RMM - Inequalities Marathon 201 - 300

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201. If $a, b \geq 0$, $a+b+c+d=0$ then:

$$4\sum a^3 \geq 3(a+b)(ac+ad+bc+bd+4cd)$$

Proposed by Daniel Sitaru – Romania

Solution by Pham Quoc Sang-Ho Chi Minh-Vietnam

We have: if $a+b+c=0$ ***then*** $a^3+b^3+c^3=3abc$ ***so***

$(a+b)+c+d=0$ ***then*** $(a+b)^3+c^3+d^3=3(a+b)cd$

We have: $3(a+b)(ac+ad+bc+bd+4cd)=3(a+b)[(a+b)(c+d)+4cd]$

$$=3(a+b)[-(a+b)^2+4cd]$$

$$=-3(a+b)^3+4\cdot 3(a+b)cd=-3(a+b)^3+4[(a+b)^3+c^3+d^3]$$

$=(a+b)^3+4(c^3+d^3)$. ***Now, we prove that***

$$4(a^3+b^3+c^3)\geq(a+b)^3+4(c^3+d^3)\Rightarrow 4(a^3+b^3)\geq(a+b)^3$$

$\Leftrightarrow (1^3+1^3)(1^3+1^3)(a^3+b^3)\geq(a+b)^3$ ***(Right because Hölder's)***

"=" a=b.

202. If $a, b, c, d > 0$ then:

$$\left(2a^2\sqrt{b^3}\sqrt[3]{c^4}\sqrt[4]{d^5}+\frac{3}{2}b^2\sqrt{c^3}\sqrt[3]{d^4}\sqrt[4]{a^5}+\frac{4}{3}c^2\sqrt{d^3}\sqrt[3]{a^4}\sqrt[4]{b^5}+\frac{5}{4}d^2\sqrt{a^3}\sqrt[3]{b^4}\sqrt[4]{c^5}\right)\left(\sum\frac{1}{a}\right)^4\geq\frac{4672}{3}$$

Proposed by Uche Eliezer Okeke-Anambra-Nigeria

Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$xa^xb^yc^zd^t+yb^xc^yd^za^t+zc^xd^ya^zb^t+td^xa^yb^zc^y\geq$$

$$\overset{AM-GM}{\geq}(x+y+z+t)(abcd)^{\frac{x+y+z+t}{x+y+z+t}}=abcd(x+y+z+t)$$

$$2a^2\sqrt{b^3}\sqrt[3]{c^4}\sqrt[4]{d^5}+\frac{3}{2}b^2\sqrt{c^3}\sqrt[3]{d^4}\sqrt[4]{a^5}+\frac{4}{3}c^2\sqrt{d^3}\sqrt[3]{a^4}\sqrt[4]{b^5}+\frac{5}{4}d^2\sqrt{a^3}\sqrt[3]{b^4}\sqrt[4]{c^5}\geq$$

$$\geq\left(2+\frac{3}{2}+\frac{4}{3}+\frac{5}{4}\right)abcd=\frac{73}{12}abcd, (1)$$

$$\left(\sum\frac{1}{a}\right)^4\overset{AM-GM}{\geq}\frac{256}{abcd}, (2)$$

$By\ multiplying\ (1), (2):$

$$\left(2a^2\sqrt{b^3}\sqrt[3]{c^4}\sqrt[4]{d^5}+\frac{3}{2}b^2\sqrt{c^3}\sqrt[3]{d^4}\sqrt[4]{a^5}+\frac{4}{3}c^2\sqrt{d^3}\sqrt[3]{a^4}\sqrt[4]{b^5}+\frac{5}{4}d^2\sqrt{a^3}\sqrt[3]{b^4}\sqrt[4]{c^5}\right)\left(\sum\frac{1}{a}\right)^4\ge$$
$$\ge\frac{73}{12}abcd\cdot\frac{256}{abcd}=\frac{4672}{3}$$

203. If $a,b,c,d>0, x,y\in\mathbb{R}$ then:

$$\frac{\sin^2x}{a}+\frac{\cos^2x}{b}+\frac{\sin^2y}{c}+\frac{\cos^2y}{d}>\frac{2}{a+b+c+d}$$

Proposed by Daniel Sitaru-Romania

Solution by Sanong Hauyrerai-Nakon Pathom-Thailand

$$\frac{\sin^2x}{a}+\frac{\cos^2x}{b}+\frac{\sin^2y}{c}+\frac{\cos^2y}{d}=\frac{\sin^4x}{a\sin^2x}+\frac{\cos^4x}{b\cos^2x}+\frac{\sin^4y}{c\sin^2y}+\frac{\cos^4y}{d\cos^2y}\ge$$
$$\overset{BERGSTROM}{\ge}\frac{(\sin^2x+\cos^2x)^2}{a\sin^2x+b\cos^2x}+\frac{(\sin^2y+\cos^2y)^2}{c\sin^2y+d\cos^2y}=$$
$$=\frac{1}{a\sin^2x+b\cos^2x}+\frac{1}{c\sin^2y+d\cos^2y}\overset{BERGSTROM}{\ge}$$
$$\ge\frac{4}{a\sin^2x+b\cos^2x+c\sin^2y+d\cos^2y}>\frac{4}{2(a+b)+2(c+d)}=\frac{2}{a+b+c+d}$$

