{S}{4R} \sum \left( \frac{1}{a \cos \frac{B-C}{2}} \right) \Rightarrow \\
 \Rightarrow \sum \frac{m_a h_a}{(b+c)^2} &\stackrel{(a)}{\geq} \frac{S}{4R} \sum \left( \frac{1}{a \cos \frac{B-C}{2}} \right) \therefore 0 < B < \pi \text{ and } -\pi < -C < 0 \therefore -\frac{\pi}{2} < \frac{B-C}{2} < \frac{\pi}{2} \Rightarrow \\
 \Rightarrow 0 < \cos \frac{B-C}{2} &\leq 1. \text{ Similarly, } 0 < \cos \frac{C-A}{2} \leq 1 \text{ and } 0 < \cos \frac{A-B}{2} \leq 1. \text{ Using the last} \\
 \text{three inequalities and (a), } \sum \frac{m_a h_a}{(b+c)^2} &\geq \frac{S}{4R} \left( \sum \frac{1}{a} \right) = \frac{rs(s^2 + 4Rr + r^2)}{4R \cdot 4Rrs} = \frac{s^2 + 4Rr + r^2}{16R^2} \stackrel{?}{\geq} \frac{9r^2}{4R^2} \Leftrightarrow \\
 \Leftrightarrow s^2 + 4Rr + r^2 &\stackrel{?}{\geq} 36r^2. \text{ Now, LHS of (1) } \stackrel{\text{Gerretsen}}{\geq} 20Rr - 4r^2 \stackrel{?}{\geq} 36r^2 \Leftrightarrow \\
 &\stackrel{?}{\geq} 20Rr \geq 40r^2 \rightarrow \text{true (Euler) (Proved)}
 \end{aligned}$$

**Solution 3 by Marian Ursarescu-Romania**

In any  $\Delta ABC$  we have  $m_a \geq \frac{b^2 + c^2}{4R} \Rightarrow m_a \geq \frac{(b+c)^2}{8R} \Rightarrow \frac{m_a}{(b+c)^2} \geq \frac{1}{8R} \Rightarrow$  we must show this:

$$\frac{1}{8R} (h_a + h_b + h_c) \geq \frac{9r^2}{4R^2} \Leftrightarrow h_a + h_b + h_c \geq \frac{18r^2}{R} \quad (1)$$



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$$\text{But } h_a + h_b + h_c = \frac{s^2 + r^2 + 4Rr}{2R} \quad (2)$$

$$\begin{aligned} \text{From (1) + (2) we must show: } & \frac{s^2 + r^2 + 4Rr}{2R} \geq \frac{18r^2}{R} \Leftrightarrow s^2 + r^2 + 4Rr \geq 3r^2 \Leftrightarrow \\ & \Leftrightarrow s^2 + 4Rr \geq 35r^2 \quad (3) \end{aligned}$$

$$\begin{aligned} \text{But } & \left. \begin{aligned} s^2 &\geq 2 + r^2 \\ R &\geq 2r \end{aligned} \right\} \Rightarrow s^2 + 4Rr \geq 35r^2 \Rightarrow (3) \text{ its true} \end{aligned}$$

**856. In  $\Delta ABC$  the following relationship holds:**

$$\sum \frac{a^2 \sin A \sin C}{a^2 + b^2} \geq \frac{2r}{R} + \frac{1}{2} \cdot \left(\frac{r}{R}\right)^2$$

*Proposed by Marian Ursărescu – Romania*

**Solution 1 by Soumava Chakraborty-Kolkata-India**

$$\begin{aligned} LHS &= \frac{1}{4R^2} \sum \frac{a^3 c}{a^2 + b^2} \geq \frac{4Rr + r^2}{2R^2} \\ &\Leftrightarrow \sum \frac{a^3 c}{a^2 + b^2} \geq 8Rr + 2r^2 \quad (1) \end{aligned}$$

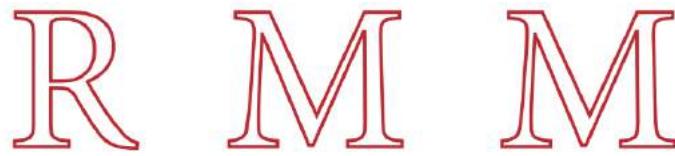
$$\begin{aligned} \text{Now, } \sum \frac{a^3 c}{a^2 + b^2} &= \sum \left( \frac{a^3 c}{a^2 + b^2} - ac + ac \right) = \sum ac \left( \frac{a^2}{a^2 + b^2} - 1 \right) + \sum ab = - \sum \frac{acb^2}{a^2 + b^2} + \sum ab \\ &\stackrel{A-G}{\geq} - \sum \frac{acb^2}{2ab} + \sum ab = - \frac{1}{2} \sum bc + \sum ab = \frac{1}{2} \sum ab \stackrel{?}{\geq} 8Rr + 2r^2 \\ &\Leftrightarrow s^2 + 4Rr + r^2 \stackrel{?}{\geq} 16Rr + 4r^2 \\ &\Leftrightarrow s^2 \stackrel{?}{\geq} 12Rr + 3r^2 \quad (2) \end{aligned}$$

$$\text{Now, LHS of (2) } \stackrel{\text{Gerretsen}}{\geq} 16Rr - 5r^2 \stackrel{?}{\geq} 12Rr + 3r^2$$

$$\Leftrightarrow 4Rr \stackrel{?}{\geq} 8r^2 \Leftrightarrow R \stackrel{?}{\geq} 2r \rightarrow \text{true (Euler)} \Rightarrow (2) \text{ is true (proved)}$$

**Solution 2 by Soumitra Mandal-Chandar Nagore-India**

$$\begin{aligned} \sum_{cyc} \frac{a^2 \sin A \sin C}{a^2 + b^2} &= \sum_{cyc} \sin A \sin C - \sum_{cyc} \frac{b^2 \sin A \sin C}{a^2 + b^2} \\ &\stackrel{AM \geq GM}{\geq} \frac{1}{4R^2} \sum_{cyc} ab - \frac{1}{2} \sum_{cyc} \frac{b}{a} \sin A \sin C = \frac{1}{4R^2} \sum_{cyc} ab - \frac{1}{8R^2} \sum_{cyc} ab \end{aligned}$$



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$$\begin{aligned}
 &= \frac{ab+bc+ca}{8R^2}, \text{ we need to prove, } \frac{ab+bc+ca}{8R^2} \geq \frac{2r}{R} + \frac{1}{2} \left( \frac{r}{R} \right)^2 \\
 \Leftrightarrow ab + bc + ca &\geq 16Rr + 4r^2 \Leftrightarrow s^2 + 4Rr + r^2 \geq 16Rr + 4r^2 \\
 \Leftrightarrow s^2 &\geq 12Rr + 3r^2 \text{ again we know } s^2 \geq 16Rr - 5r^2 \text{ hence we need to show} \\
 16Rr - 5r^2 &\geq 12Rr + 3r^2 \Leftrightarrow 12Rr + 3r^2 \Leftrightarrow 4Rr \geq 8r^2 \Leftrightarrow R \geq 2r \text{ which is true} \\
 &\text{Hence proved}
 \end{aligned}$$

**Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia**

$$\begin{aligned}
 s^2 \geq 12Rr + 3r^2 \text{ (True)} &\Leftrightarrow s^2 + 4Rr + r^2 \geq 16Rr + 4r^2 \Leftrightarrow \sum ab \geq 16Rr + 4r^2 \\
 \Leftrightarrow \sum ab &\geq 8R^2 \left( \frac{2r}{R} + \frac{1}{2} \left( \frac{r}{R} \right)^2 \right) \Leftrightarrow \frac{\sum ab}{8R^2} \geq \frac{2r}{R} + \frac{1}{2} \left( \frac{r}{R} \right)^2 \quad (*) \\
 (*) \Rightarrow \frac{\sum ab}{8R^2} &= \frac{abc}{4R^2} \cdot \frac{\sum ab}{2abc} = \frac{S}{R} \cdot \sum \frac{1}{2a} = \frac{S}{R} \cdot \sum \left( \frac{1}{a} - \frac{1}{2a} \right) = \\
 &= \frac{S}{R} \cdot \sum \left( \frac{1}{a} - \frac{b}{2ab} \right) \stackrel{Mg \leq Ma}{\leq} \frac{S}{R} \cdot \sum \left( \frac{1}{a} - \frac{b}{a^2 + b^2} \right) = \frac{S}{R} \left( \sum \left( \frac{1}{b} - \frac{b}{a^2 + b^2} \right) \right) = \\
 &= \frac{S}{R} \cdot \sum \frac{a^2}{b(a^2 + b^2)} = \frac{2R^2 \sin A \sin B \sin C}{R} \cdot \sum \frac{a^2}{2R \cdot \sin B(a^2 + b^2)} = \sum \frac{a^2 \sin A \sin C}{a^2 + b^2}.
 \end{aligned}$$

**857. In  $\Delta ABC$  the following relationship holds:**

$$3(h_a - 2r)w_a + 3(h_b - 2r)w_b + 3(h_c - 2r) \geq r \left( 2 \sum m_a + \sum w_a \right)$$

*Proposed by Bogdan Fustei – Romania*

**Solution by Soumava Chakraborty-Kolkata-India**

$$\begin{aligned}
 \sum 3(h_a - 2r)w_a &= \sum \left\{ 3 \left( \frac{2\Delta}{a} - \frac{2\Delta}{s} \right) w_a \right\} = \sum \left\{ 32rs \left( \frac{1}{a} - \frac{1}{s} \right) w_a \right\} = \\
 &= 3r \left\{ \sum \frac{2(s-a)}{a} w_a \right\} = 3r \left( \sum \frac{b+c-a}{a} w_a \right) = 2r \sum \frac{b+c}{a} w_a - 3r \sum w_a \\
 &\geq r \left( 2 \sum m_a + \sum w_a \right) \Leftrightarrow 3 \sum \frac{b+c}{a} w_a \stackrel{(1)}{\geq} 4 \sum w_a + 2 \sum m_a
 \end{aligned}$$



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$$\left. \begin{array}{l} \text{Now, } (\sum m_a)^2 \stackrel{(a)}{\leq} 4s^2 - 16Rr + 5r^2 (X. G. Chu, X. Z. Yang) \\ (\sum w_a)^2 \stackrel{(b)}{\leq} (4R + r)(\sum h_a) \text{ (Bogdan Fustei),} \\ \sum \left( \frac{b+c}{a} \right) w_a \stackrel{(c)}{\geq} 2s\sqrt{3} \text{ (Bogdan Fustei)} \end{array} \right\} \text{Now,}$$

$$2 \sum w_a \leq 2 \sum \sqrt{s(s-a)} \stackrel{C-B-S}{\leq} 2\sqrt{3}\sqrt{s}\sqrt{s} = 2\sqrt{3}s \stackrel{\text{by (c)}}{\leq} \sum \left( \frac{b+c}{a} \right) w_a \Rightarrow$$

$$\Rightarrow 2 \sum w_a \stackrel{(i)}{\leq} \sum \left( \frac{b+c}{a} \right) w_a$$

$$(i) \Rightarrow \text{in order to prove (1), it suffices to prove: } 2 \sum \frac{b+c}{a} w_a \geq 2 \sum w_a + 2 \sum m_a \Leftrightarrow$$

$$\Leftrightarrow \sum \frac{b+c}{a} w_a \stackrel{(2)}{\geq} \sum w_a + \sum m_a. \text{ Now, LHS of (2) } \stackrel{CBS}{\leq}_{(m)} \sqrt{2} \sqrt{(\sum w_a)^2 + (\sum m_a)^2}$$

$$\stackrel{\text{by (a),(b)}}{\leq} \sqrt{\frac{(4R+r)(s^2 + 4Rr + r^2)}{2R} + 4s^2 - 16Rr + 5r^2} =$$

$$= \sqrt{\frac{(4R+r)(s^2 + 4Rr + r^2) + 2R(4s^2 - 16Rr + 5r^2)}{R}}$$

$$\text{Again, LHS of (2) } \stackrel{\text{by (c)}}{\leq} 2s\sqrt{3}$$

(m), (n)  $\Rightarrow$  in order to prove (2), it suffices to prove:

$$2s\sqrt{3} \geq \sqrt{\frac{(12R+r)s^2 + r(4R+r)^2 - 2R(16Rr - 5r^2)}{R}}$$

$$\Leftrightarrow 12Rs^2 \geq 12Rs^2 + rs^2 + r(4R+r)^2 - 2R(16Rr - 5r^2) \Leftrightarrow$$

$$\Leftrightarrow r\{2R(16R - 5r) - (4R+r)^2\} \geq rs^2 \Leftrightarrow s^2 \stackrel{(3)}{\leq} 16R^2 - 18Rr - r^2$$

$$\text{Now, LHS of (3) } \stackrel{\text{Gerretsen}}{\leq} 4R^2 + 4Rr + 3r^2 \stackrel{?}{\leq} 16R^2 - 18Rr - r^2$$

$$\Leftrightarrow 6R^2 - 11Rr - 2r^2 \stackrel{?}{\geq} 0 \Leftrightarrow (R-2r)(6R+r)^2 \stackrel{?}{\geq} 0 \rightarrow \text{true} \because R \stackrel{\text{Euler}}{\geq} 2r$$

$\Rightarrow (3)$  is true  $\Rightarrow (2)$  is true (Proved)



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**858. In  $\Delta ABC$  the following relationship holds:**

$$(a + w_a)^2 + (b + w_b)^2 + (c + w_c)^2 \leq (s + 3R)^2$$

*Proposed by Mehmet Sahin-Ankara-Turkey*

*Solution by Daniel Sitaru-Romania*

$$\begin{aligned} \sum (a + w_a)^2 &= \sum a^2 + 2 \sum aw_a + \sum w_a^2 \leq \\ &\stackrel{CBS}{\leq} \sum a^2 + 2 \sqrt{\sum a^2 \sum w_a^2} + \sum w_a^2 \leq \sum a^2 + 2 \sqrt{\sum a^2 \sum s(s-a)} + \sum s(s-a) = \\ &= 2(s^2 - r^2 - 4Rr) + 2s \sqrt{\sum a^2} + s^2 \stackrel{LEIBNIZ}{\leq} 2(s^2 - r^2 - 4Rr) + 2s\sqrt{9R^2} + s^2 = \\ &= 3s^2 - 2r^2 - 8Rr + 6sR \stackrel{GERRETSEN}{\leq} s^2 + 2(4R^2 + 4Rr + 3r^2) - 2r^2 - 8Rr + 6sR = \\ &= s^2 + 8R^2 + 4r^2 + 6sR \stackrel{EULER}{\leq} s^2 + 8R^2 + 4 \cdot \frac{R^2}{4} + 6sR = s^2 + 6sR + 9R^2 = (s + 3R)^2 \end{aligned}$$

**859. In  $\Delta ABC$  the following relationship holds:**

$$\frac{s^2 - r_a^2}{s^2 + r_a^2} + \frac{s^2 - r_b^2}{s^2 + r_b^2} + \frac{s^2 - r_c^2}{s^2 + r_c^2} \geq \frac{3r}{R}$$

*Proposed by Daniel Sitaru – Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\begin{aligned} \because r_a = s \tan \frac{A}{2}, \text{ etc, } \therefore \sum \frac{s^2 - r_a^2}{s^2 + r_a^2} &= \sum \left( \frac{s^2 - s^2 \tan^2 \frac{A}{2}}{s^2 + s^2 \tan^2 \frac{A}{2}} \right) = \sum \left( \frac{1 - \tan^2 \frac{A}{2}}{1 + \tan^2 \frac{A}{2}} \right) = \sum \left( \frac{1 - \tan^2 \frac{A}{2}}{\sec^2 \frac{A}{2}} \right) = \\ &= \sum \left( \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2} \right) = \sum \cos A = 1 + \frac{r}{R} \stackrel{Euler}{\geq} \frac{2r}{R} + \frac{r}{R} = \frac{3r}{R} \text{ (Proved)} \end{aligned}$$

**860. If in  $\Delta ABC, I$  – incenter then:**

$$\frac{m_a}{AI} + \frac{m_b}{BI} + \frac{m_c}{CI} \geq \frac{h_a + h_b + h_c}{2r}$$