204. If $a,b,c,d,e,f>0$ then:

$$\frac{(a^3+b^3)^4}{(c^6+d^6)^5}\cdot\frac{(c^5+d^5)^6}{(e^8+f^8)^7}\cdot\frac{(e^7+f^7)^8}{(a^4+b^4)^3}>1$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Sanong Hauyrerai-Nakon Pathom-Thailand

$$x,y>0, n\in\mathbb{N}^*\to\frac{n+1}{n}>1$$
$$(x^n+y^n)^{\frac{n+1}{n}}>(x^n)^{\frac{n+1}{n}}+(y^n)^{\frac{n+1}{n}}\to(x^n+y^n)^{\frac{n+1}{n}}>x^{n+1}+y^{n+1}$$
$$(x^n+y^n)^{n+1}>(x^{n+1}+y^{n+1})^n\to\frac{(x^n+y^n)^{n+1}}{(x^{n+1}+y^{n+1})^n}>1$$

$$\begin{cases}\dfrac{(a^3+b^3)^4}{(a^4+b^4)^3}>1\\ \dfrac{\left(c^5+d^5\right)^6}{(c^6+d^6)^5}>1\\ \dfrac{(e^7+f^7)^8}{(e^8+f^8)^7}>1\end{cases}\overset{by\ multipying}{\Rightarrow}\frac{(a^3+b^3)^4}{(c^6+d^6)^5}\cdot\frac{\left(c^5+d^5\right)^6}{(e^8+f^8)^7}\cdot\frac{(e^7+f^7)^8}{(a^4+b^4)^3}>1$$

Solution 2 by Ravi Prakash-New Delhi-India

$$WLOG: y\le x\to t=\frac{y}{x}\le 1, k\in\mathbb{N}$$

$$\left(x^k+y^k\right)^{k+1}-\left(x^{k+1}+y^{k+1}\right)^k=x^{k(k+1)}\left(\left(1+t^k\right)^{k+1}-\left(1+t^{k+1}\right)^k\right)=x^{k(k+1)}f(t)$$

$$f(t)=\left(1+t^k\right)^{k+1}-\left(1+t^{k+1}\right)^k=$$

$$=\sum_{r=0}^{k+1}\binom{r}{k+1}\left(t^k\right)^r-\sum_{r=0}^{k}\binom{r}{k}\left(t^{k+1}\right)^r=\sum_{r=1}^{k}\left(\binom{r-1}{k}+\binom{r}{k}\right)t^{kr}-\sum_{r=1}^{k-1}\binom{r}{k}t^{kr+r}=$$

$$=\sum_{r=1}^{k}\binom{r-1}{k}t^{kr}+\binom{k+1}{k}\left(t^k\right)^k+\sum_{r=1}^{k-1}\binom{k}{r}t^{kr}(1-t^r)>0;\ \frac{\left(x^k+y^k\right)^{k+1}}{(x^{k+1}+y^{k+1})^k}>1$$

$$\begin{cases}\dfrac{(a^3+b^3)^4}{(a^4+b^4)^3}>1\\ \dfrac{\left(c^5+d^5\right)^6}{(c^6+d^6)^5}>1\\ \dfrac{(e^7+f^7)^8}{(e^8+f^8)^7}>1\end{cases}\overset{by\ multipying}{\Rightarrow}\frac{(a^3+b^3)^4}{(c^6+d^6)^5}\cdot\frac{\left(c^5+d^5\right)^6}{(e^8+f^8)^7}\cdot\frac{(e^7+f^7)^8}{(a^4+b^4)^3}>1$$

205. If $0<a\le b$ then:

$$\left(1+\frac{a+3b}{4}\right)^{3a+b}\le\left(1+\frac{3a+b}{4}\right)^{a+3b}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$f(x)=(1+x)^{\frac{1}{x}}, x>0, lnf(x)=\frac{ln(1+x)}{x}$$

$$\frac{f'(x)}{f(x)}=\frac{x-(x+1)ln(x+1)}{x^2(x+1)}, g(x)=x-(x+1)ln(x+1)$$

$$g'(x)=-ln(x+1)<0, x>0\to g(x)<g(0)=0, \forall x>0$$

$$f'(x)<0, \forall x>0, f-strictly\ decreasing$$

$$0<a\le b\to\frac{a+3b}{4}\ge\frac{3a+b}{4}\to f\left(\frac{a+3b}{4}\right)\le f\left(\frac{3a+b}{4}\right)$$

$$\left(1+\frac{a+3b}{4}\right)^{\frac{4}{a+3b}}\le\left(1+\frac{4a+3}{4}\right)^{\frac{4}{4a+b}}\to\left(1+\frac{a+3b}{4}\right)^{3a+b}\le\left(1+\frac{3a+b}{4}\right)^{a+3b}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$f(m)=\frac{ln(1+m)}{m}, m>0$$

$$f'(x)=\frac{1}{m(1+m)}-\frac{1}{m^2}\left(\frac{m}{1+m}-ln(1+m)\right)<0$$

$$\left(\frac{m}{1+m}\le ln(1+m)\le m, m\ge 0\right)$$

$$f-decreasing, a\le b\leftrightarrow\frac{a+3b}{4}\ge\frac{3a+b}{4}\to f\left(\frac{a+3b}{4}\right)\le f\left(\frac{3a+b}{4}\right)$$

$$\frac{ln\left(1+\frac{a+3b}{4}\right)}{a+3b}\ge\frac{ln\left(1+\frac{3a+b}{4}\right)}{3a+b}\to$$

$$\left(1+\frac{a+3b}{4}\right)^{\frac{1}{a+3b}}\le\left(1+\frac{4a+3}{4}\right)^{\frac{1}{4a+b}}\to\left(1+\frac{a+3b}{4}\right)^{3a+b}\le\left(1+\frac{3a+b}{4}\right)^{a+3b}$$

206. If $0\le x\le y\le z$ then:

$$\frac{(2+e^x)^2}{(2+e^y)(2+e^z)}\ge\frac{(1+e^x+e^{2x})^2}{(1+e^y+e^{2y})(1+e^z+e^{2z})}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

Let $e^x=a, e^y=b, e^z=c, 0\le x\le y\le z\to 1\le a\le b\le c$

$$\frac{(2+e^x)^2}{(2+e^y)(2+e^z)}\ge\frac{(1+e^x+e^{2x})^2}{(1+e^y+e^{2y})(1+e^z+e^{2z})}\leftrightarrow$$

$$\frac{(1+b+b^2)(1+c+c^2)}{(2+b)(2+c)} \ge \frac{(1+a+a^2)^2}{(2+a)^2}, (1)$$

$$1+b+b^2 \ge 1+a+a^2 \leftrightarrow (b-a)(1+b+a) \ge 0, (2)$$

$$b \le c \to 2+b \le 2+c \to \frac{1}{2+b} \ge \frac{1}{2+c} \to \frac{1}{(2+b)(2+c)} \ge \frac{1}{(2+c)^2}, (3)$$

$$\frac{(1+b+b^2)(1+c+c^2)}{(2+b)(2+c)} \overset{(2),(3)}{\ge} \frac{(1+a+a^2)(1+c+c^2)}{(2+c)^2} \ge \frac{(1+a+a^2)^2}{(2+a)^2} \leftrightarrow$$

$$\leftrightarrow \frac{1+c+c^2}{(2+c)^2} \ge \frac{1+a+a^2}{(2+a)^2}, (4);\ f(t) = \frac{1+t+t^2}{(2+t)^2}, \forall t \ge 1, f'(t) = \frac{3t}{(2+t)^2} > 0, \forall t \ge 1$$

$$f - increasing \to f(c) \ge f(a)$$

Solution 2 by Athanasios Mplegiannis-Greece

$f(x) = \frac{1+e^x+e^{2x}}{2+e^x}, x \in \mathbb{R};\ f'(x) = \frac{(e^x+2e^{2x})(2+e^x)-e^x(1+e^x+e^{2x})}{(2+e^x)^2} = \frac{e^{3x}+4e^{2x}+e^x}{(2+e^x)^2} > 0, \forall x \in \mathbb{R}$***, so,***

f is a stricIty increasing function.

$$\begin{cases} 0 < f(x) \le f(y) \\ 0 < f(x) \le f(z) \end{cases} \Rightarrow f^2(x) \le f(y)f(z) \Rightarrow \left(\frac{1+e^x+e^{2x}}{2+e^x}\right)^2 \le$$

$$\le \frac{1+e^y+e^{2y}}{2+e^y} \cdot \frac{1+e^z+e^{2z}}{2+e^z} \Rightarrow \left(\frac{1+e^x+e^{2x}}{2+e^x}\right)^2 \le \frac{1+e^y+e^{2y}}{2+e^y} \cdot \frac{1+e^x+e^{2z}}{2+e^z} \Rightarrow$$

$\Rightarrow \frac{(1+e^x+e^{2x})^2}{(1+e^y+e^{2y})(1+e^z+e^{2z})} \le \frac{(2+e^x)^2}{(2+e^y)(2+e^z)}$. ***Equality holds for $x = y = z$.***

207. **If $a, b, c, d \in \mathbb{N}^*, 1 \le a \le b \le c \le d$ then:**

$$4\log_{a+1} a \le \log_{a+1} a + \log_{b+1} b + \log_{c+1} c + \log_{d+1} d \le 4\log_{d+1} d$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Le Van-Ho Chi Minh-Vietnam

Put $f(x) = \frac{\ln x}{\ln(x+1)}, x \ge 1$. ***Then*** $f'(x) \cdot [\ln(x+1)]^2 = \frac{\ln(x+1)}{x} - \frac{\ln x}{x+1} =$

$= \frac{[(x+1)\ln(x+1) - x\ln(x)]}{x(x+1)} > 0$. ***Then $f(x)$ is a positive function, which gives us***

$$4f(a) \le f(a) + f(b) + f(c) + f(d) \le 4f(d)$$

→ *Q.E.D. Equality holds when* $a = b = c = d = 1$.