*Proposed by Bogdan Fustei-Romania*



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*Solution by Soumava Chakraborty-Kolkata-India*

$$m_a \geq \frac{b+c}{2} \cos \frac{A}{2} \therefore \frac{m_a}{AI} \stackrel{(1)}{\geq} \frac{b+c}{2} \cos \frac{A}{2} \times \frac{\sin \frac{A}{2}}{r} = \frac{b+c}{4r} (\sin A) = \frac{b+c}{4r} \cdot \frac{a}{2R} = \frac{a(b+c)}{8Rr}$$

*Similarly,*  $\frac{m_b}{BI} \stackrel{(2)}{\geq} \frac{b(c+a)}{8Rr}$  &  $\frac{m_c}{CI} \stackrel{(3)}{\geq} \frac{c(a+b)}{8Rr}$

$$(1)+(2)+(3) \Rightarrow LHS \geq \frac{2 \sum ab}{8Rr} = \frac{1}{2r} \left( \frac{\sum ab}{2R} \right) = \frac{\sum h_a}{2r}; \left( \because h_a = \frac{bc}{2R}, etc \right) \text{ (Proved)}$$

**861.** In  $\Delta ABC$  the following relationship holds:

$$8 \cos(11\pi - 32A) \cos(11\pi - 32B) \cos(11\pi - 32C) \leq 1$$

*Proposed by Daniel Sitaru – Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

*Given inequality*  $\Leftrightarrow \cos(11\pi - 32A) \cos(11\pi - 32B) \cos(11\pi - 32C) \leq \frac{1}{8}$ . Now

$$\because x \leq |x| \therefore \cos(11\pi - 32A) \cos(11\pi - 32B) \cos(11\pi - 32C) \leq$$

$$\leq |\cos(11\pi - 32A) \cos(11\pi - 32B) \cos(11\pi - 32C)| =$$

$$= |(-\cos 32A)(-\cos 32B)(-\cos 32C)| = |\cos 32A||\cos 32B||\cos 32C| \stackrel{?}{\leq} \frac{1}{8}$$

$$\Leftrightarrow \ln(|\cos 32A||\cos 32B||\cos 32C|) \stackrel{?}{\leq} -3 \ln 2 \Leftrightarrow \sum \ln|\cos 32A| \stackrel{?}{\leq} -3 \ln 2 \stackrel{(1)}{\leq}$$

$$\text{Let } f(x) = \ln|\cos 32x| \quad \forall x \in (0, \pi) \because f''(x) = -1024 \sec^2(32x) < 0$$

*∴  $f(x)$  is concave*  $\Rightarrow$

$$\Rightarrow \sum \ln|\cos 32A| \stackrel{\text{Jensen}}{\leq} 3 \ln \left| \cos \frac{32\pi}{3} \right| = 3 \ln \left| -\frac{1}{2} \right| = 3 \ln \frac{1}{2} = -3 \ln 2 \Rightarrow$$

$$\Rightarrow \sum \ln|\cos 32A| \leq -3 \ln 2 \Rightarrow (1) \text{ is true (Proved)}$$

**862.** In  $\Delta ABC$  the following relationship holds:

$$3\sqrt{3}r \leq \frac{m_a}{\sqrt{r_a}} + \frac{m_b}{\sqrt{r_b}} + \frac{m_c}{\sqrt{r_c}} \leq \frac{3\sqrt{3}R}{2\sqrt{r}}$$

*Proposed by Mehmet Sahin-Ankara-Turkey*



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**Solution 1 by Marian Ursarescu-Romania**

$$\begin{aligned} r_a = \frac{s}{s-a} &\Rightarrow \frac{m_a\sqrt{s-a}}{\sqrt{s}} + \frac{m_b\sqrt{s-b}}{\sqrt{s}} + \frac{m_c\sqrt{s-c}}{\sqrt{s}} \geq 3\sqrt{3}r \Leftrightarrow \\ &\Leftrightarrow m_a\sqrt{s-a} + m_b\sqrt{s-b} + m_c\sqrt{s-c} \geq 3\sqrt{3}s \cdot r \quad (1) \end{aligned}$$

But in any  $\Delta ABC$  we have:  $m_a \geq \frac{b+c}{2} \cdot \cos \frac{A}{2} \Rightarrow$

$$m_a \geq \sqrt{bc} \cdot \sqrt{\frac{s(s-a)}{bc}} \Rightarrow m_a \geq \sqrt{s(s-a)} \quad (2)$$

From (1)+(2) we must show:

$$\sqrt{s}(s-a+s-b+s-c) \geq 3\sqrt{3}sr \Leftrightarrow s \geq 3\sqrt{3}r \quad (\text{true})$$

Now, let  $a \leq b \leq c \Rightarrow m_a \geq m_b \geq m_c$  and  $r_a \leq r_b \leq r_c$ . From Cebyshev inequality  $\Rightarrow$

$$\begin{aligned} \frac{m_a}{\sqrt{r_a}} + \frac{m_b}{\sqrt{r_b}} + \frac{m_c}{\sqrt{r_c}} &\leq \frac{1}{3} (m_a + m_b + m_c) \left( \frac{1}{\sqrt{r_a}} + \frac{1}{\sqrt{r_b}} + \frac{1}{\sqrt{r_c}} \right) \Rightarrow \text{we must show this:} \\ (m_a + m_b + m_c) \left( \frac{1}{\sqrt{r_a}} + \frac{1}{\sqrt{r_b}} + \frac{1}{\sqrt{r_c}} \right) &\leq \frac{9\sqrt{3}R}{2\sqrt{r}} \quad (3) \end{aligned}$$

But from Cauchy's inequality  $\Rightarrow$

$$\begin{aligned} m_a + m_b + m_c &\leq \sqrt{3(m_a^2 + m_b^2 + m_c^2)} \leq \sqrt{\frac{9}{4}(a^2 + b^2 + c^2)} \Rightarrow \\ m_a + m_b + m_c &\leq \frac{3}{2}\sqrt{a^2 + b^2 + c^2} \leq \frac{3}{2}\sqrt{9R^2} = \frac{9}{2}R \quad (4) \end{aligned}$$

$$\frac{1}{\sqrt{r_a}} + \frac{1}{\sqrt{r_b}} + \frac{1}{\sqrt{r_c}} \leq \sqrt{3 \left( \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} \right)} = \frac{\sqrt{3}}{\sqrt{r}} \quad (5)$$

From (4) + (5)  $\Rightarrow$  (3) its true.

**Solution 2 by Soumava Chakraborty-Kolkata-India**

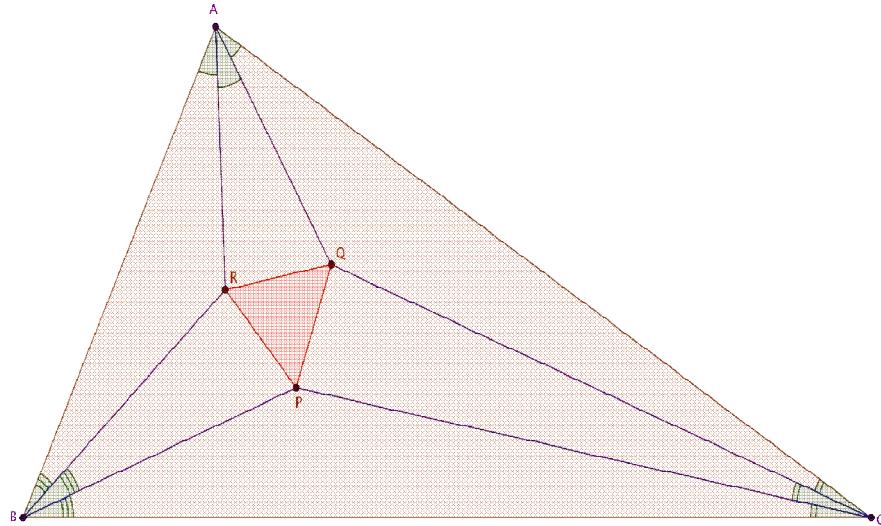
$$\sum \frac{m_a}{\sqrt{r_a}} \stackrel{A-G}{\geq} 3 \sqrt[3]{\frac{\prod m_a}{\prod r_a}} \stackrel{m_a \geq \sqrt{s(s-a)}}{\geq} 3 \sqrt[3]{\frac{srs}{\sqrt{rs^2}}} = 3 \sqrt[3]{s\sqrt{r}} \stackrel{s \geq 3\sqrt{3}r}{\geq} 3 \sqrt[3]{3\sqrt{3}r\sqrt{r}} = 3\sqrt{3}\sqrt{r} = 3\sqrt{3}r$$

$$\text{Also, } \sum \frac{m_a}{\sqrt{r_a}} \stackrel{C-B-S}{\leq} \sqrt{\sum m_a^2} \sqrt{\sum \frac{1}{r_a}} = \sqrt{\frac{3}{4} \sum a^2 \left(\frac{1}{r}\right)} \stackrel{\text{Leibnitz}}{\leq} \sqrt{\frac{27R^2}{4r}} = \frac{3\sqrt{3}R}{2\sqrt{r}} \quad (\text{Done})$$

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863.

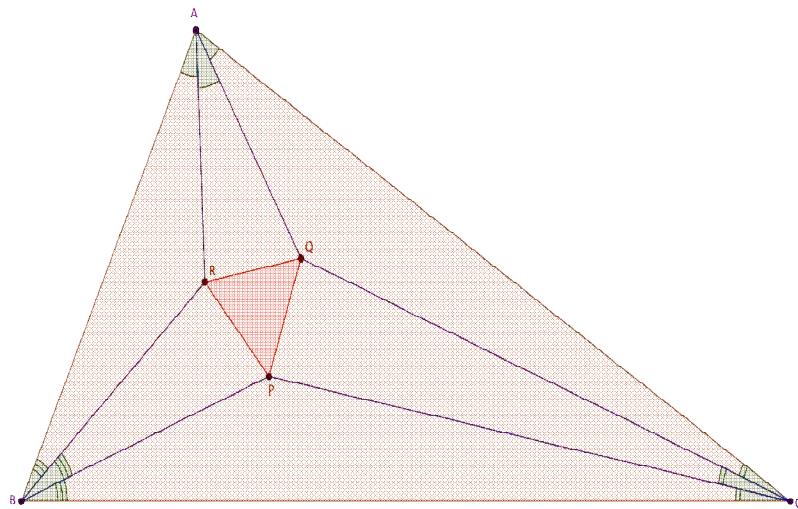


**Prove that:**

$$a \cdot RA \cdot QA + b \cdot RB \cdot QB + c \cdot RC \cdot QC \geq 24\sqrt{3}r^3$$

*Proposed by Daniel Sitaru – Romania*

*Solution by Marian Ursărescu – Romania*



**For sine law we have:**  $\frac{AR}{\sin \frac{B}{3}} = \frac{AB}{\sin \left( \frac{A+B}{3} \right)} \Rightarrow AR = \frac{c \sin \frac{B}{3}}{\sin \left( \frac{\pi-C}{3} \right)}$ , similarly,  $AQ = \frac{b \cdot \sin \frac{C}{3}}{\sin \left( \frac{\pi-B}{3} \right)}$  (1)



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**From (1) we must show this:**  $abc \sum \frac{\sin \frac{B}{3} \cdot \sin \frac{C}{3}}{\sin(\frac{\pi-B}{3}) \cdot \sin(\frac{\pi-C}{3})} \geq 24\sqrt{3}r^3 \quad (2)$

**But**  $abc \geq 24\sqrt{3}r^3 \quad (3)$ . **From (2)+(3) we must show:**  $\sum \frac{\sin \frac{B}{3} \cdot \sin \frac{C}{3}}{\sin(\frac{\pi-B}{3}) \cdot \sin(\frac{\pi-C}{3})} \geq 1 \quad (4)$

**But**  $\sum \frac{\sin \frac{B}{3} \cdot \sin \frac{C}{3}}{\sin(\frac{\pi-B}{3}) \cdot \sin(\frac{\pi-C}{3})} \geq 3 \sqrt[3]{\frac{\sin^2 \frac{A}{3} \cdot \sin^2 \frac{B}{3} \cdot \sin^2 \frac{C}{3}}{\sin^2(\frac{\pi-A}{3}) \cdot \sin^2(\frac{\pi-B}{3}) \cdot \sin^2(\frac{\pi-C}{3})}} \quad (5)$

**From (4)+(5) we must show:**  $\frac{\sin \frac{A}{3} \cdot \sin \frac{B}{3} \cdot \sin \frac{C}{3}}{\sin(\frac{\pi-A}{3}) \cdot \sin(\frac{\pi-B}{3}) \cdot \sin(\frac{\pi-C}{3})} \geq \frac{1}{3\sqrt{3}} \quad (6)$

**But**  $\frac{\sin \frac{A}{3}}{\sin(\frac{\pi-A}{3})} \geq \frac{1}{3}$  because  $\Leftrightarrow \sqrt{3} \sin \frac{A}{3} \geq \sin \left( \frac{\pi}{3} - \frac{A}{3} \right) \Leftrightarrow \sqrt{3} \sin \frac{A}{3} \geq \frac{\sqrt{3}}{2} \cos \frac{A}{3} - \frac{1}{2} \sin \frac{A}{3} \Leftrightarrow \Leftrightarrow \sin \left( \frac{\pi+A}{3} \right) \geq 0$  true  $\Rightarrow (6)$  its true.

**864. In  $\Delta ABC$  the following relationship holds:**

$$2(\sqrt{m_a h_a} + \sqrt{m_b h_b} + \sqrt{m_c h_c}) \leq R \left( 6 + \frac{h_a}{r_a} + \frac{h_b}{r_b} + \frac{h_c}{r_c} \right)$$

*Proposed by Bogdan Fustei – Romania*

**Solution 1 by Myagmarsuren Yadamsuren-Darkhan-Mongolia**

$$1) 2 \sum \sqrt{m_a \cdot h_a} \stackrel{h_a \leq m_a}{\leq} 2 \sum m_a \leq 2(4R + r) \quad (1) LHS$$

$$\begin{aligned} 2) 2(4R + r) &= 2(5R + r - R) \stackrel{\text{Euler}}{\leq} 2(5 + r - 2r) = 2(5R - r) = \\ &= \frac{4r(5R - r)}{2r} = \frac{20Rr - 4r^2}{2r} = \frac{16Rr - 5r^2 + 4Rr + r^2}{2r} \stackrel{\text{GERRETSEN}}{\leq} \frac{s^2 + 4Rr + r^2}{2r} = \\ &= \frac{\sum ab}{2r} = 2Rs \cdot \frac{\sum ab}{4sRr} = 2Rs \cdot \frac{\sum ab}{abc} = R \left( 2 \sum \frac{s}{a} \right) = R \left( 6 + 2 \sum \frac{s-a}{a} \right) = \\ &= R \left( 6 + 2 \left( \frac{h_a}{r_a} + \frac{h_b}{r_b} + \frac{h_c}{r_c} \right) \right) \end{aligned}$$

$$1) 2 \cdot \sum \sqrt{m_a h_a} \leq 2 \sum m_a \leq 2 \sum r_a \quad (1)$$

$$2) R \left( 6 + \sum \frac{h_a}{r_a} \right) = \frac{\sum ab}{2r} \quad (2)$$



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$$\begin{aligned}
 (1); (2) \Rightarrow 2 \sum r_a &\leq \frac{\sum ab}{2r} (K) \quad (\text{ASSURE}) \\
 : \sum a^2 &\geq \sum ab; -\sum a^2 &\leq -\sum ab \quad (\text{TRUE}) \\
 -\sum a^2 + 2 \sum ab &\leq \sum ab \quad (*) \\
 -\sum a^2 + 2 \sum ab &= 4 \sum (s-a)(s-b) \quad (**) \\
 (*) ; (**) \quad 4 \sum (s-a)(s-b) &\leq \sum ab \\
 4 \sum \frac{\Delta^2}{s(s-c)} &\leq \sum ab \\
 \frac{4\Delta}{s} \cdot \sum \frac{\Delta}{s-c} &\leq \sum ab \Rightarrow 2 \sum r_c \leq \frac{\sum ab}{2r} \quad (K)
 \end{aligned}$$