Solution 2 by Priyanka Banerjee-India

Let $f(x) = \log_{x+1} x$; $f(x) = \frac{\log_e x}{\log_e(x+1)} \Rightarrow f(x) = \frac{\frac{1}{x}\log_e(x+1) - \frac{1}{x+1}\log_e 2}{(\log_e(x+1))^2}$

Now, $(x+1)\log_e(x+1) > x\log_e x$. *Hence* $f(x) > 0$. *As* $a \le b \le c \le d$

$$\log_{a+1} a \le \log_{b+1} b \le \log_{c+1} c \le \log_{d+1} d$$

Hence $4\log_{a+1} a \le \log_{a+1} a + \log_{b+1} b + \log_{c+1} c + \log_{d+1} d \le 4\log_{d+1} d$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

Because $1 \le a \le b \le c \le d, a, b, c, d \in N$, *we get*

(1) $\log_a b \ge \log_{a+1}(b+1), \log_a c \ge \log_{(a+1)}(c+1), \log_a d \ge \log_{(a+1)}(d+1)$

and (2) $\log_d a \le \log_{(d+1)}(a+1), \log_d b \le \log_{(d+1)}(b+1), \log_d c \le \log_{(d+1)}(c+1)$

from (1), we obtain

$$(\log_a b)(\log_a e)(\log_a d) \ge \left(\log_{(a+1)}(b+1)\right)\left(\log_{(a+1)}(c+1)\right)\left(\log_{(a+1)}(b+1)\right)$$

$$\Rightarrow \frac{(\log_a b)(\log_a c)(\log_a d)}{\left(\log_{(a+1)}(b+1)\right)\left(\log_{(a+1)}(c+1)\right)\left(\log_{(a+1)}(d+1)\right)} \ge 1$$

$$\Rightarrow \frac{\left(\log_{(a+1)} b\right)\left(\log_{(a+1)} e\right)\left(\log_{(a+1)} d\right)}{\left(\log_{(a+1)}(b+1)\right)\left(\log_{(a+1)}(c+1)\right)\left(\log_{(a+1)}(d+1)\right)} \ge (\log_{a+1} a)^3$$

$$\Rightarrow \sqrt[3]{\frac{(\log_{a+1} b)(\log_{a+1} e)(\log_{a+1} d)}{(\log_{a+1} b + 1)(\log_{a+1} c + 1)(\log_{a+1} d + 1)}} \ge \log_{a+1} a$$

$$\Rightarrow 3\sqrt{\frac{(\log_{a+1} b)(\log_{a+1} c)(\log_{a+1} d)}{(\log_{a+1} b + 1)(\log_{a+1} c + 1)(\log_{a+1} d + 1)}} \ge 3\log_{a+1} a$$

$$\rightarrow \frac{\log_{a+1} b}{\log_{a+1} b + 1} + \frac{\log_{a+1} c}{\log_{a+1} c + 1} + \frac{\log_{a+1} d}{\log_{a+1} d + 1} \ge 3\log_{a+1} a$$

$$\rightarrow \log_{(b+1)} b + \log_{(c+1)} c + \log_{(d+1)} d \ge 3\log_{a+1} a$$

$$\rightarrow \log_{a+1} a + \log_{b+1} b + \log_{c+1} c + \log_{d+1} d \ge 4\log_{a+1} a$$

and (2) can show it such like this. Therefore it is to be true.

208. **If $a, b, c, d > 0$ then:**

$$\frac{\left(\sqrt{a}+\sqrt{b}\right)^2}{a+b}+\frac{\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)^3}{a+b+c}+\frac{\left(\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}+\sqrt[4]{d}\right)^4}{a+b+c+d}\le 75$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Lazaros Zachariadis-Thessaloniki-Greece

$$\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2\le\frac{a+b}{2}\Rightarrow\frac{\left(\sqrt{a}+\sqrt{b}\right)^2}{2^2}\le\frac{a+b}{2}\Rightarrow\frac{\left(\sqrt{a}+\sqrt{b}\right)^2}{a+b}\le 2\quad \textbf{(1)}$$

$$\left(\frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{3}\right)^3\le\frac{a+b+c}{3}\Rightarrow\frac{\left(\sum_{cyc}\sqrt[3]{a}\right)^3}{3^3}\le\frac{a+b+c}{3}\Rightarrow\frac{\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)^3}{a+b+c}\le 9\quad \textbf{(2)}$$

$$\left(\frac{\sum_{cyc}\sqrt[4]{a}}{4}\right)^4\le\frac{a+b+c+d}{4}\Rightarrow\frac{\left(\sum_{cyc}\sqrt[4]{a}\right)^4}{\sum_{cyc}a}\le 4^3=64\quad \textbf{(3)}$$

$$\textbf{(1)+(2)+(3)}\Rightarrow\frac{\left(\sum_{cyc}\sqrt{a}\right)^2}{b+a+c}+\frac{\left(\sum_{cyc}\sqrt[3]{a}\right)^3}{a+b+c}+\frac{\left(\sum_{cyc}\sqrt[4]{a}\right)^4}{a+b+c+d}\le 2+9+64=75$$