*Solution 2 by Soumava Chakraborty-Kolkata-India*

$$\begin{aligned}
 RHS &= R \left\{ 6 + \sum \left( \frac{2\Delta}{a} \times \frac{s-a}{\Delta} \right) \right\} = R \left( 6 + 2 \sum \frac{s-a}{a} \right) = R \left( 6 + 2s \sum \frac{1}{a} - 6 \right) = \\
 &= R \cdot \frac{2s \sum ab}{4Rrs} = \frac{\sum ab}{2r} \\
 LHS &\stackrel{CBS}{\leq} 2 \sqrt{\sum m_a} \sqrt{\sum h_a} \stackrel{?}{\leq} RHS \stackrel{(1)}{=} \frac{\sum ab}{2r} \Leftrightarrow 4 \left( \sum m_a \right) \left( \frac{\sum ab}{2R} \right) \stackrel{?}{\leq} \frac{(\sum ab)^2}{4r^2} \Leftrightarrow \\
 &\Leftrightarrow R \sum ab \stackrel{?}{\geq} \frac{8r^2}{(a)} (\sum m_a). \text{ Now, RHS of } (a) \leq 8r^2(4R+r) \stackrel{?}{\leq} R(s^2 + 4Rr + r^2) \\
 &\Leftrightarrow Rs^2 + (Rr - 8r^2)(4R+r) \stackrel{?}{\geq} 0. \text{ Now, LHS of } (b) \stackrel{\text{Gerretsen}}{\geq} R(16Rr - 5r^2) + \\
 &+ (Rr - 8r^2)(4R+r) \stackrel{?}{\geq} 0 \Leftrightarrow 5r^2 - 9Rr - 2r^2 \stackrel{?}{\geq} 0 \Leftrightarrow (5R+r)(R-2r) \stackrel{?}{\geq} 0 \\
 &\rightarrow \text{true} \because R \stackrel{\text{Euler}}{\geq} 2r \quad (\text{Proved})
 \end{aligned}$$

**865. In  $\triangle ABC$  the following relationship holds:**

$$\frac{h_a}{m_a} + \frac{h_b}{m_b} + \frac{h_c}{m_c} \leq 2 \left( \frac{h_b h_c}{h_b^2 + h_c^2} + \frac{h_c h_a}{h_c^2 + h_a^2} + \frac{h_a h_b}{h_a^2 + h_b^2} \right)$$

*Proposed by Bogdan Fustei-Romania*



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*Solution by Daniel Sitaru-Romania*

$$\begin{aligned} \sum \frac{h_a}{m_a} &\stackrel{\text{TERESHIN}}{\geq} \sum \frac{h_a}{b^2 + c^2} = 8RS \sum \frac{1}{b^2 + c^2} = 2abc \sum \frac{1}{a(b^2 + c^2)} = \\ &= 2 \sum \frac{bc}{b^2 + c^2} = 2 \sum \frac{\frac{2S}{b} \cdot \frac{2S}{c}}{\frac{4S^2}{b^2} + \frac{4S^2}{c^2}} = 2 \left( \frac{h_b h_c}{h_b^2 + h_c^2} + \frac{h_c h_a}{h_c^2 + h_a^2} + \frac{h_a h_b}{h_a^2 + h_b^2} \right) \end{aligned}$$

866. In  $\Delta ABC, \Delta A'B'C'$  the following relationship holds:

$$\sum \frac{c\sqrt{a'b'} + c'\sqrt{ab}}{(a+b)(a'+b')} < 2$$

*Proposed by Daniel Sitaru – Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\begin{aligned} LHS &= \sum \frac{c\sqrt{a'b'}}{(a+b)(a'+b')} + \sum \frac{c'\sqrt{ab}}{(a+b)(a'+b')} \stackrel{A-G}{\geq} \text{(1)} \\ &\geq \sum \frac{c\sqrt{a'b'}}{(a+b)(2\sqrt{a'b'})} + \sum \frac{c'\sqrt{ab}}{(2\sqrt{ab})(a'+b')} = \frac{1}{2} \sum \frac{c}{a+b} + \frac{1}{2} \sum \frac{c'}{a'+b'} \\ \text{Now, } \frac{1}{2} \sum \frac{c}{a+b} &= \frac{1}{2} \cdot \frac{\sum c(b+c)(c+a)}{(a+b)(b+c)(c+a)} = \frac{1}{2} \cdot \frac{(\sum ab)(\sum a) + \sum a^3}{2abc + \sum ab(2s-c)} = \frac{1}{2} \cdot \frac{2s \sum ab + 3abc + 2s(\sum a^2 - \sum ab)}{2s(s^2 + 4Rr + r^2)} = \\ &= \frac{1}{2} \cdot \frac{2s(2s^2 - 8Rr - 2r^2) + 12Rrs}{2s(s^2 + 2Rr + r^2)} = \frac{1}{2} \cdot \frac{2s^2 - 2Rr - 2r^2}{s^2 + 2Rr + r^2} = \frac{s^2 - Rr - r^2}{s^2 + 2Rr + r^2} = \\ &= \frac{s^2 + 2Rr + r^2 - 3Rr - 2r^2}{s^2 + 2Rr + r^2} = 1 - \frac{3Rr + 2r^2}{s^2 + 2Rr + r^2} < 1 \Rightarrow \frac{1}{2} \sum \frac{c}{a+b} \stackrel{(2)}{<} 1 \\ \text{Similarly, } \frac{1}{2} \sum \frac{c'}{a'+b'} &\stackrel{(3)}{<} 1; \text{ (1), (2), (3)} \Rightarrow LHS < 2 \text{ (Proved)} \end{aligned}$$

867. If in  $\Delta ABC, O$  – circumcenter,  $I$  – incenter then:

$$\sqrt{m_a^2 - w_a^2} + \sqrt{m_b^2 - w_b^2} + \sqrt{m_c^2 - w_c^2} \leq 2\sqrt{3} \cdot OI$$



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*Proposed by Rovsen Pirguliyev-Sumgait-Azerbaijan*

**Solution 1 by Soumava Chakraborty-Kolkata-India**

$$\begin{aligned}
 \sum_{\Delta} \sqrt{\frac{2(b^2 + c^2) - a^2}{4} - \frac{bc((b+c)^2 - a^2)}{(b+c)^2}} &= \sum_{\Delta} \sqrt{\frac{b^2 + c^2}{2} - \frac{a^2}{4} - bc + \frac{a^2 \cdot bc}{(b+c)^2}} \stackrel{Mg \leq M_a}{\leq} \\
 &\leq \sum_{\Delta} \sqrt{\frac{b^2 + c^2}{2} - \frac{a^2}{4} - bc + \frac{a^2(b+c)^2}{4(b+c)^2}} = \sum_{\Delta} \sqrt{\frac{b^2 + c^2}{2} - bc - \frac{a^2}{4} + \frac{a^2}{4}} \stackrel{CBS}{\leq} \\
 &\leq \sqrt{3 \left( \sum a^2 - \sum ab \right)} = \sqrt{3(2s^2 - 8Rr - 2r^2 - s^2 - 4Rr - r^2)} = \\
 &= \sqrt{3(s^2 - 12Rr - 3r^2)} \stackrel{GERRETSEN}{\leq} \sqrt{3(4R^2 - 8Rr)} = 2\sqrt{3}\sqrt{R(R - 2r)} = 2\sqrt{3} \cdot OI
 \end{aligned}$$

**Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia**

$$\begin{aligned}
 \sqrt{m_a^2 - w_a^2} + \sqrt{m_b^2 - w_b^2} + \sqrt{m_c^2 - w_c^2} &\stackrel{(i)}{\leq} 2\sqrt{3}OI \\
 m_a^2 - w_a^2 &= \frac{2b^2 + 2c^2 - a^2}{4} - \frac{4b^2c^2}{(b+c)^2} \cdot \frac{s(s-a)}{bc} = \frac{2b^2 + 2c^2 - a^2}{4} - \\
 &- \frac{bc(b+c+a)(b+c-a)}{(b+c)^2} = \frac{2b^2 + 2c^2 - a^2}{4} - \frac{bc\{(b+c)^2 - a^2\}}{(b+c)^2} = \frac{2b^2 + 2c^2 - a^2}{4} - \\
 &- bc + \frac{a^2bc}{(b+c)^2} = \frac{2(b-c)^2 - a^2}{4} + \frac{a^2bc}{(b+c)^2} = \frac{(b-c)^2}{2} - a^2 \left\{ \frac{1}{4} - \frac{bc}{(b+c)^2} \right\} = \\
 &= \frac{(b-c)^2}{2} - \frac{a^2}{4(b+c)^2} \{(b+c)^2 - 4bc\} = \frac{(b-c)^2}{2} = \frac{a^2(b-c)^2}{4(b+c)^2} \stackrel{(1)}{\leq} \frac{(b-c)^2}{2} \\
 &\left( \because \frac{a^2(b-c)^2}{4(b+c)^2} \geq 0 \right). \text{ Similarly, } m_b^2 - w_b^2 \stackrel{(2)}{\leq} \frac{(c-a)^2}{2} \text{ & } m_c^2 - w_c^2 \stackrel{(3)}{\leq} \frac{(a-b)^2}{2} \\
 (1) + (2) + (3) \Rightarrow \sum(m_a^2 - w_a^2) &\stackrel{(a)}{\leq} \frac{\sum(a-b)^2}{2} = \sum a^2 - \sum ab. \text{ Now, LHS of (i)} \stackrel{CBS}{\leq} \\
 &\leq \sqrt{3} \sqrt{\sum(m_a^2 - w_a^2)} \stackrel{(a)}{\leq} \sqrt{3} \sqrt{\sum a^2 - \sum ab} \stackrel{(i)}{\leq} 2\sqrt{3}OI = 2\sqrt{3}\sqrt{R(R - 2r)} \Leftrightarrow \\
 &\Leftrightarrow s^2 - 12Rr - 3r^2 \stackrel{(i)}{\leq} 4R(R - 2r) \Leftrightarrow s^2 \stackrel{(i)}{\leq} 4R^2 + 4Rr + 3r^2 \rightarrow \text{true} \\
 &\quad (\text{Gerretsen}) \quad (\text{Proved})
 \end{aligned}$$



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**868. In  $\Delta ABC$  the following relationship holds:**

$$\sum m_a r_a \sqrt{r_b + r_c} \geq \frac{s\sqrt{2r}(h_a + h_b + h_c)(r_a + r_b + r_c)}{m_a + m_b + m_c}$$

*Proposed by Bogdan Fustei – Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

*By Bogdan Fustei,  $\frac{b+c}{2} \stackrel{(1)}{\geq} \sqrt{2r(r_b + r_c)}$ .*

$$\text{Proof of (1): } 2r(r_b + r_c) = 2rs \left( \frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} + \frac{\sin \frac{C}{2}}{\cos \frac{C}{2}} \right) = 2rs \frac{\cos^2 \frac{A}{2}}{\pi \cos \frac{A}{2}} \stackrel{(a)}{=} 8Rr \cos^2 \frac{A}{2}.$$

*Using (a), (1)  $\Leftrightarrow$*

$$\begin{aligned} \Leftrightarrow \frac{(b+c)^2}{4} &\geq 8Rr \cos^2 \frac{A}{2} \Leftrightarrow \frac{(b+c)^2}{4} \geq 8 \frac{abc}{4\Delta} \cdot \frac{\Delta}{s} \cdot \frac{s(s-a)}{bc} = 2a(s-a) = a(b+c-a) \\ &\Leftrightarrow (b+c-2a)^2 \geq 0 \rightarrow \text{true} \Rightarrow (1) \text{ is true.} \end{aligned}$$

$$\begin{aligned} \because m_a &\geq \frac{b+c}{2} \cos \frac{A}{2}, \text{ etc, } \therefore \sum m_a \stackrel{\text{by (1)}}{\geq} \sum \sqrt{2r(r_b + r_c)} \cos \frac{A}{2} \stackrel{(a)}{=} \sum \sqrt{8Rr \cos^2 \frac{A}{2}} \cos \frac{A}{2} = \\ &= \sum \sqrt{2Rr} (1 + \cos A) = \sqrt{2Rr} \left( 3 + 1 + \frac{r}{R} \right) = \sqrt{2Rr} \left( \frac{4R+r}{R} \right) \Rightarrow \sum m_a \stackrel{(2)}{\geq} \sqrt{\frac{2r}{R}} \left( \sum r_a \right) \\ &\quad \sum m_a r_a \sqrt{r_b + r_c} \stackrel{m_a \geq \frac{b+c}{2} \cos \frac{A}{2}, \text{ etc}}{\geq} \sum \frac{b+c}{2} \cdot s \tan \frac{A}{2} \sqrt{4R \cos^2 \frac{A}{2}} = \\ &= \sqrt{Rs} \sum \frac{(b+c)}{2} \cdot \frac{a}{2R} = \sqrt{Rs} \frac{2 \sum ab}{4R} = \sqrt{Rs} \left( \frac{\sum ab}{2R} \right) = \sqrt{Rs} \left( \sum h_a \right) \left( \because \frac{bc}{2R} = h_a, \text{ etc} \right) \\ &\Rightarrow \sum m_a r_a \sqrt{r_b + r_c} \stackrel{(3)}{\geq} s \sqrt{R} \left( \sum h_a \right) \\ (2), (3) \Rightarrow (\sum m_a r_a \sqrt{r_b + r_c})(\sum m_a) &\geq s \sqrt{2r} (\sum h_a) (\sum r_a) \Rightarrow \\ \sum m_a r_a \sqrt{r_b + r_c} &\geq \frac{s \sqrt{2r} (\sum h_a) (\sum r_a)}{\sum m_a} \text{ (Proved)} \end{aligned}$$



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**869. In  $\Delta ABC$  the following relationship holds:**

$$\frac{s^2r}{m_a m_b m_c} \leq \frac{R}{2r}$$

*Proposed by Seyran Ibrahimov-Maasili-Azerbaijan*

*Solution by Daniel Sitaru-Romania*

$$\begin{aligned} m_a &\geq \sqrt{s(s-a)} \rightarrow \prod m_a \geq \prod \sqrt{s(s-a)} = sS = s^2r \\ \frac{1}{m_a m_b m_c} &\leq \frac{1}{s^2r} \rightarrow \frac{s^2r}{m_a m_b m_c} \leq \frac{s^2r}{s^2r} = 1 \leq \frac{R}{2r} \stackrel{\text{EULER}}{\leq} 2r \end{aligned}$$

**870. In  $\Delta ABC$  the following relationship holds:**

$$\sum \frac{(r_a + r_b)(r_a + r_c)}{a} \geq \frac{a^3 + b^3 + c^3}{ab + bc + ca}$$

*Proposed by Adil Abdullayev-Baku-Azerbaijan*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\begin{aligned} \sum a^3 &= 3abc + 2s \left( \sum a^2 - \sum ab \right) = 12Rrs + 2s(s^2 - 12Rr - 3r^2) \stackrel{(1)}{=} \\ &= 2s(s^2 - 6Rr - 3r^2) \\ \sum \frac{(r_a + r_b)(r_a + r_c)}{a} &= \sum \frac{\left( \frac{\Delta}{s-a} + \frac{\Delta}{s-b} \right) \left( \frac{\Delta}{s-a} + \frac{\Delta}{s-c} \right)}{a} = \\ &= \Delta^2 \sum \frac{bc}{a(s-a)^2(s-b)(s-c)} = \frac{r^2 s^2}{r^2 s} \sum \frac{bc}{a(s-a)} = \frac{s}{4Rrs} \sum \frac{b^2 c^2}{s-a} \\ &\stackrel{\text{Bergstrom}}{\geq} \frac{(\sum ab)^2}{4Rr \sum (s-a)} = \frac{(\sum ab)^2}{4Rrs} \stackrel{?}{\geq} \frac{\sum a^3}{\sum ab} \Leftrightarrow \\ &\Leftrightarrow \left( s^2 + r(4R + r) \right)^2 \stackrel{?}{\geq} 8Rrs^2(s^2 - 6Rr - 3r^2) \text{ (using (1))} \\ &\Leftrightarrow s^6 + r^3(4R + r)^3 + 3s^4r(4R + r) + 3s^2r^2(4R + r)^2 \stackrel{?}{\geq} 8Rrs^2(s^2 - 6Rr - 3r^2) \\ &\Leftrightarrow s^6 + 3s^4r(4R + r) + r^3(4R + r)^3 \stackrel{?}{\geq} s^2r(8R(s^2 - 6Rr - 3r^2) - 3r(4R + r)^2) \end{aligned}$$