Solution 2 by Rajsekhar Azaad-India

$$m^{th}\text{ power of AM}\le\text{AM of }m^{th}\text{ power}\Rightarrow\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2\le\frac{a+b}{2}\Rightarrow\frac{\left(\sqrt{a}+\sqrt{b}\right)^2}{a+b}\le 2\quad \textbf{(i)}$$

$$\left(\frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{3}\right)^3\le\frac{a+b+c}{3}\Rightarrow\frac{\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)^3}{a+b+c}\le 3^2\quad \textbf{(ii)}$$

$$\left(\frac{\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}+\sqrt[4]{d}}{4}\right)^4\le\frac{a+b+c+d}{4}\Rightarrow\frac{\left(\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}+\sqrt[4]{d}\right)^4}{a+b+c+d}\le 4^3\quad \textbf{(iii)}$$

Adding (i), (ii), (iii): $\frac{\left(\sqrt{a}+\sqrt{b}\right)^2}{a+b}+\frac{\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)^3}{a+b+c}+\frac{\left(\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}+\sqrt[4]{d}\right)^4}{a+b+c+d}\le 2+3^2+4^3=75$

Solution 3 by Ravi Prakash-New Delhi-India

For $a,b,c,d\ge 0$

$$2(a+b)-\left(a^{\frac{1}{2}}+b^{\frac{1}{2}}\right)^2=\left(\sqrt{a}-\sqrt{b}\right)^2\ge 0\Rightarrow\frac{\left(\sqrt{a}+\sqrt{b}\right)^2}{a+b}\le 2\quad \textbf{(1)}$$

$$\textit{and }\ 9(a+b+c)-\left(a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}}\right)^3=$$

$$=8(a+b+c)-3\left(a^{\frac{1}{3}}b^{\frac{2}{3}}+a^{\frac{2}{3}}b^{\frac{1}{3}}+a^{\frac{1}{3}}c^{\frac{2}{3}}+a^{\frac{2}{3}}c^{\frac{1}{3}}+b^{\frac{1}{3}}c^{\frac{2}{3}}+b^{\frac{2}{3}}c^{\frac{1}{3}}+2a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}\right)$$

$$=\left(2a+b-3a^{\frac{2}{3}}b^{\frac{1}{3}}\right)+\left(a+2b-3a^{\frac{1}{3}}b^{\frac{2}{3}}\right)+\left(2a+c-3a^{\frac{2}{3}}c^{\frac{1}{3}}\right)+$$

$$+\left(a+2c-3a^{\frac{1}{3}}c^{\frac{2}{3}}\right)+\left(2b+c-3b^{\frac{2}{3}}c^{\frac{1}{3}}\right)+\left(b+2c-3b^{\frac{1}{3}}c^{\frac{2}{3}}\right)+$$

$$+2\left(a+b+c-3a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}\right)\geq 0\ [\because AM\geq GM]$$

$$\left[2a+b\geq 3(a^2b)^{\frac{1}{3}}\ etc.\right]\Rightarrow \frac{\left(a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}}\right)^3}{a+b+c}\leq 9\quad \textit{(2). Also,}$$

$$\frac{a+b+c+d}{4}\geq\left(\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}}{4}\right)^2\geq\left[\frac{a^{\frac{1}{4}}+b^{\frac{1}{4}}+c^{\frac{1}{4}}+d^{\frac{1}{4}}}{4}\right]^4\Rightarrow\frac{\left(a^{\frac{1}{4}}+b^{\frac{1}{4}}+c^{\frac{1}{4}}+d^{\frac{1}{4}}\right)^4}{a+b+c+d}\leq 64\quad (3)$$

Adding (1), (2), (3) we get the desired inequality.

Solution 4 by Soumava Chakraborty-Kolkata-India

$$\forall a,b,c,d>0,\textit{ we have: }\underbrace{\frac{(\sqrt{a}+\sqrt{b})^2}{a+b}}_{e_1}+\underbrace{\frac{(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c})^3}{a+b+c}}_{e_2}+\frac{(\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}+\sqrt[4]{d})^4}{a+b+c+d}\leq 75$$

$$(\sqrt[4]{a})^4+(\sqrt[4]{b})^4+(\sqrt[4]{c})^4+(\sqrt[4]{d})^4\overset{Chebyshev}{\geq}\frac{1}{4^3}(\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}+\sqrt[4]{d})^4$$

$$\Rightarrow e_3\leq 64\quad (1)$$

$$\textit{Again, }(\sqrt[3]{a})^3+(\sqrt[3]{b})^3+(\sqrt[3]{c})^3\overset{Chebyshev}{\geq}\frac{1}{3^2}(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c})^3$$

$$\Rightarrow e_2\leq 9\quad (2)$$

$$\textit{Also, }(\sqrt{a})^2+(\sqrt{b})^2\overset{Chebyshev}{\geq}\frac{1}{2}(\sqrt{a}+\sqrt{b})^2$$

$$\Rightarrow e_1\leq 2\quad (3)$$

$$(1)+(2)+(3)\Rightarrow LHS\leq 75$$

209. If $a,b,c\geq 0$ then:

$$\frac{a+\sqrt{ab}+\sqrt[3]{abc}}{3}\leq\sqrt[3]{a\left(\frac{a+b}{2}\right)\left(\frac{a+b+c}{3}\right)}$$