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$$\begin{aligned}
 & \Leftrightarrow s^6 + 3s^4r(4R + r) - 8Rrs^4 + r^3(4R + r)^3 + s^2r(8R(6Rr + 3r^2) + 3r(4R + r)^2) \stackrel{?}{\geq} 0 \\
 & \Leftrightarrow s^6 + s^4(4Rr + 3r^2) + r^3(4R + r)^3 + s^2r(8R(6Rr + 3r^2) + 3r(4R + r)^2) \stackrel{?}{\underset{(2)}{\geq}} 0 \\
 & \text{Now, LHS of (2) } \stackrel{\text{Gerretsen}}{\geq} s^4(20Rr - 2r^2) + r^3(4R + r)^3 + \\
 & \quad + s^2r(8R(6Rr + 3r^2) + 3r(4R + r)^2) \stackrel{?}{\geq} 0 \\
 & \Leftrightarrow s^4(20R - 2r) + r^2(4R + r)^3 + s^2(8R(6Rr + 3r^2) + 3r(4R + r)^2) \stackrel{?}{\underset{(3)}{\geq}} 0 \\
 & \text{Now, LHS of (3) } \stackrel{\text{Gerretsen}}{\geq} s^2(16Rr - 5r^2)(20R - 2r) + r^2(4R + r)^3 + \\
 & \quad + s^2(8R(6Rr + 3r^2) + 3r(4R + r)^2) \stackrel{?}{\geq} 0 \Leftrightarrow s^2(416R - 84Rr + 13r^2) + r(4R + r)^3 \stackrel{?}{\underset{(4)}{\geq}} 0 \\
 & \text{Now, LHS of (4) } \stackrel{\text{Gerretsen}}{\geq} (16Rr - 5r^2)(416R^2 - 84Rr + 13r^2) + r(4R + r)^3 \stackrel{?}{\geq} 0 \\
 & \Leftrightarrow 420t^3 - 211t^2 + 40t - 4 \stackrel{?}{\geq} 0 \Leftrightarrow (t-2)(420t^2 + 629t + 1298) + 2592 \stackrel{?}{\geq} 0 \\
 & \quad \rightarrow \text{true } \because t \geq 2 \text{ (Euler) (Proved)}
 \end{aligned}$$

**871.** In  $\Delta ABC$  the following relationship holds:

$$\left(\frac{1}{b} + \frac{1}{c}\right)\frac{h_a}{r_a} + \left(\frac{1}{c} + \frac{1}{a}\right)\frac{h_b}{r_b} + \left(\frac{1}{a} + \frac{1}{b}\right)\frac{h_c}{r_c} \leq \frac{9R}{S}$$

*Proposed by Mehmet Sahin-Ankara-Turkey*

*Solution by Daniel Sitaru-Romania*

$$\begin{aligned}
 \sum \left(\frac{1}{b} + \frac{1}{c}\right)\frac{h_a}{r_a} &= \sum \frac{b+c}{bc} \cdot \frac{\frac{2S}{a}}{\frac{S-a}{s-a}} = 2 \sum \frac{(b+c)(s-a)}{abc} = \frac{2}{abc} \sum (2s-a)(s-a) = \\
 &= \frac{2}{4RS} \sum (2s^2 - 3as + a^2) = \frac{1}{2RS} \left(6s^2 - 6s^2 + \sum a^2\right) = \frac{1}{2RS} \cdot 2(s^2 - r^2 - 4Rr) \leq \\
 &\stackrel{\text{GERRETSSEN}}{\leq} \frac{1}{RS} (4R^2 + 4Rr + 3r^2 - r^2 - 4Rr) = \frac{4R^2 + 2r^2}{RS} \stackrel{\text{EULER}}{\leq} \frac{4R^2 + 2\left(\frac{R}{2}\right)^2}{RS} = \frac{9R}{S}
 \end{aligned}$$



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**872. In  $\Delta ABC$  the following relationship holds:**

$$m_a \sqrt{\frac{r_a}{h_a}} + m_b \sqrt{\frac{r_b}{h_b}} + m_c \sqrt{\frac{r_c}{h_c}} \geq \frac{(h_a + h_b + h_c)(r_a + r_b + r_c)}{m_a + m_b + m_c}$$

*Proposed by Bogdan Fustei – Romania*

**Solution by Soumava Chakraborty-Kolkata-India**

By Bogdan Fustei,  $\frac{b+c}{2} \stackrel{(1)}{\geq} \sqrt{2r(r_b + r_c)}$ . Proof of (1):  $2r(r_b + r_c) = 2rs \left( \frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} + \frac{\sin \frac{C}{2}}{\cos \frac{C}{2}} \right) = 2rs \frac{\cos^2 \frac{A}{2}}{\prod \cos \frac{A}{2}} \stackrel{(a)}{=} 8Rr \cos^2 \frac{A}{2}$ . Using (a), (1)  $\Leftrightarrow \frac{(b+c)^2}{4} \geq 8Rr \cos^2 \frac{A}{2} \Leftrightarrow \frac{(b+c)^2}{4} \geq \frac{8abc}{4\Delta} \cdot \frac{\Delta}{s} \cdot \frac{s(s-a)}{bc} = 2a(s-a) = a(b+c-a) \Leftrightarrow (b+c-2a)^2 \geq 0 \rightarrow \text{true} \Rightarrow (1) \text{ is true}$

$$\because m_a \geq \frac{b+c}{2} \cos \frac{A}{2} \text{ etc, } \therefore m_a \stackrel{\text{by (1)}}{\geq} \sqrt{2r(r_a + r_c)} \cos \frac{A}{2} \stackrel{\text{by (a)}}{=} \sqrt{8Rr \cos^2 \frac{A}{2}} \cos \frac{A}{2} = \sqrt{2Rr}(1 + \cos A) \Rightarrow m_a \stackrel{(i)}{\geq} \sqrt{2Rr}(1 + \cos A)$$

Similarly,  $m_b \stackrel{(ii)}{\geq} \sqrt{2Rr}(1 + \cos B)$  &  $m_c \stackrel{(iii)}{\geq} \sqrt{2Rr}(1 + \cos C)$ ; (i)+(ii)+(iii)  $\Rightarrow \sum m_a \geq \sqrt{2Rr} \left( 3 + 1 + \frac{r}{R} \right) = \sqrt{2Rr} \left( \frac{4R+r}{R} \right) = \sqrt{\frac{2r}{R}} (4R+r) \Rightarrow \sum m_a \stackrel{(2)}{\geq} \sqrt{\frac{2r}{R}} (4R+r)$ . Now,  $\sum m_a \sqrt{\frac{r_a}{h_a}} = \sum m_a \sqrt{\frac{s \tan \frac{A}{2} a}{2rs}} = \sum m_a \sqrt{\frac{4R \sin^2 \frac{A}{2}}{2r}} \stackrel{m_a \geq \frac{b+c}{2} \cos \frac{A}{2}}{\geq} \sum \frac{b+c}{2} \cos \frac{A}{2} \sin \frac{A}{2} \sqrt{\frac{4R}{2r}} = \sqrt{\frac{R}{2r}} \sum \frac{b+c}{2} \cdot \sin A = \sqrt{\frac{R}{2r}} \sum \frac{b+c}{2} \cdot \frac{a}{2R} = \sqrt{\frac{R}{2r}} \left( \frac{2 \sum ab}{4R} \right) = \sqrt{\frac{R}{2r}} \left( \sum h_a \right) \left( \because \frac{bc}{2R} = h_a, \text{etc} \right) \Rightarrow \sum m_a \sqrt{\frac{r_a}{h_a}} \stackrel{(3)}{\geq} \sqrt{\frac{R}{2r}} \left( \sum h_a \right)$

$$(2), (3) (\sum m_a) \left( \sum m_a \sqrt{\frac{r_a}{h_a}} \right) \geq \sqrt{\frac{2r}{R}} (\sum r_a) \sqrt{\frac{R}{2r}} (\sum h_a) = (\sum r_a) (\sum h_a) \Rightarrow \sum m_a \sqrt{\frac{r_a}{h_a}} \geq \frac{(\sum h_a)(\sum r_a)}{\sum m_a} \text{ (Hence proved)}$$



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**873. In  $\Delta ABC$ :**

$$m_a + m_b + m_c + \frac{1}{\sin \frac{A}{2}} + \frac{1}{\sin \frac{B}{2}} + \frac{1}{\sin \frac{C}{2}} < \left(3 + \frac{2}{r}\right) \max(a, b, c)$$

*Proposed by Daniel Sitaru – Romania*

**Solution by Soumava Chakraborty-Kolkata-India**

*WLOG, we may assume  $\max(a, b, c) = a \Leftrightarrow a \geq b, a \geq c \Rightarrow$*

$$3a \geq a + b + c \Rightarrow 3a \geq 2s \Rightarrow \max(a, b, c) \stackrel{(1)}{\geq} \frac{2s}{3} \Leftrightarrow m_a < \frac{b+c}{2},$$

$$\begin{aligned} \therefore \sum m_a &< \frac{4s}{2} = 2s = 3 \cdot \frac{2s}{3} \stackrel{(1)}{\leq} 3 \max(a, b, c) \Rightarrow \sum m_a \stackrel{(2)}{<} 3 \max(a, b, c) \\ \sum \frac{1}{\sin \frac{A}{2}} &= \sum \sqrt{\frac{bc}{(s-b)(s-c)}} = \sum \sqrt{\frac{bc(s-a)}{(s-a)(s-b)(s-c)}} = \frac{1}{r\sqrt{s}} \sum \sqrt{bc(s-a)} \stackrel{CBS}{\leq} \\ &\leq \frac{1}{r\sqrt{s}} \sqrt{\sum ab} \sqrt{\sum (s-a)} = \frac{\sqrt{\sum ab}}{r} = \frac{\sqrt{3 \sum ab}}{\sqrt{3}r} < \frac{\sqrt{(\sum a)^2}}{\sqrt{3}r} = \frac{2s}{\sqrt{3}r} = \frac{2s}{3} \cdot \frac{\sqrt{3}}{r} < \frac{4s}{3r} = \\ &= \frac{2}{r} \cdot \frac{2s}{3} = \frac{2}{r} \max(a, b, c) \Rightarrow \sum \frac{1}{\sin \frac{A}{2}} \stackrel{(3)}{<} \frac{2}{r} \max(a, b, c) \\ (2)+(3) \Rightarrow LHS &< \left(3 + \frac{2}{r}\right) \max(a, b, c) \quad (\text{Proved}) \end{aligned}$$

**874. In  $\Delta ABC$  the following relationship holds:**

$$\sqrt{m_a m_b m_c w_a w_b w_c} < abc$$

*Proposed by Bogdan Fustei – Romania*

**Solution Soumava Chakraborty-Kolkata-India**

$$m_a < \frac{b+c}{2} \text{ etc} \Rightarrow \prod m_a \stackrel{(1)}{<} \frac{\prod(a+b)}{8}; w_a w_b w_c = \prod \left( \frac{2bc}{b+c} \cos \frac{A}{2} \right) = \frac{8 \cdot 16R^2 r^2 s^2}{\prod(a+b)} \cdot \frac{s}{4R} \stackrel{(2)}{=} \frac{32Rr^2 s^3}{\prod(a+b)}$$

$$(1), (2) \Rightarrow m_a m_b m_c w_a w_b w_c \stackrel{?}{<} 4Rr^2 s^3 \stackrel{?}{<} 16R^2 r^2 s^2 \Leftrightarrow s \stackrel{?}{<} 4R \Leftrightarrow s^2 \stackrel{?}{<} 16R^2$$

$$\text{But } s^2 \leq \frac{27R^2}{4} \stackrel{?}{<} 16R^2 \Leftrightarrow 64 \stackrel{?}{>} 27 \rightarrow \text{true (proved)}$$



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**875. In  $\Delta ABC$  the following relationship holds:**

$$\left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \left( \frac{ab}{rb + 4Rc} + \frac{bc}{rc + 4Ra} + \frac{ca}{ra + 4Rb} \right) \geq \frac{9}{r_a + r_b + r_c}$$

*Proposed by Daniel Sitaru – Romania*

**Solution by Soumava Chakraborty-Kolkata-India**

$$\begin{aligned} \frac{ab}{rb + 4Rc} + \frac{bc}{rc + 4Ra} + \frac{ca}{ra + 4Rb} &= \frac{a^2b^2}{rab^2 + 4Rabc} + \frac{b^2c^2}{rbc^2 + 4Rabc} + \frac{c^2a^2}{rc^2a + 4Rabc} \\ &\stackrel{\text{Bergstrom}}{\geq} \frac{(\sum ab)^2}{r(\sum ab^2) + 12R \cdot 4Rrs} \stackrel{\sum ab^2 \leq \sum a^3}{\stackrel{(1)}{\geq}} \frac{(\sum ab)^2}{r(\sum a^3) + 48R^2rs} \end{aligned}$$

$$\begin{aligned} \text{Now, } \sum a^3 &= 3abc + 2s(\sum a^2 - \sum ab) = 12Rrs + 2s(s^2 - 12Rr - 3r^2) \stackrel{(2)}{=} \\ &= 2s(s^2 - 6Rr - 3r^2) \end{aligned}$$

$$(1), (2) \Rightarrow \sum \frac{ab}{rb + 4Rc} \stackrel{(3)}{\geq} \frac{(\sum ab)^2}{2rs(s^2 - 6Rr - 3r^2) + 48R^2rs} = \frac{(\sum ab)^2}{2rs(24R^2 - 6Rr - 3r^2 + s^2)}$$

$$(3) \Rightarrow LHS \geq \frac{(\sum ab)^3}{8Rr^2s^2(s^2 + 24R^2 - 6Rr - 3r^2)} \stackrel{?}{\geq} \frac{9}{4R+r} \Leftrightarrow$$

$$\Leftrightarrow (s^2 + 4Rr + r^2)^3(4R + r) \stackrel{(4)}{\stackrel{?}{\geq}} 72Rr^2s^2(s^2 + 24R^2 - 6Rr - 3r^2)$$

$$LHS \text{ of (4)} \stackrel{\text{Gerretsen}}{\geq} (20Rr - 4r^2)(4R + r)$$

$$(s^4 + r^2(4R + r)^2 + 2s^2(4Rr + r^2)) \stackrel{?}{\geq} 72Rr^2s^2(s^2 + 24R^2 - 6Rr - 3r^2) \Leftrightarrow$$

$$\begin{aligned} &\Leftrightarrow s^4(20R^2 - 17Rr - r^2) + s^2r\{2(4R + r)^2(5R - r) - 18R(24R^2 - 6Rr - 3r^2)\} + \\ &+ r^2(4R + r)^3(5R - r)(20R^2 - 17Rr - r^2) \stackrel{(5)}{\stackrel{?}{\geq}} 0. \text{ Now, } LHS \text{ of (5)} \geq s^2r(16R - 5r) + \end{aligned}$$

$$+ s^2r\{2(4R + r)^2(5R - r) - 18R(24R^2 - 6Rr - 3r^2)\} + r^2(4R + r)^3(5R - r) \stackrel{?}{\geq} 0$$

$$\Leftrightarrow s^2(48R^3 - 216R^2r + 117Rr^2 + 3r^3) + r(4R + r)^3(5R - r) \stackrel{?}{\geq} 0$$

$$\Leftrightarrow s^2(48R^3 + 117Rr^2 + 3r^3) + r(4R + r)^3(5R - r) \stackrel{(6)}{\stackrel{?}{\geq}} 216R^2rs^2$$

$$LHS \text{ of (6)} \stackrel{\text{Gerretsen}}{\stackrel{(a)}{\geq}} (16Rr - 5r^2)(48R^3 + 117Rr^2 + 3r^3) + r(4R + r)^3(5R - r) \&$$



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$$\text{RHS of (6)} \stackrel{\substack{\text{Gerretten} \\ (\bar{b})}}{\geq} 216R^2r(4R^2 + 4Rr + 3r^2)$$

(a), (b)  $\Rightarrow$  in order to prove (6), it suffices to prove:

$$\begin{aligned}
 & (16R - 5r)(48R^3 + 117Rr^2 + 3r^3) + \\
 & + (4R + r)^3(5R - r) \geq 216R^2(4R^2 + 4Rr + 3r^2) \\
 & \Leftrightarrow 56t^4 - 232t^3 + 309t^2 - 136t - 4 \geq 0 \quad \left(t = \frac{R}{r}\right) \Leftrightarrow \\
 & \Leftrightarrow (t - 2)[(t - 2)\{(t - 2)(56t + 104) + 261\} + 108] \geq 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2 \text{ (proved)}
 \end{aligned}$$

**876.** In  $\Delta ABC$  the following relationship holds:

$$(3h_a - 7r)w_a + (3h_b - 7r)w_b + (3h_c - 7r)w_c \geq 2r(m_a + m_b + m_c)$$

Proposed by Bogdan Fustei – Romania

*Solution by Soumava Chakraborty-Kolkata-India*

$$\begin{aligned}
 \sum (3h_a - 7r)w_a &= \sum (3h_a - 6r)w_a - r \sum w_a = \sum \left(\frac{6\Delta}{a} - \frac{6\Delta}{s}\right)w_a - r \sum w_a = \\
 &= \sum \frac{6rs}{as}(s-a)w_a - r \sum w_a = 3r \sum \frac{b+c-a}{a}w_a - r \sum w_a = \\
 &= 3r \sum \frac{b+c}{a}w_a - 4r \sum w_a \geq 2r \left(\sum m_a\right) \Leftrightarrow 3 \sum \frac{b+c}{a}w_a \stackrel{(1)}{\geq} 4 \sum w_a + 2 \sum m_a
 \end{aligned}$$

Now,  $\left(\sum m_a\right)^2 \stackrel{(a)}{\leq} 4s^2 - 16Rr + 5r^2$  (*X.G.Chu, X.Z.Yang*) }  
 $\left(\sum w_a\right)^2 \stackrel{(b)}{\leq} (4R + r) \left(\sum h_a\right)$  (*Bogdan Fustei*, }  
 $\sum \left(\frac{b+c}{a}\right)w_a \stackrel{(c)}{\leq} 2s\sqrt{3}$  (*Bogdan Fustei*)

$$\begin{aligned}
 & \text{Now, } 2 \sum w_a \leq 2 \sum \sqrt{s(s-a)} \stackrel{C-B-S}{\leq} 2\sqrt{3}\sqrt{s}\sqrt{s} = \\
 & = 2\sqrt{3}s \stackrel{\text{by (c)}}{\leq} \sum \left(\frac{b+c}{a}\right)w_a \Rightarrow 2 \sum w_a \stackrel{(i)}{\leq} \sum \left(\frac{b+c}{a}\right)w_a
 \end{aligned}$$

(i)  $\Rightarrow$  in order to prove (1), it suffices to prove:  $2 \sum \frac{b+c}{a}w_a \geq 2 \sum w_a + 2 \sum m_a \Leftrightarrow$



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$$\begin{aligned}
 & \Leftrightarrow \sum \frac{b+c}{a} w_a \stackrel{(2)}{\geq} \sum w_a + \sum m_a. \text{ Now, LHS of (2) } \\
 & \stackrel{CBS}{\leq_{(m)}} \sqrt{2} \sqrt{(\sum w_a)^2 + (\sum m_a)^2} \stackrel{by (a),(b)}{\leq} \sqrt{2} \sqrt{\frac{(4R+r)(s^2+4Rr+r^2)}{2R} + 4s^2 - 16Rr + 5r^2} \\
 & = \sqrt{\frac{(4R+r)(s^2+4Rr+r^2) + 2R(4s^2-16Rr+5r^2)}{R}} \\
 & \text{Again, LHS of (2) } \stackrel{by (c)}{\geq} 2s\sqrt{3}; (m), (n) \Rightarrow \text{in order to prove (2), it suffices to prove:} \\
 & 2s\sqrt{3} \geq \sqrt{\frac{(12R+r)s^2 + r(4R+r)^2 - 2R(16Rr-5r^2)}{R}} \Leftrightarrow \\
 & \Leftrightarrow 12Rs^2 \geq 12Rs^2 + rs^2 + r(4R+r)^2 - 2R(16Rr-5r^2) \Leftrightarrow \\
 & \Leftrightarrow r\{2R(16R-5r) - (4R+r)^2\} \geq rs^2 \stackrel{(3)}{\Leftrightarrow} s^2 \leq 16R^2 - 18Rr - r^2 \\
 & \text{Now, LHS of (3) } \stackrel{Gerretsen}{\leq} 4R^2 + 4Rr + 3r^2 \stackrel{(?)}{\leq} 16R^2 - 18Rr - r^2 \Leftrightarrow \\
 & \Leftrightarrow 6R^2 - 11Rr - 2r^2 \stackrel{?}{\geq} 0 \Leftrightarrow (R-2r)(6R+r) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because R \stackrel{\text{Euler}}{\geq} 2r \Rightarrow \\
 & \Rightarrow (3) \text{ is true } \Rightarrow (2) \text{ is true (proved)}
 \end{aligned}$$

**877. In  $\Delta ABC$  the following relationship holds:**

$$\left( \frac{a^2}{4r} + \frac{h_a^2}{r} \right) \left( \frac{b^2}{4r} + \frac{h_b^2}{r} \right) \left( \frac{c^2}{4r} + \frac{h_c^2}{r} \right) \geq 64h_a h_b h_c$$

*Proposed by Daniel Sitaru – Romania*

*Solution by Marian Ursărescu – Romania*

$$\begin{aligned}
 \frac{a^2}{4r} + \frac{4a^2}{r} &= \frac{1}{r} \left( \frac{a^3}{4} + \frac{4S^2}{a^2} \right) = \frac{1}{r} \left( \frac{a^2}{4} + \frac{4S^2}{3a^2} + \frac{4S^2}{3a^2} + \frac{4S^2}{3a^2} \right) \geq \frac{1}{r} \cdot 4 \sqrt[4]{\frac{a^2}{4} \cdot \frac{4^3 \cdot S^6}{27a^6}} \Rightarrow \\
 &\Rightarrow \frac{a^2}{4r} + \frac{4a^2}{r} \geq \frac{8S}{3ar} \cdot \sqrt[4]{3S^2} \quad (1)
 \end{aligned}$$

$$From (1) \Rightarrow \left( \frac{a^2}{4r} + \frac{h_a^2}{r} \right) \left( \frac{b^2}{4r} + \frac{h_b^2}{r} \right) \left( \frac{c^2}{4r} + \frac{h_c^2}{r} \right) \geq \frac{8^3 S^3}{27abc r^3} \sqrt[4]{27S^6} \quad (2)$$

*From (2) we must show:*



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$$\begin{aligned} \frac{8^3 S^4}{27abc r^3} \sqrt[4]{27S^2} &\geq 64 h_a h_b h_c \Leftrightarrow \frac{8S^4}{27abc r^3} \sqrt[4]{27 \cdot S^2} \geq \frac{8S^3}{abc} \Leftrightarrow \frac{S \sqrt[4]{27S^2}}{27r^3} \geq 1 \Leftrightarrow \\ &\Leftrightarrow S \sqrt[4]{27S^2} \geq 3^3 r^3 \Leftrightarrow S^4 \cdot 27S^2 \geq 3^{12} r^{12} \Leftrightarrow S^6 \geq 3^9 r^{12} \Leftrightarrow S \geq 3\sqrt{3}r^2 \\ &\text{Which its true, because } S \geq 3\sqrt{3} \frac{s^2}{s^2} \Leftrightarrow s^2 \geq 3\sqrt{3}s \Leftrightarrow s^4 \geq 27s^2 \Leftrightarrow \\ &\Leftrightarrow s^3 \geq 27(s-a)(s-b)(s-c) \Leftrightarrow s \geq 3\sqrt[3]{(s-a)(s-b)(s-c)} \text{ true} \end{aligned}$$

**878.** In  $\Delta ABC$  the following relationship holds:

$$\frac{a}{r_b^2 + r_c^2} + \frac{b}{r_c^2 + r_a^2} + \frac{c}{r_a^2 + r_b^2} \leq \frac{2R - r}{s}$$

*Proposed by Mehmet Sahin-Ankara-Turkey*

*Solution by Daniel Sitaru-Romania*

$$\begin{aligned} \sum \frac{a}{r_b^2 + r_c^2} &= \sum \frac{a}{\left(\frac{s}{s-b}\right)^2 + \left(\frac{s}{s-c}\right)^2} = \frac{1}{s^2} \sum \frac{a}{\frac{(s-b)^2 + (s-c)^2}{(s-b)^2(s-c)^2}} = \\ &= \frac{1}{srs} \sum \frac{a(s-b)^2(s-c)^2}{(s-b)^2 + (s-c)^2} \stackrel{AM-GM}{\leq} \frac{1}{srs} \sum \frac{a(s-b)^2(s-c)^2}{2(s-b)(s-c)} = \frac{1}{2srs} \cdot 2rs(2R-r) = \frac{2R-r}{s} \end{aligned}$$

**879.** If in  $\Delta ABC$ ,  $K$  – Lemoine's point then:

$$AK^2 + BK^2 + CK^2 \geq \frac{abc(a+b+c)}{a^2 + b^2 + c^2}$$

*Proposed by Marian Ursărescu – Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

Let  $AK_A$  be the symmedian from  $A$  to  $BC$ . Now,  $\frac{KK_A}{AK} = \frac{a^2}{b^2+c^2}$  (Honsberger)  $\Rightarrow$   
 $\Rightarrow \frac{KK_A}{AK} + 1 = \frac{\sum a^2}{b^2+c^2} \Rightarrow \frac{AK_A}{AK} = \frac{\sum a^2}{b^2+c^2} \Rightarrow AK = \frac{(b^2+c^2)}{\sum a^2} AK_A$ . Again,  $\frac{BK_A}{CK_A} = \frac{c^2}{b^2} \Rightarrow \frac{m}{n} = \frac{c^2}{b^2}$ ,

where  $m = BK_A$ ,  $n = CK_A$ . Stewart's theorem with cevian

$$AK_A \Rightarrow b^2m + c^2n \stackrel{(2)}{=} a(d^2 + mn) \text{ (where } d = AK_A) \because \frac{m}{n} = \frac{c^2}{b^2} \text{ & } m + n = a$$



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$$\begin{aligned}
 & \therefore \frac{m+n}{n} = \frac{b^2+c^2}{b^2} \Rightarrow \frac{a}{n} = \frac{b^2+c^2}{b^2} \Rightarrow n \stackrel{(a)}{=} \frac{ab^2}{b^2+c^2} \text{ & } m \stackrel{(b)}{=} \frac{ac^2}{b^2+c^2}. \text{ Using (a), (b), (2) becomes:} \\
 & \frac{b^2ac^2 + c^2ab^2}{b^2 + c^2} = a \left( d^2 + \frac{a^2b^2c^2}{(b^2 + c^2)^2} \right) \Rightarrow \frac{2b^2c^2}{b^2 + c^2} = \left\{ \frac{d^2(b^2 + c^2) + a^2b^2c^2}{(b^2 + c^2)^2} \right\} \Rightarrow \\
 & \Rightarrow 2b^2c^2(b^2 + c^2) = d^2(b^2 + c^2)^2 + a^2b^2c^2 \Rightarrow b^2c^2(2b^2 + 2c^2 - a^2) = d^2(b^2 + c^2)^2 \Rightarrow \\
 & \Rightarrow d^2 = \frac{4b^2c^2}{(b^2 + c^2)^2} \left( \frac{2b^2 + 2c^2 - a^2}{4} \right) = \frac{4b^2c^2}{(b^2 + c^2)^2} m_a^2 \Rightarrow d = AK_A \stackrel{(3)}{=} \frac{2bc}{b^2 + c^2} \cdot m_a \\
 & (1), (3) \Rightarrow AK = \frac{2bc}{\sum a^2} m_a \Rightarrow AK^2 \stackrel{(i)}{=} \frac{4b^2c^2}{(\sum a^2)^2} m_a^2. \text{ Similarly, } BK^2 \stackrel{(ii)}{=} \frac{4c^2a^2}{(\sum a^2)^2} m_b^2 \text{ &} \\
 & CK^2 \stackrel{(iii)}{=} \frac{4a^2b^2}{(\sum a^2)^2} m_c^2; (i) + (ii) + (iii) \Rightarrow \sum AK^2 = \frac{4}{(\sum a^2)^2} \sum b^2c^2 m_a^2 \stackrel{m_a^2 \geq s(s-a), \text{etc}}{\geq} \\
 & \geq \frac{4s}{(\sum a^2)^2} \sum b^2c^2(s-a) = \frac{4s}{(\sum a^2)^2} \left\{ s \sum a^2b^2 - abc \left( \sum ab \right) \right\} \geq \\
 & \geq \frac{4s}{(\sum a^2)^2} \{ sabc(2s) - abc(s^2 + 4Rr + r^2) \} = \frac{4sabc}{(\sum a^2)^2} (2s^2 - s^2 - 4Rr - r^2) = \\
 & = \frac{abc(a+b+c)\{2(s^2 - 4Rr - r^2)\}}{(\sum a^2)^2} = \frac{abc(a+b+c)(\sum a^2)}{(\sum a^2)^2} = \frac{abc(a+b+c)}{\sum a^2} \quad (\text{proved})
 \end{aligned}$$

**880. In  $\Delta ABC$  the following relationship holds:**

$$\frac{a^4 + b^4 + c^4}{2r} \geq w_a w_b w_c \left( 5 + \frac{h_a}{r_a} + \frac{h_b}{r_b} + \frac{h_c}{r_c} \right)$$

*Proposed by Bogdan Fustei – Romania*

**Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia**

$$\begin{aligned}
 1) LHS: \frac{a^4 + b^4 + c^4}{2r} & \stackrel{\text{Chebyshev}}{\geq} \frac{(a+b+c)(a^3 + b^3 + c^3)}{6r} = \frac{4s^2(s^2 - 6Rr - 3r^2)}{6r} = \frac{2s^2(s^2 - 6Rr - 3r^2)}{3r} \quad (*) \\
 2) \prod w_a \left( 5 + \sum \frac{h_a}{r_a} \right) & \leq s \cdot S \cdot \left( 5 + \sum \frac{2(s-a)}{a} \right) = s \cdot S \cdot \left( 5 - 6 + 2s \sum \frac{1}{a} \right) = \\
 & = s \cdot S \cdot \left( \frac{2s \sum ab}{abc} - 1 \right) = \\
 & = s \cdot S \left( \frac{s^2 + 2Rr + r^2}{2Rr} \right) = r \cdot S^2 \cdot \frac{(s^2 + 2Rr + r^2)}{2Rr} = \frac{s^2(s^2 + 2Rr + r^2)}{2R} \quad (**)
 \end{aligned}$$



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$$(*) ; (**) \Rightarrow \frac{2s^2(s^2 - 6Rr - 3r^2)}{3r} \geq \frac{s^2(s^2 + 2Rr + r^2)}{2R} \quad (\text{ASSURE})$$

$$4R(s^2 - 6Rr - 3r^2) \geq 3r(s^2 + 2Rr + r^2)$$

$$(4R - 3r)s^2 - 24R^2r - 18Rr^2 = 3r^3 \geq 0 \Rightarrow 16Rr - 5r^2 \leq s^2$$

$$(4R - 3r)(16Rr - 5r^2) - 24R^2r - 18Rr^2 - 3r^3 \geq 0$$

$$(4R - 3r)(16R - 5r) - 24R^2 - 18Rr - 3r^2 \geq 0$$

$$64R^2 - 68Rr + 15r^2 - 24R^2 - 18Rr - 3r^2 \geq 0$$

$$40R^2 - 86Rr + 12r^2 \geq 0; \quad 20R^2 - 43Rr + 6r^2 \geq 0$$

$$\begin{matrix} (20R - 3r) \cdot (R - 2r) \geq 0 \\ >0 \\ \Rightarrow \text{Euler} \end{matrix}$$