Proposed by Kiran Kedlaya-Berkeley-California-USA

Solution by Soumitra Mandal-Chandar Nagore-India

$$\frac{a+\sqrt{ab}+\sqrt[3]{abc}}{3}\le\sqrt[3]{a\left(\frac{a+b}{2}\right)\left(\frac{a+b+c}{3}\right)}\Leftrightarrow\sqrt[3]{\frac{1}{a}\cdot\frac{2}{a+b}\cdot\frac{3}{a+b+c}}\left(a+\sqrt{ab}+\sqrt[3]{abc}\right)\le 3$$

$$\Leftrightarrow\sqrt[3]{\frac{2a}{a+b}\cdot\frac{3a}{a+b+c}\cdot 1}+\sqrt[3]{\frac{2\sqrt{ab}}{a+b}\cdot\frac{3b}{a+b+c}\cdot 1}+\sqrt[3]{\frac{2b}{a+b}\cdot\frac{3c}{a+b+c}\cdot 1}\le 3$$

Now, $\sqrt[3]{\frac{2a}{a+b}\cdot\frac{3a}{a+b+c}\cdot 1}\overset{AM\ge GM}{\le}\frac{\frac{2a}{a+b}+\frac{3a}{a+b+c}+1}{3}$

$$\sqrt[3]{\frac{2\sqrt{ab}}{a+b}\cdot\frac{3b}{a+b+c}\cdot 1}\overset{AM\ge GM}{\le}\frac{\frac{2\sqrt{ab}}{a+b}+\frac{3b}{a+b+c}+1}{3}\le\frac{2+\frac{3b}{a+b+c}}{3}$$ **and**

$$\sqrt[3]{\frac{2b}{a+b}\cdot\frac{3c}{a+b+c}\cdot 1}\overset{AM\ge GM}{\le}\frac{\frac{2b}{a+b}+\frac{3c}{a+b+c}+1}{3}$$

$$\therefore\sqrt[3]{\frac{1}{a}\cdot\frac{2}{a+b}\cdot\frac{3}{a+b+c}}\left(a+\sqrt{ab}+\sqrt[3]{abc}\right)$$

$$\le\frac{4+2+3}{3}=3\Rightarrow\frac{a+\sqrt{ab}+\sqrt[3]{abc}}{3}\le\sqrt[3]{a\left(\frac{a+b}{2}\right)\left(\frac{a+b+c}{3}\right)}$$

210. Prove that if $x,y,z>0$ then:

$$\sqrt{\frac{x}{y}}+2\sqrt{\frac{y}{z}}+3\sqrt{\frac{z}{x}}\le\sqrt{6\left(\frac{x}{y}+\frac{2y}{z}+\frac{3z}{x}\right)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Le Minh Cuong-Ho Chi Minh-Vietnam

Apply Schwarz we get: $(LHS)^2=\left(\sqrt{\frac{x}{y}}+\sqrt{2}\sqrt{\frac{2y}{z}}+\sqrt{3}\sqrt{\frac{3z}{x}}\right)^2$

$\le\left(1^2+(\sqrt{2})^2+(\sqrt{3})^2\right)\left(\frac{x}{y}+\frac{2y}{z}+\frac{3z}{x}\right)\le 6\left(\frac{x}{y}+\frac{2y}{z}+\frac{3z}{x}\right)\le(RSH)^2$**. So** $HS\le RHS$ **!**

Solution 2 by Pham Quoc Sang-Ho Chi Minh-Vietnam

$$\sqrt{\frac{x}{y}}+2\sqrt{\frac{y}{z}}+3\sqrt{\frac{z}{x}}\le\sqrt{6\left(\frac{x}{y}+\frac{2y}{z}+\frac{3z}{x}\right)}\Leftrightarrow\left(\sqrt{\frac{x}{y}}+2\sqrt{\frac{y}{z}}+3\sqrt{\frac{z}{x}}\right)^2\le 6\left(\frac{x}{y}+\frac{2y}{z}+\frac{3z}{x}\right)$$

$$\Leftrightarrow 5\frac{x}{y}+8\frac{y}{z}+9\frac{z}{x}\geq 4\sqrt{\frac{x}{2}}+6\sqrt{\frac{z}{y}}+12\sqrt{\frac{y}{x}}$$ ***(1).Let*** $a=\sqrt{\frac{x}{y}},b=\sqrt{\frac{y}{z}},c=\sqrt{\frac{z}{x}}\Rightarrow abc=1$