**881. In  $\Delta ABC$  the following relationship holds:**

$$r_a(h_b + h_c)^2 + r_b(h_c + h_a)^2 + r_c(h_a + h_b)^2 \geq 12sS$$

*Proposed by Mehmet Sahin-Ankara-Turkey*

**Solution 1 by Marian Ursarescu-Romania**

$$(h_b + h_c)^2 \geq 4h_bh_c \Rightarrow r_a(h_b + h_c)^2 \geq 4r_a h_b h_c \Rightarrow \text{We must show this:}$$

$$4 \sum r_a h_b h_c \geq 12sS$$

$$\Leftrightarrow \sum \frac{r_a h_b h_c}{h_a} \geq 3sS \Leftrightarrow h_a h_b h_c \cdot \sum \frac{r_a}{h_a} \geq 3sS \quad (1)$$

$$\text{But } h_a h_b h_c = \frac{2s^2 r^2}{R} \quad (2) \text{ and } \sum \frac{r_a}{h_a} = \sum \frac{\frac{s-a}{2s}}{\frac{a}{s-a}} = \frac{1}{2} \sum \frac{a}{s-a} = \frac{1}{2} \cdot \frac{2(2R-r)}{r} = \frac{2R-r}{r} \quad (3)$$

$$\text{From (1)+(2)+(3) we must show: } \frac{2s^2 r^2}{R} \cdot \frac{2R-r}{r} \geq 3sS \Leftrightarrow 2(2R-r) \geq 3R \Leftrightarrow$$

$$\Leftrightarrow 4R - 2r \geq 3R \Leftrightarrow R \geq 2r \text{ (true from Euler)}$$

**Solution 2 by Soumava Chakraborty-Kolkata-India**

$$\begin{aligned} \text{Firstly, } \sum a^3 &= 3abc + 2s(\sum a^2 - \sum ab) = 12Rrs + 2s(s^2 - 12Rr - 3r^2) \stackrel{(1)}{=} \\ &= 2s(s^2 - 6Rr - 3r^2). \text{ Also, } \sum \frac{1}{s-a} = \frac{\sum(s-b)(s-c)}{sr^2} = \frac{\sum(s^2 - s(b+c) + bc)}{sr^2} = \frac{3s^2 - 4s^2 + s^2 + 4Rr + r^2}{sr^2} = \\ &\stackrel{(2)}{=} \frac{4R + r}{sr} \end{aligned}$$



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$$\begin{aligned}
\sum r_a(h_b + h_c)^2 &= \sum \left( \frac{S}{s-a} \right) \left( \frac{ca}{2R} + \frac{ab}{2R} \right)^2 = \left( \frac{S}{4R^2} \right) \left( \sum \frac{a^2(b+c)^2}{s-a} \right) = \\
&= \left( \frac{S}{4R^2} \right) \left( \sum \frac{a^2(s+s-a)^2}{s-a} \right) = \left( \frac{S}{4R^2} \right) \left( \sum \frac{a^2(s^2 + (s-a)^2 + 2s(s-a))}{s-a} \right) = \\
&= \frac{Ss^2}{4R^2} \sum \frac{a^2}{s-a} + \frac{S}{4R^2} \sum a^2(s-a) + \frac{Ss}{2R^2} \sum a^2 = \frac{Ss^2}{4R^2} \sum \frac{a^2 - s^2 + s^2}{s-a} + \frac{Ss}{4R^2} \sum a^2 - \\
&\quad - \frac{S}{4R^2} \sum a^3 + \frac{Ss}{2R^2} \sum a^2 \stackrel{\text{by (1)}}{=} - \frac{Ss^2}{4R^2} \sum (a+s) + \frac{Ss^4}{4R^2} \sum \frac{1}{s-a} + \\
&\quad + \frac{6Ss}{4R^2} (s^2 - 4Rr - r^2) - \frac{2Ss}{4R^2} (s^2 - 6Rr - 3r^2) \stackrel{\text{by (2)}}{=} - \frac{5rs^4}{4R^2} + \frac{(4R+r)s^4}{4R^2} + \\
&\quad + \frac{rs^2}{2R^2} (3(s^2 - 4Rr - r^2) - (s^2 - 6Rr - 3r^2)) = \frac{(R-r)s^4}{R^2} + \frac{rs^2}{R^2} (s^2 - 3Rr) = \\
&= \frac{(R-r)s^4 + rs^2(s^2 - 3Rr)}{R^2} = \frac{s^2(s^2 - 3r^2)}{R} \geq 12sS \Leftrightarrow s^2 - 3r^2 \geq 12Rr \Leftrightarrow \\
&\Leftrightarrow s^2 \stackrel{(3)}{\geq} 12Rr + 3r^2. \text{ Now, LHS of (3) } \stackrel{\text{Gerretsen}}{\geq} 16Rr - 5r^2 \stackrel{?}{\geq} 12Rr + 3r^2 \Leftrightarrow 4Rr \stackrel{?}{\geq} 8r^2 \\
&\rightarrow \text{true (Euler) (proved)}
\end{aligned}$$

**882. In  $\Delta ABC$  the following relationship holds:**

$$w_a + w_b + w_c \leq \sqrt{(r_a + r_b + r_c)(h_a + h_b + h_c)}$$

*Proposed by Bogdan Fustei – Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\begin{aligned}
\sum w_a &= \sum \frac{2bc}{b+c} \cos \frac{A}{2} = \sum \frac{2\sqrt{bc}}{b+c} \sqrt{s(s-a)} \stackrel{C-B-S}{\leq} 2\sqrt{s} \sqrt{\sum bc} \sqrt{\sum \frac{s-a}{(b+c)^2}} \leq \\
&\stackrel{A-G}{\leq} 2\sqrt{s} \sqrt{\sum ab} \sqrt{\sum \frac{s-a}{4bc}} = \sqrt{s} \sqrt{\sum ab} \sqrt{\sum \frac{a(s-a)}{4Rrs}} = \\
&= \sqrt{\sum ab} \sqrt{\frac{s(2s) - 2(s^2 - 4Rr - r^2)}{4Rr}} = \sqrt{\sum ab} \sqrt{\frac{4R+r}{2R}} = \sqrt{\frac{\sum ab}{2R}} \sqrt{\sum r_a} =
\end{aligned}$$



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$$= \sqrt{(\sum h_a)(\sum r_a)} \quad \left( \because \frac{ab}{2R} = h_c, \text{etc} \right) \quad (\text{proved})$$

**883. In  $\Delta ABC$  the following relationship holds:**

$$r_a \sqrt{\frac{h_a}{m_a}} + r_b \sqrt{\frac{h_b}{m_b}} + r_c \sqrt{\frac{h_c}{m_c}} \leq \sqrt{24R^2 - 15r^2}$$

*Proposed by Mehmet Sahin-Ankara-Turkey*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\begin{aligned} \because m_a \geq h_a, \text{etc}, \therefore LHS \leq \sum r_a &= 4R + r \stackrel{?}{\leq} \sqrt{24R^2 - 15r^2} \Leftrightarrow R^2 - Rr - 2r^2 \stackrel{?}{\geq} 0 \Leftrightarrow \\ &\Leftrightarrow (R - 2r)(R + r) \stackrel{?}{\geq} 0 \rightarrow \text{true (Euler) (proved)} \end{aligned}$$

**884. In  $\Delta ABC$  the following relationship holds:**

$$\frac{a}{s_a} + \frac{b}{s_b} + \frac{c}{s_c} \geq 8\sqrt{3} \left(\frac{r}{R}\right)^2$$

*Proposed by Mehmet Sahin-Ankara-Turkey*

*Solution 1 by Marian Ursarescu-Romania*

*In any  $\Delta ABC$  we have:  $s_a \leq m_a \Rightarrow \frac{1}{s_a} \geq \frac{1}{m_a} \Rightarrow$  we must show:*

$$\frac{a}{m_a} + \frac{b}{m_b} + \frac{c}{m_c} \geq 8\sqrt{3} \left(\frac{r}{R}\right)^2 \quad (1)$$

*But  $\frac{a}{m_a} + \frac{b}{m_b} + \frac{c}{m_c} = \frac{a^2}{am_a} + \frac{b^2}{bm_b} + \frac{c^2}{cm_c} \geq \frac{(a+b+c)^2}{am_a+bm_b+cm_c}$  (2) (from Bergertröm inequality).*

$$\begin{aligned} \text{From Cauchy's inequality we have: } (am_a + bm_b + cm_c)^2 &\leq \\ \leq (a^2 + b^2 + c^2)(m_a^2 + m_b^2 + m_c^2) &\Rightarrow (am_a + bm_b + cm_c)^2 \leq \frac{3}{4}(a^2 + b^2 + c^2)^2 \Rightarrow \\ \frac{1}{am_a+bm_b+cm_c} &\geq \frac{2}{\sqrt{3}} \cdot \frac{1}{a^2+b^2+c^2} \quad (3). \text{ From (1) + (2) + (3)} \\ \text{we must show: } \frac{8p^2}{\sqrt{3}(a^2+b^2+c^2)} &\geq 8\sqrt{3} \frac{r^2}{R^2} \end{aligned}$$



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$$\Leftrightarrow p^2 R^2 \geq 3r^2(a^2 + b^2 + c^2) \quad (4). \text{ But } p^2 \geq 27r^2 \text{ and } R^2 \geq \frac{a^2+b^2+c^2}{9} \Rightarrow \\ \Rightarrow p^2 R^2 \geq 3r^2(a^2 + b^2 + c^2) \Rightarrow (4) \text{ its true.}$$

**Solution 2 by Soumava Chakraborty-Kolkata-India**

$$\begin{aligned} \sum \frac{a}{s_a} &= \sum \frac{a}{\frac{2bc}{b^2+c^2} m_a} = \frac{1}{2} \sum \frac{a(b^2+c^2)}{bcm_a} = \frac{1}{2} \sum \frac{a^2(b^2+c^2)}{abcm_a} = \frac{1}{2abc} \sum \frac{a^2(b^2+c^2)}{m_a} = \\ &= \frac{1}{2abc} \left( \sum \frac{a^2b^2}{m_a} + \sum \frac{a^2c^2}{m_a} \right) \stackrel{\text{Bergström}}{\geq} \left( \frac{2}{2abc} \right) \frac{(\sum ab)^2}{\sum m_a} \stackrel{\sum m_a \leq 4R+r}{\geq} \frac{(\sum ab)^2}{4RS(4R+r)} \stackrel{\text{Gordon}}{\geq} \\ &\geq \frac{4\sqrt{3}S(\sum ab)}{4RS(4R+r)} = \frac{\sqrt{3}(\sum ab)}{R(4R+r)} \stackrel{?}{\geq} 8\sqrt{3} \left( \frac{r}{R} \right)^2 \Leftrightarrow R(s^2 + 4Rr + r^2) \stackrel{?}{\geq} (1) 8r^2(4R+r) \\ \text{Now, LHS of (1)} &\stackrel{\text{Gerretsen}}{\geq} R(20Rr - 4r^2) \stackrel{?}{\geq} 8r^2(4R+r) \Leftrightarrow 5R^2 - 9Rr - 2r^2 \stackrel{?}{\geq} 0 \Leftrightarrow \\ &\Leftrightarrow (R - 2r)(5R + r) \stackrel{?}{\geq} 0 \rightarrow \text{true (Euler)} \Rightarrow (1) \text{ is true (proved)} \end{aligned}$$

**885. In  $\Delta ABC$  the following relationship holds:**

$$\frac{m_a^2}{w_a} + \frac{m_b^2}{w_b} + \frac{m_c^2}{w_c} \geq \frac{2r}{R} \sqrt{\frac{(r_a + r_b + r_c)^3}{h_a + h_b + h_c}}$$

*Proposed by Bogdan Fustei – Romania*

**Solution by Soumava Chakraborty-Kolkata-India**

By Bogdan Fustei,  $\frac{b+c}{2} \stackrel{(1)}{\geq} \sqrt{2r(r_b + r_c)}$ .

$$\begin{aligned} \text{Proof of (1): } 2r(r_b + r_c) &= 2rs \left( \tan \frac{B}{2} + \tan \frac{C}{2} \right) = \\ &= 2rs \left( \frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} + \frac{\sin \frac{C}{2}}{\cos \frac{C}{2}} \right) = 2rs \frac{\sin \left( \frac{B+C}{2} \right)}{\cos \frac{B}{2} \cos \frac{C}{2}} = \frac{2rs \cos^2 \frac{A}{2}}{\prod \cos \frac{A}{2}} = \frac{2rs \cos^2 \frac{A}{2}}{\frac{s}{4R}} \stackrel{(a)}{=} 8Rr \cos^2 \frac{A}{2} \end{aligned}$$

$$\begin{aligned} \text{Using (a), (1)} &\Leftrightarrow \frac{(b+c)^2}{4} \geq 8Rr \cos^2 \frac{A}{2} \Leftrightarrow \frac{(b+c)^2}{4} \geq \frac{8abc}{4\Delta} \cdot \frac{\Delta}{s} \cdot \frac{s(s-a)}{bc} = \frac{2a(s-a)}{1} \Leftrightarrow \\ &\Leftrightarrow (b+c)^2 \geq 4a(b+c-a) = 4a(b+c) - 4a^2 \Leftrightarrow (b+c)^2 + 4a^2 - 4a(b+c) \Leftrightarrow \\ &\Leftrightarrow (b+c-2a)^2 \geq 0 \rightarrow \text{true} \Rightarrow (1) \text{ is true. Now, } \because m_a \geq w_a, \text{ etc, LHS of (i)} \geq \end{aligned}$$



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$$\begin{aligned}
 &\geq \sum m_a \stackrel{m_a \geq \frac{b+c}{2} \cos \frac{A}{2}, etc}{\geq} \sum \frac{b+c}{2} \cos \frac{A}{2} \stackrel{by (1)}{\geq} \sum \sqrt{2r(r_b + r_c)} \cos \frac{A}{2} \stackrel{by (a)}{=} \sum \sqrt{8Rr} \cos^2 \frac{A}{2} \\
 &= \sum \sqrt{2Rr} (1 + \cos A) = \sqrt{2Rr} \left\{ \sum (1 + \cos A) \right\} = \sqrt{2Rr} \left( 3 + 1 + \frac{r}{R} \right) = \sqrt{2Rr} \left( \frac{4R+r}{R} \right) \\
 &\stackrel{?}{\geq} \frac{2r}{R} \sqrt{\frac{(4R+r)^2 R}{\sum ab}} \quad (\text{RHS of (i)}) \Leftrightarrow 4R + r \stackrel{?}{\geq} (4R + r) \sqrt{\frac{4r(4R+r)}{s^2 + 4Rr + r^2}} \Leftrightarrow \\
 &s^2 + 4Rr + r^2 \stackrel{?}{\geq} 16Rr + 4r^2 \Leftrightarrow s^2 \stackrel{?}{\geq}_{(ii)} 12Rr + 3r^2. \\
 &\text{Now, LHS of (ii)} \stackrel{\text{Gerretsen}}{\geq} 16Rr - 5r^2 \stackrel{?}{\geq} 12Rr + 3r^2 \\
 &\Leftrightarrow 4Rr \stackrel{?}{\geq} 8r^2 \Leftrightarrow R \stackrel{?}{\geq} 2r \rightarrow \text{true (proved)}
 \end{aligned}$$

**886. In  $\Delta ABC$  the following relationship holds:**

$$m_a + m_b + m_c \geq \frac{h_a h_b}{h_c} + \frac{h_b h_c}{h_a} + \frac{h_c h_a}{h_b}$$

*Proposed by Bogdan Fustei-Romania*

*Solution by Daniel Sitaru-Romania*

$$\begin{aligned}
 \sum m_a &\stackrel{\text{TERESHIN}}{\geq} \sum \frac{b^2 + c^2}{4R} = \frac{1}{4R} \cdot 2 \sum a^2 = \frac{2S}{4RS} \sum a^2 = \frac{2S}{abc} \sum a^2 = \\
 &= 2S \sum \frac{a}{bc} = \sum \frac{\frac{2S}{b} \cdot \frac{2S}{c}}{\frac{2S}{a}} = \sum \frac{h_b h_c}{h_a}
 \end{aligned}$$

**887. In  $\Delta ABC$  the following relationship holds:**

$$\frac{m_a^2}{bc} + \frac{m_b^2}{ca} + \frac{m_c^2}{ab} \geq 2 + \frac{r}{2R}$$

*Proposed by Adil Abdullayev-Baku-Azerbaijan*

*Solution 1 by Bogdan Fustei-Romania*

$$\frac{1}{2R} \cdot \sum \frac{m_a^2}{h_a} = \sum \frac{m_a^2}{bc} \geq \frac{4R + r}{2R} \Rightarrow \sum \frac{m_a^2}{h_a} \geq 4R + r = r_a + r_b + r_c$$



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$$m_a^2 = \frac{2(b^2 + c^2) - a^2}{4} \quad (\text{and analogs})$$

$$m_a^2 = \frac{2(b^2 + c^2) - a^2 + 4bc - 4bc}{4} = \frac{(b^2 + c^2 + 2bc) + (b^2 + c^2 + 2bc) - a^2 - 4bc}{4}$$

$$m_a^2 = \frac{[(b+c)^2 - a^2] + (b+c)^2 - 4bc}{4} = \frac{(a+b+c)(b+c-a) + (b^2 + c^2 - 2bc)}{4}$$

$$m_a^2 = \frac{2p(p-a)+(b-c)^2}{4} = p(p-a) + \frac{(b-c)^2}{4} \quad (\text{and analogs})$$

$$\frac{m_a^2}{h_a} = \frac{p(p-a)}{h_a} + \frac{(b-c)^2}{4h_a}; \frac{p(p-a)}{h_a} = \frac{ap(p-a)}{ah_a} = \frac{a(p-a)p}{2s}$$

$$\sum \frac{p(p-a)}{h_a} = \sum \frac{ap(p-a)}{2s} = \sum \frac{ap^2 - a^2p}{2s} = \frac{p^2(a+b+c) - p(a^2 + b^2 + c^2)}{2s}$$

$$\sum \frac{p(p-a)}{h_a} = \frac{2p^3 - 2p(p^2 - r^2 + 4Rr)}{2s}; a^2 + b^2 + c^2 = 2(p^2 - r^2 - 4Rr)$$

$$\sum \frac{p(p-a)}{h_a} = \frac{2p(p^2 - p^2 + r^2 + 4Rr)}{2sr} = \frac{r(4R+r)}{r} = 4R + r; \text{So, } \sum \frac{m_a^2}{h_a} = \sum \frac{p(p-a)}{h_a} + \frac{1}{4} \cdot \sum \frac{(b-c)^2}{h_a}$$

$\sum \frac{p(p-a)}{h_a} = 4R + r$ , from the proven above the problem is proved.