(1) $\Leftrightarrow 5a^2+8b^2+9c^2\geq \frac{4}{c}+\frac{6}{b}+\frac{12}{a}$ ***or*** $5a^2+8b^2+9c^2\geq 4ab+6ac+12bc$

$$\Leftrightarrow 2(a-b)^2+3(a-c)^2+6(b-c)^2\geq 0$$

$$\text{“=”}\ a=b=c=1\Leftrightarrow x=y=z.$$

Solution 3 by Ravi Prakash-New Delhi-India

Put $x=a\sin\alpha, b=a\sin\beta, c=a\sin\gamma\ \left(-\frac{\pi}{2}\leq\alpha,\beta,\gamma\leq\frac{\pi}{2}\right)$. ***Now,***

$$\sqrt{7(a^2-x^2)}+\sqrt{7(a^2-y^2)}+\sqrt{7(a^2-z^2)}+a(xyz)^{\frac{1}{3}}$$

$$=\sqrt{7}a(\cos\alpha+\cos\beta+\cos\gamma)+3\left\{3(\sin\alpha\sin\beta+\sin\gamma)^{\frac{1}{3}}\right\}a$$

$$\leq 7a[\cos\alpha+\cos\beta+\cos\gamma]+3\left[3\{|\sin\alpha||\sin\beta||\sin\gamma|\}^{\frac{1}{3}}\right]a$$

$$[\because t\leq |t|]$$

$$\leq\sqrt{7}a[\cos\alpha+\cos\beta+\cos\gamma]+3a[|\sin\alpha|+|\sin\beta|+|\sin\gamma|]$$

$$[AM\geq GM]$$

$$=a\left[\sqrt{7}\cos\alpha+3|\sin\alpha|\right]+\sqrt{7}\cos\beta+3|\sin\beta|+\sqrt{7}\cos\gamma+3|\sin\gamma|$$

$$\leq a\left[\sqrt{7+a}+\sqrt{7+a}+\sqrt{7+a}\right]$$

$$=12a\quad\left[\max(a\sin\alpha+b\cos\alpha)=\sqrt{a^2+b^2}\right]$$

Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

Give $a=\frac{x}{y}; b=\frac{y}{z}; c=\frac{z}{x}$. ***Hence*** $\sqrt{a}+2\sqrt{b}+3\sqrt{c}=\sqrt{a}+\sqrt{b}+\sqrt{b}+\sqrt{c}+\sqrt{c}+\sqrt{c}$

$\leq\sqrt{6}(a+b+b+c+c+c)=\sqrt{6}(a+2b+3c)$. ***Therefore it is to be true.***

211. Prove that if $a,b,c>0$ then:

$$\sqrt{\frac{a}{b+c}}+2\sqrt{\frac{b}{c+a}}+4\sqrt{\frac{c}{a+b}}\leq\sqrt{7\left(\frac{a}{b+c}+\frac{2b}{c+a}+\frac{4c}{a+b}\right)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Pham Quoc Sang-Ho Chi Minh-Vietnam

Let $x=\sqrt{\frac{a}{b+c}},y=\sqrt{\frac{b}{c+a}},z=\sqrt{\frac{c}{a+b}}$***. Now, we prove that***

$$x+2y+4z\le\sqrt{7(x^2+2y^2+4z)^2}$$

$$\Leftrightarrow x^2+4y^2+16z^2+4xy+8xz+16yz\le 7(x^2+2y^2+4z^2)$$

$$\Leftrightarrow 6x^2+10y^2+12z^2\ge 4xy+8xz+16yz\Leftrightarrow 2(x-y)^2+4(x-z)^2+8(y-z)^2\ge 0$$

"=" $x=y=z$ ***or*** $a=b=c$.

Solution 2 by Ravi Prakash-New Delhi-India

We know, $x_1,x_2,y_1,y_2,z_1,z_2\in\mathbb{R}$ ***then***

$|x_1x_2+y_1y_2+z_1z_2|\le\sqrt{(x_1^2+y_1^2+z_1^2)(x_2^2+y_2^2+z_2^2)}$. ***Take*** $x_1=1,y_1=\sqrt{2},z_1=2$

$x_2=\sqrt{\frac{a}{b+c}},y_2=\sqrt{2}\sqrt{\frac{b}{c+a}},z_2=2\sqrt{\frac{c}{a+b}}$ ***to obtain desired inequality.***

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

Give $x=\frac{a}{b+c};y=\frac{b}{c+a};z=\frac{c}{a+b}$.