**Solution 2 by Soumava Chakraborty-Kolkata-India**

$$\begin{aligned} LHS &\stackrel{m_a \geq \sqrt{s(s-a)}}{\geq} s \sum \frac{s-a}{bc} = \frac{s}{4Rrs} \sum a(s-a) = \frac{1}{4Rr} \{s(2s) - 2(s^2 - 4Rr - r^2)\} = \\ &= \frac{4R+r}{2R} = 2 + \frac{r}{2R} \quad (\text{Done}) \end{aligned}$$

**888. In  $\Delta ABC$  the following relationship holds:**

$$\cos \frac{A-B}{2} + \cos \frac{B-C}{2} + \cos \frac{C-A}{2} > \frac{1}{2} \left( \frac{h_b + h_c}{a} + \frac{h_c + h_a}{b} + \frac{h_a + h_b}{c} \right)$$

*Proposed by Bogdan Fustei – Romania*

**Solution by Soumava Chakraborty-Kolkata-India**

$$\begin{aligned} RHS &= \sum \frac{1}{2} \left( \frac{ca+ab}{2Ra} \right) = \frac{1}{4R} \sum (b+c) = \frac{2 \sum a}{4R} = \frac{s}{R} = \frac{\sum a}{2R} = \sum \sin A = \frac{1}{2} \sum (\sin A + \sin B) = \\ &= \frac{1}{2} \sum 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} = \sum \cos \frac{C}{2} \cos \frac{A-B}{2} \leq \sum \cos \frac{A-B}{2} \left( \because \cos \frac{C}{2}, \text{etc} \leq 1 \right) \\ &\quad (\text{proved}) \end{aligned}$$



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**889. In  $\Delta ABC$  the following relationship holds:**

$$w_a + w_b + w_c \leq \sqrt{(r_a + r_b + r_c)(h_a + h_b + h_c)}$$

*Proposed by Bogdan Fustei – Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\begin{aligned} \sum w_a &= \sum \frac{2bc}{b+c} \cos \frac{A}{2} = \sum \frac{2\sqrt{bc}}{b+c} \sqrt{s(s-a)} \stackrel{c-b-s}{\leq} 2\sqrt{s} \sqrt{\sum bc} \sqrt{\sum \frac{s-a}{(b+c)^2}} \leq \\ &\stackrel{A-G}{\leq} 2\sqrt{s} \sqrt{\sum ab} \sqrt{\sum \frac{s-a}{4bc}} = \sqrt{s} \sqrt{\sum ab} \sqrt{\sum \frac{a(s-a)}{4Rrs}} = \\ &= \sqrt{\sum ab} \sqrt{\frac{s(2s) - 2(s^2 - 4Rr - r^2)}{4Rr}} = \sqrt{\sum ab} \sqrt{\frac{4R+r}{2R}} = \sqrt{\frac{\sum ab}{2R}} \sqrt{\sum r_a} = \\ &= \sqrt{(\sum h_a)(\sum r_a)} \quad \left( \because \frac{ab}{2R} = h_c, \text{etc} \right) \quad (\text{proved}) \end{aligned}$$

**890. In  $\Delta ABC$  the following relationship holds:**

$$bc \geq w_a^2 \left( 1 + \left( \frac{a}{b+c} \right)^2 \right)$$

*Proposed by Bogdan Fustei – Romania*

*Solution 1 by Radu Butelca-Romania*

$$\begin{aligned} w_a &= \frac{2bc}{b+c} \cos \frac{A}{2} \\ \cos \frac{A}{2} &= \sqrt{\frac{s(s-a)}{bc}} \end{aligned} \Rightarrow w_a^2 = \frac{bc[(b+c)^2 - a^2]}{(b+c)^2} \quad (1)$$

$$w_a^2 \left[ \frac{(b+c)^2 + a^2}{(b+c)^2} \right] \stackrel{(1)}{=} \frac{bc[(b+c)^2 - a^2][(b+c)^2 + a^2]}{(b+c)^4} \leq bc$$

$$[(b+c)^2 - a^2][(b+c)^2 + a^2] \leq (b+c)^4 \Leftrightarrow (b+c)^4 - a^4 \leq (b+c)^4$$

$$\Leftrightarrow a^4 \geq 0, \text{ which is true because } a \text{ is lenght of a triangle.}$$

*Solution 2 by Rajsekhar Azaad-India*

$$w_a^2 = bc \left[ 1 - \frac{a^2}{(b+c)^2} \right] \Rightarrow w_a^2 \cdot \left( 1 + \frac{a^2}{(b+c)^2} \right) = bc \left[ 1 - \frac{a^2}{(b+c)^2} \right] \left[ 1 + \frac{a^2}{(b+c)^2} \right] =$$



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$$= bc \left[ 1 - \frac{a^4}{(b+c)^4} \right] \leq bc \Leftrightarrow bc \geq w_a^2 \left( 1 + \frac{a^2}{(b+c)^2} \right) \quad (\text{proved})$$

**891.** In  $\Delta ABC$  the following relationship holds:

$$15r_a^2 + 10r_b^2 + 7r_c^2 > 270r^2$$

*Proposed by Daniel Sitaru – Romania*

**Solution 1 by Marian Ursărescu – Romania**

$$\text{From Cauchy's inequality} \Rightarrow 15r_a^2 + 10r_b^2 + 7r_c^2 > \frac{(\sqrt{15}r_a + \sqrt{10}r_b + \sqrt{7}r_c)^2}{3} \quad (1)$$

$$\text{From (1) the inequality becomes: } (\sqrt{15}r_a + \sqrt{10}r_b + \sqrt{7}r_c)^2 > 3 \cdot 270r^2 \Leftrightarrow$$

$$\Leftrightarrow \sqrt{15}r_a + \sqrt{10}r_b + \sqrt{7}r_c > 9\sqrt{10}r \quad (2)$$

$$\text{We have: } r_a = s \tan \frac{A}{2} \text{ and } r = s \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2} \quad (3)$$

*From (2)+(3) we must show this:*

$$\sqrt{15}s \tan \frac{A}{2} + \sqrt{10}s \tan \frac{B}{2} + \sqrt{7}s \tan \frac{C}{2} > 9\sqrt{10}s \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2} \Leftrightarrow$$

$$\sqrt{15} \tan \frac{A}{2} + \sqrt{10} \tan \frac{B}{2} + \sqrt{7} \tan \frac{C}{2} > 9\sqrt{10} \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2} \quad (4)$$

$$\text{But } \sqrt{15} \tan \frac{A}{2} + \sqrt{10} \tan \frac{B}{2} + \sqrt{7} \tan \frac{C}{2} > 3\sqrt[3]{5\sqrt{42} \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}} \quad (5)$$

*From (4)+(5) we must show:*

$$3\sqrt[3]{5\sqrt{42} \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}} > 9\sqrt{10} \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2} \Leftrightarrow$$

$$\Leftrightarrow \sqrt[3]{5\sqrt{42} \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}} > 3\sqrt{10} \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2} \Leftrightarrow$$

$$5\sqrt{42} \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2} > 3\sqrt{3} \cdot 10\sqrt{10} \tan^3 \frac{A}{2} \tan^3 \frac{B}{2} \tan^3 \frac{C}{2} \Leftrightarrow$$

$$\sqrt{7} \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2} > 6\sqrt{5} \tan^3 \frac{A}{2} \tan^3 \frac{B}{2} \tan^3 \frac{C}{2} \Leftrightarrow$$

$$\Leftrightarrow \tan^2 \frac{A}{2} \tan^2 \frac{B}{2} \tan^2 \frac{C}{2} < \frac{\sqrt{7}}{6\sqrt{5}} \quad (6)$$



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*But in any  $\Delta ABC$  we have relation:  $\sum \tan \frac{A}{2} \tan \frac{B}{2} = 1 \Rightarrow$*

$$\begin{aligned} & \Rightarrow 1 = \sum \tan \frac{A}{2} \tan \frac{B}{2} \geq 3 \sqrt[3]{\tan^2 \frac{A}{2} \tan^2 \frac{B}{2} \tan^2 \frac{C}{2}} \Rightarrow \\ & \Rightarrow \sqrt[3]{\tan^2 \frac{A}{2} \tan^2 \frac{B}{2} \tan^2 \frac{C}{2}} \leq \frac{1}{3} \Leftrightarrow \tan^2 \frac{A}{2} \tan^2 \frac{B}{2} \tan^2 \frac{C}{2} \leq \frac{1}{27} \Leftrightarrow \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2} \leq \\ & \leq \frac{1}{3\sqrt{3}} < \frac{\sqrt{7}}{6\sqrt{5}} \Rightarrow (20 < 21) \\ & 6 \text{ its true. Remark: } s = \frac{a+b+c}{2} \end{aligned}$$

**Solution 2 by Soumava Chakraborty-Kolkata-India**

$$\begin{aligned} \text{Given inequality} & \Leftrightarrow s^2 \left\{ \frac{15}{(s-a)^2} + \frac{10}{(s-b)^2} + \frac{7}{(s-c)^2} \right\} > \frac{270s^2}{s^2} \Leftrightarrow \frac{15}{(s-a)^2} + \frac{10}{(s-b)^2} + \frac{7}{(s-c)^2} \stackrel{(1)}{>} \frac{270}{s^2} \\ \text{Now, LHS of (1)} & = \frac{\left(\sqrt[3]{15}\right)^3}{(s-a)^2} + \frac{\left(\sqrt[3]{10}\right)^2}{(s-b)^2} + \frac{\left(\sqrt[3]{7}\right)^2}{(s-c)^2} \stackrel{\text{Radon}}{>} \frac{\left(\sqrt[3]{15} + \sqrt[3]{10} + \sqrt[3]{7}\right)^3}{(\sum(s-a))^2} > \frac{278}{s^2} > \frac{270}{s^2} \\ & \Rightarrow (1) \text{ is true (Proved)} \end{aligned}$$

**892. In acute  $\Delta ABC$  the following relationship holds:**

$$a^2 + b^2 + c^2 \geq 6abc \sqrt{\frac{6 \cos A \cos B \cos C}{a^2 + b^2 + c^2}}$$

*Proposed by Daniel Sitaru – Romania*

**Solution 1 by Serban George Florin-Romania**

$$\begin{aligned} a^2 + b^2 + c^2 & \geq 6abc \sqrt{\frac{6 \cos A \cos B \cos C}{a^2 + b^2 + c^2}}^2 \\ (a^2 + b^2 + c^2)^2 & \geq \frac{6^2 a^2 b^2 c^2 \cdot 6 \cos A \cos B \cos C}{a^2 + b^2 + c^2} \\ (a^2 + b^2 + c^2)^3 & \geq 6^3 a^2 b^2 c^2 \cos A \cos B \cos C \\ (a^2 + b^2 + c^2)^3 & \stackrel{Ma \geq Mg}{\geq} \left(3\sqrt[3]{a^2 b^2 c^2}\right)^3 = 3^3 a^2 b^2 c^2 \geq 6^3 a^2 b^2 c^2 \cos A \cos B \cos C \end{aligned}$$



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$$\Rightarrow \cos A \cos B \cos C \leq \frac{1}{8}, \sqrt[3]{\cos A \cos B \cos C} \leq \frac{1}{2}$$

If  $\Delta ABC$  is obtuse triangle  $\Rightarrow \cos A \cos B \cos C < 0 < \frac{1}{8}$  (A)

If  $\Delta ABC$  is acute-angled.

$$\begin{aligned} \sqrt[3]{\cos A \cos B \cos C} &\stackrel{(Mg \leq Ma)}{\leq} \frac{\cos A + \cos B + \cos C}{3} \leq \frac{1}{2} \Rightarrow \cos A + \cos B + \cos C \leq \frac{3}{2} \Rightarrow \\ &\Rightarrow 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \leq \frac{3}{2} \Rightarrow \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \leq \frac{1}{8} \\ &\sin \frac{A}{2} \leq \frac{a}{2\sqrt{bc}}, \prod \sin \frac{A}{2} \leq \prod \frac{a}{2\sqrt{bc}} = \frac{1}{8} \quad (A) \end{aligned}$$

*Solution 2 by Marian Ursarescu-Romania*

$$\text{Inequality} \Leftrightarrow \sqrt{(a^2 + b^2 + c^2)^3} \geq 6abc\sqrt{6 \cos A \cos B \cos C} \Leftrightarrow$$

$$(a^2 + b^2 + c^2)^3 \geq 6^3 \cdot a^2 b^2 c^2 \cdot \cos A \cos B \cos C \quad (1)$$

$$a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr) \quad (2) \quad abc = 4sRr \quad (3)$$

$$\cos A \cos B \cos C = \frac{s^2 - (2R+r)^2}{4R^2} = \frac{s^2 - 4R^2 - 4Rr - r^2}{4R^2} \quad (4)$$

From (1) + (2) + (3) + (4) we must show:

$$8(s^2 - r^2 - 4Rr)^3 \geq 6^3 \cdot 16s^2 R^2 r^2 \cdot \frac{s^2 - 4R^2 - 4Rr - r^2}{4R^2} \Leftrightarrow$$

$$\Leftrightarrow (s^2 - r^2 - 4Rr)^3 \geq 108s^2 r^2 (s^2 - 4R^2 - 4Rr - r^2) \quad (5)$$

Now, use the Gerretsen inequality  $\Rightarrow$

$$16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2 \quad (6)$$

From (5) + (6) we must show:  $(2R - r)^3 \geq s^2 r$  (7)

Now, from Gergonne's inequality:  $s^2 \leq \frac{R(4R+r)^2}{2(2R-r)}$  (8)

From (7) + (8) we must show:

$$(2R - r)^3 \geq \frac{R(4R+r)^2}{2(2R-r)} \Leftrightarrow (2R - r)^4 \geq \frac{Rr}{2} (4R + r)^2 \quad (9)$$

From Euler  $R \geq 2r$  (10). From (9) + (10)  $\Leftrightarrow (2R - r)^4 \geq r^2 (4R + r)^2 \Leftrightarrow$

$$\Leftrightarrow (2Rr - r^2)^2 \geq r(4R + r) \Leftrightarrow R \geq 2r \quad (\text{true})$$

*Solution 3 by Soumava Chakraborty-Kolkata-India*



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$$\begin{aligned} \because \prod \cos A \leq \frac{1}{8}, \therefore RHS \leq 6abc \sqrt{\frac{3}{4 \sum a^2}} &\stackrel{?}{\leq} \sum a^2 \Leftrightarrow \left(\sum a^2\right)^2 \stackrel{?}{\geq} 36a^2b^2c^2 \cdot \frac{3}{4 \sum a^2} \Leftrightarrow \\ &\Leftrightarrow (\sum a^2)^3 \geq 27a^2b^2c^2 \rightarrow \text{true (AM-GM) (Done)} \end{aligned}$$

**Solution 4 by Myagmarsuren Yadamsuren-Darkhan-Mongolia**

$m_a \geq \sqrt{s(s-a)}$ ,  $m_b \geq \sqrt{s(s-b)}$  and  $m_c \geq \sqrt{s(s-c)}$  and  $abc = 4Rrs$  then

$$\begin{aligned} \sum_{cyc} \frac{m_a}{bc} &\geq 3 \sqrt[3]{\frac{m_a m_b m_c}{a^2 b^2 c^2}} \geq 3 \sqrt[3]{\frac{s\Delta}{16R^2 r^2 s^2}} = 3 \sqrt[3]{\frac{s^2 r}{16R^2 r^2 s^2}} \geq 3 \sqrt[3]{\frac{1}{8R^2 \cdot R}} [\because R \geq 2r] \\ &= \frac{3}{2R} \quad (\text{proved}) \end{aligned}$$

**Solution 5 by Soumitra Mandal-Chandar Nagore-India**

$$a^2 + b^2 + c^2 \geq 6abc \sqrt{\frac{6 \cos A \cos B \cos C}{a^2 + b^2 + c^2}}$$

$$(a^2 + b^2 + c^2)^3 \geq 27a^2b^2c^2 \quad (\text{true})$$

$$(a^2 + b^2 + c^2)^2 \geq \frac{27a^2b^2c^2}{a^2 + b^2 + c^2}$$

$$a^2 + b^2 + c^2 \geq 3abc \sqrt{\frac{3}{a^2 + b^2 + c^2}} = 6abc \cdot \sqrt{\frac{\frac{6}{27} \cdot \left(\frac{3}{2}\right)^3}{a^2 + b^2 + c^2}} =$$

$$= 6abc \sqrt{\frac{6 \left(\frac{1 + \frac{1}{2}}{3}\right)^3}{a^2 + b^2 + c^2}} \stackrel{\text{Euler}}{\geq} 6abc \sqrt{\frac{6 \left(\frac{1 + \frac{r}{R}}{3}\right)^3}{a^2 + b^2 + c^2}} =$$

$$= 6abc \sqrt{\frac{6 \left(\frac{\cos A + \cos B + \cos C}{3}\right)^3}{a^2 + b^2 + c^2}} \stackrel{Ma \geq Mg}{\geq} 6abc \sqrt{\frac{6 \cos A \cos B \cos C}{a^2 + b^2 + c^2}}$$

$$\cos A + \cos B + \cos C = 1 + \frac{r \cdot \frac{1}{2}}{R'} \geq \frac{r}{R} - \text{Euler}$$

**Solution 6 by Sanong Huayrerai-Nakon Pathom-Thailand**

$$a^2 = b^2 + c^2 - 2bccosA$$



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$$\begin{aligned}
 b^2 &= c^2 + a^2 - 2cacosB \\
 c^2 &= a^2 + b^2 - 2abcosC \\
 a^2 + b^2 + c^2 &= 2(bccosA + cacosB + abcosC) \\
 (a^2 + b^2 + c^2)^3 &= (2(bccosA + cacosB + abcosC))^3 \stackrel{AM-GM}{\geq} \\
 &\geq 8 \cdot 27abc \cdot abccosAcosBcosC = 6^3(abc)^2cosAcosBcosC \\
 (a^2 + b^2 + c^2)^2 &\geq \frac{6^3(abc)^2cosAcosBcosC}{a^2 + b^2 + c^2} \\
 a^2 + b^2 + c^2 &\geq 6abc \sqrt{\frac{6cosAcosBcosC}{a^2 + b^2 + c^2}}
 \end{aligned}$$

**893. In  $\Delta ABC$  the following relationship holds:**

$$\frac{h_a}{h_b + h_c} + \frac{h_b}{h_c + h_a} + \frac{h_c}{h_a + h_b} > \frac{1}{2} \left( \frac{w_a}{a} + \frac{w_b}{b} + \frac{w_c}{c} \right)$$

*Proposed by Bogdan Fustei – Romania*

*Solution by Serban George Florin – Romania*

$$\begin{aligned}
 \sum \frac{h_a}{h_b + h_c} &= \sum \frac{\frac{2S}{a}}{\frac{2S}{b} + \frac{2S}{c}} = \sum \frac{bc}{a(b+c)} \\
 \frac{1}{2} \sum \frac{w_a}{a} &= \frac{1}{2} \sum \frac{2bc \cos \frac{A}{2}}{a(b+c)} = \sum \frac{b \cos \frac{A}{2}}{a(b+c)} \\
 \frac{1}{2} \sum \frac{w_a}{a} &= \sum \frac{bc \cos \frac{A}{2}}{a(b+c)} < \sum \frac{bc}{a(b+c)} = \sum \frac{h_a}{h_b + h_c} \\
 &\quad \left( \cos \frac{A}{2}, \cos \frac{B}{2}, \cos \frac{C}{2} < 1 \right) \text{ true}
 \end{aligned}$$

**894. If  $m > 0$  then in  $\Delta ABC$  the following relationship holds:**

$$\cot^{m+1} \frac{A}{2} + \cot^{m+1} \frac{B}{2} + \cot^{m+1} \frac{C}{2} \geq 3^{\frac{m+3}{2}}$$

*Proposed by D.M.Batinetu-Giurgiu, Neculai Stanciu-Romania*



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*Solution by Soumava Chakraborty-Kolkata-India*

$$\begin{aligned}
 & \cot^{m+1} \frac{A}{2} + \cot^{m+1} \frac{B}{2} + \cot^{m+1} \frac{C}{2} = \sum \frac{\cot^{m+1} \frac{A}{2}}{1^m} \stackrel{\text{RADON}}{\geq} \\
 & \geq \frac{\left( \sum \cot \frac{A}{2} \right)^{m+1}}{3^m} \stackrel{\text{JENSEN}}{\geq} \frac{\left( 3 \cot \frac{\pi}{6} \right)^{m+1}}{3^m} = \frac{3^{m+1} \cdot (\sqrt{3})^{m+1}}{3^m} = \\
 & = \frac{3^m \cdot 3^{\frac{m+1}{2}}}{3^m} = 3^{\frac{m+3}{2}} ; (f: (0, \pi) \rightarrow \mathbb{R}, f(x) = \cot \frac{x}{2}, f - \text{convexe})
 \end{aligned}$$

895. In  $\Delta ABC$  the following relationship holds:

$$\frac{bc}{aw_a} + \frac{ca}{bw_b} + \frac{ab}{cw_c} \leq \frac{9R^2}{2S}$$

*Proposed by Mehmet Sahin-Ankara-Turkey*

*Solution by Daniel Sitaru-Romania*

$$\begin{aligned}
 \frac{1}{aw_a} &= \frac{1}{\frac{2abc}{b+c} \cos \frac{A}{2}} = \frac{1}{\frac{bc}{2R \cos \frac{B-C}{2}}} = \frac{2R \cos \frac{B-C}{2}}{bc} = \frac{\cos \frac{B-C}{2}}{2S} \leq \frac{1}{2S} \quad (1) \\
 \sum \frac{bc}{aw_a} &\stackrel{(1)}{\geq} \sum \frac{bc}{2S} \leq \frac{1}{2S} \sum a^2 \stackrel{\text{LEIBNIZ}}{\geq} \frac{9R^2}{2S}
 \end{aligned}$$

896. If in  $\Delta ABC$ ,  $\sin A \sin B \sin C = \frac{1}{8R^3}$  then:

$$\frac{ab}{a^5 + b^5 + c^2} + \frac{bc}{b^5 + c^5 + a^2} + \frac{ca}{c^5 + a^5 + b^2} \leq \left( \frac{a+b+c}{3} \right)^{10}$$

*Proposed by Nguyen Van Nho-Nghe An-Vietnam*

*Solution by Daniel Sitaru-Romania*

$$\sin A \sin B \sin C = \frac{1}{8R^3} \rightarrow abc = 1 ; \sum \frac{ab}{a^5 + b^5 + c^2} \stackrel{\text{AM-GM}}{\leq} \sum \frac{ab}{3\sqrt[3]{a^5 b^5 c^2}} =$$



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$$= \frac{1}{3} \sum \frac{ab}{ab\sqrt[3]{(abc)^2}} = \frac{1}{3} \cdot 3 = 1 = (\sqrt[3]{1})^{10} = (\sqrt[3]{abc})^{10} \stackrel{AM-GM}{\geq} \left(\frac{a+b+c}{3}\right)^{10}$$

**897.** In  $\Delta ABC$  the following relationship holds:

$$\frac{a^2}{h_b + h_c} + \frac{b^2}{h_c + h_a} + \frac{c^2}{h_a + h_b} \geq 6r$$

*Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan*

**Solution by Daniel Sitaru-Romania**

$$\begin{aligned} \sum \frac{a^2}{h_b + h_c} &= \frac{4R(s^2 - r^2 - Rr)}{s^2 + r^2 + 2Rr} \stackrel{GERRETSEN}{\geq} \frac{4R(16Rr - 5r^2 - r^2 - Rr)}{4R^2 + 4Rr + 3r^2 + r^2 + 2Rr} = \\ &= \frac{4R(15Rr - 6r^2)}{4R^2 + 6Rr + 4r^2} \stackrel{EULER}{\geq} \frac{4R(15Rr - 6r \cdot \frac{R}{2})}{4R^2 + 6R \cdot \frac{R}{2} + 4 \cdot \frac{R^2}{4}} = \frac{48Rr}{8R^2} = 6r \end{aligned}$$

**898.** In  $\Delta ABC$  the following relationship holds:

$$\frac{m_a}{bc} + \frac{m_b}{ca} + \frac{m_c}{ab} \geq \frac{3}{2R}$$

*Proposed by Daniel Sitaru – Romania*

**Solution 1 by Bogdan Fustei-Romania**

$bc = 2Rh_a$  (and the analogs); The inequality from enunciation becomes:

$$\frac{1}{2R} \left( \frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c} \right) \geq \frac{3}{2R} \Rightarrow \frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c} \geq 3$$

$m_a \geq h_a$  (and the analogs)  $\Rightarrow \frac{m_a}{h_a} \geq 1$  (and the analogs) so,  $\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c} \geq 3$  true

and the inequality from enunciation is proved

**Solution 2 by Mehmet Sahin-Ankara-Turkey**

$\frac{m_a}{bc} + \frac{m_b}{ca} + \frac{m_c}{ab} = \frac{am_a + bm_b + cm_c}{abc}$ . Let  $T = am_b + bm_c + cm_a$ ;  $T \geq 3\sqrt[3]{abc \cdot m_a m_b m_c}$  and

$$m_a \geq \sqrt{s(s-a)}; T \geq 3\sqrt[3]{4R\Delta\sqrt{s(s-a)} \cdot \sqrt{s(s-a)} \cdot \sqrt{s(s-a)}}; T \geq 3\sqrt[3]{4R\Delta \cdot s\Delta}$$



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$$T \geq 3\sqrt[3]{4 \cdot 2r \cdot r \cdot s \cdot r \cdot r \cdot s} = 6rs; \frac{T}{abc} \geq \frac{6s}{4R\Delta} = \frac{6}{4R} = \frac{3}{2R} \therefore$$

*Solution 3 by Seyran Ibrahimov-Maasilli-Azerbaijan*

$$\sum \frac{m_a}{bc} \geq \frac{3}{2R} \quad (1)$$

$$m_a \geq \frac{b^2+c^2}{4R} \quad (\text{Tereshin})$$

$$(1) \Rightarrow \frac{1}{4R} \sum \frac{b^2+c^2}{bc} \stackrel{AM-GM}{\geq} \frac{1}{4R} \cdot 6 = \frac{3}{2R} \quad (\text{proved})$$

**899.** If  $M \in \text{Int}(\Delta ABC)$ ,  $AM = x$ ,  $BM = y$ ,  $CM = z$  then:

$$\frac{ax}{ax + by + 98cz} + \frac{by}{by + cz + 98ax} + \frac{cz}{cz + ax + 98by} \geq \frac{3}{100}$$

*Proposed by Daniel Sitaru-Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$ax = u, by = v, cz = w$$

*The inequality to prove can be written:*

$$\sum \frac{u}{u+v+98w} \geq \frac{3}{100} \leftrightarrow 100 \sum u(v+w+98u)(w+u+98v) \geq 3 \prod (u+v+98w)$$

$$9506 \sum u^3 + 931491 \sum u^2v \geq 2794764uvw + 9409 \sum uv^2 \quad (a)$$

$$u^3 + v^3 + w^3 \stackrel{AM-GM}{\geq} 3uvw^2$$

$$v^3 + w^3 + u^3 \stackrel{AM-GM}{\geq} 3vwu^2$$

$$w^3 + u^3 + v^3 \stackrel{AM-GM}{\geq} 3wu^2$$

$$\sum u^3 \geq \sum uv^2 \rightarrow 9506 \sum u^3 \geq 9506 \sum uv^2 \quad (1)$$

$$97 \sum uv^2 \stackrel{AM-GM}{\geq} 291uvw \quad (2)$$

$$931491 \sum u^2v \stackrel{AM-GM}{\geq} 2794473uvw \quad (3)$$



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*By adding (1), (2), (3)  $\rightarrow (a)$*

**900.** In  $\Delta ABC$  the following relationship holds:

$$\sqrt{2} \left( \frac{1}{a} \cos \frac{A}{2} + \frac{1}{b} \cos \frac{B}{2} + \frac{1}{c} \cos \frac{C}{2} \right) \leq \sqrt{s \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \left( \frac{1}{w_a^2} + \frac{1}{w_b^2} + \frac{1}{w_c^2} \right)}$$

*Proposed by Bogdan Fustei – Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\begin{aligned}
& \sqrt{2} \sum \frac{\cos \frac{A}{2}}{a} \leq \sqrt{s \left( \sum \frac{1}{a} \right) \left( \sum \frac{1}{w_a^2} \right)} \\
& w_a^2 \leq s(s-a), \text{ etc, } \therefore \frac{1}{w_a^2} \geq \frac{1}{s(s-a)}, \text{ etc.} \\
& \therefore \sum \frac{1}{w_a^2} \stackrel{(1)}{\geq} \frac{1}{s} \sum \frac{1}{s-a} = \frac{\sum (s-b)(s-c)}{r^2 s^2} = \frac{3s^2 - s(4s) + s^2 + 4Rr + r^2}{r^2 s^2} = \\
& = \frac{4R + r}{rs^2} \\
& \therefore \left( \sum \frac{1}{a} \right) \left( \sum \frac{1}{w_a^2} \right) \stackrel{\text{by (1)}}{\geq} \frac{\sum ab}{4Rrs} \cdot \frac{4R + r}{rs^2} = \frac{(\sum ab)(4R + r)}{4Rr^2 s^3} \Rightarrow \\
& \Rightarrow RHS = \sqrt{s \left( \sum \frac{1}{a} \right) \left( \sum \frac{1}{w_a^2} \right)} \stackrel{(a)}{\geq} \sqrt{\frac{(\sum ab)(4R+r)}{4Rr^2 s^2}}. \text{ Now, LHS} \stackrel{CBS}{\leq} \sqrt{2} \sqrt{\sum \frac{1}{a^2}} \sqrt{\sum \cos^2 \frac{A}{2}} \leq \\
& \stackrel{\text{Goldstone}}{\stackrel{(b)}{\leq}} \sqrt{2 \cdot \frac{1}{4r^2}} \sqrt{\frac{1}{2} \sum (1 + \cos A)} = \sqrt{\frac{3 + 1 + \frac{r}{R}}{4r^2}} = \sqrt{\frac{4R + r}{4Rr^2}}
\end{aligned}$$

(a), (b)  $\Rightarrow$  it suffices to prove:  $\frac{1}{4Rr^2} \leq \frac{\sum ab}{4Rr^2 s^2} \Leftrightarrow \sum ab \geq s^2 \Leftrightarrow 4Rr + r^2 \geq 0 \rightarrow \text{true}$

*(proved)*



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*Its nice to be important but more important its to be nice.*

*At this paper works a TEAM.*

*This is RMM TEAM.*

*To be continued!*

*Daniel Sitaru*