Hence $\sqrt{x}+2\sqrt{y}+4\sqrt{z}=\sqrt{x}+\sqrt{y}+\sqrt{y}+\sqrt{z}+\sqrt{z}+\sqrt{z}+\sqrt{z}$

$\le\sqrt{7}(x+y+y+z+z+z+z)=\sqrt{7}(x+2y+4z)$. ***Therefore it is to be true.***

212. If $0\le a\le b\le c$ then:

$$(a-b)c\sqrt{c}+(b-c)a\sqrt{a}+(c-a)b\sqrt{b}\le 0$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

Let $a=x^2,b=y^2,c=z^2,x,y,z\ge 0$. ***Also*** $0\le a\le b\le c\Rightarrow 0\le x\le y\le z$

$$(a-b)c\sqrt{c}+(b-c)a\sqrt{a}+(c-a)b\sqrt{b}=(x^2-y^2)z^3+(y^2-z^2)x^3+(z^2-x^2)y^3$$

$$=\begin{vmatrix}x^3 & y^3 & z^3\\ x^2 & y^2 & z^2\\ 1 & 1 & 1\end{vmatrix}=\begin{vmatrix}x^3-y^3 & y^3-z^3 & z^3\\ x^2-y^2 & y^2-z^2 & z^2\\ 0 & 0 & 1\end{vmatrix}\begin{bmatrix}use\ C_1\to C_1-C_2\\ C_2\to C_2-C_3\end{bmatrix}$$

$$=(x-y)(y-z)\begin{vmatrix}x^2+y^2+xy & y^2+z^2+yz\\ x+y & y+z\end{vmatrix}$$

$$= (x-y)(y-z)\begin{vmatrix} x^2-z^2+(x-z)y & y^2+z^2+yz \\ x-z & y+z \end{vmatrix}$$

$$= (x-y)(y-z)(x-z)\begin{vmatrix} x+y+z & y^2+z^2+yz \\ 1 & y+z \end{vmatrix}$$

$$= (x-y)(y-z)(x-z)\begin{vmatrix} x+y & y^2 \\ 1 & y+z \end{vmatrix} = (x-y)(y-z)(x-z)(xy+yz+zx) \le 0$$

since $x \le y \le z$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Lemma: Let $f:[a,b] \to \mathbb{R}$ ***be convex, then for any*** $x \in (a,b)$

$$\frac{f(x)-f(a)}{x-a} \le \frac{f(b)-f(a)}{b-a} \le \frac{f(b)-f(x)}{b-x}$$

$$(a-b)c\sqrt{c}+(b-c)a\sqrt{a}+(c-a)b\sqrt{b} \le 0, 0 \le a \le b \le c$$

$$\Leftrightarrow \frac{c\sqrt{c}-b\sqrt{b}}{c-b} \ge \frac{b\sqrt{b}-a\sqrt{a}}{b-a} \quad (1)$$

Let $f(x)=\sqrt{x^3}$ ***for all*** $x \in [a,c]$, $f''(x)=\frac{3}{4\sqrt{x}} > 0$ ***for all*** $x > 0$

f ***is convex over*** $[a,c]$, ***hence by the above lemma*** $\frac{c\sqrt{c}-b\sqrt{b}}{c-b} \ge \frac{c\sqrt{c}-a\sqrt{a}}{c-a} \ge \frac{b\sqrt{b}-a\sqrt{a}}{b-a}$

$\Rightarrow \frac{c\sqrt{c}-b\sqrt{b}}{c-b} \ge \frac{b\sqrt{b}-a\sqrt{a}}{b-a} \therefore (a-b)c\sqrt{c}+(b-c)a\sqrt{a}+(c-a)b\sqrt{b} \le 0$ ***(proved)***

213. If a,b,c be positive real number such that $a \le b \le c$ then

$$2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+3 \ge (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Proposed by Pham Quoc Sang-Ho Chi Minh-Vietnam

Solution by Ravi Prakash-New Delhi-India

$$2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+3 \ge (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$$\Leftrightarrow 2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+3 \ge 1+1+1+\frac{a}{b}+\frac{a}{c}+\frac{b}{a}+\frac{b}{c}+\frac{c}{a}+\frac{c}{b}$$

$$\Leftrightarrow \frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge \frac{b}{a}+\frac{c}{b}+\frac{a}{c} \Leftrightarrow \frac{a-c}{b}+\frac{b-a}{c}+\frac{c-b}{a} \ge 0$$

$$\Leftrightarrow ac(a-c)+ab(b-a)+bc(c-b)\geq 0$$

$$\Leftrightarrow \begin{vmatrix} bc & ac & ab \\ 1 & 1 & 1 \\ a & b & c \end{vmatrix} \geq 0 \Leftrightarrow (a-b)(b-c)(c-a)\geq 0 \text{ which is true as } a\leq b\leq c$$

214. If $a,b,c,d,e>0, 2b=a+c, 2c=b+d$ then:

$$a^2+b^2+c^2+d^2\geq 4\sqrt[8]{e}\left(a+d-\sqrt[8]{e}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

$2b=a+c, 2c=b+d \Rightarrow a,b,c,d$ are in A.P. with common difference $\frac{1}{3}(d-a)$

$$\therefore a^2+b^2+c^2+d^2=a^2+\left\{a+\frac{1}{3}(d-a)\right\}^2+\left\{a+\frac{2}{3}(d-a)\right\}^2+d^2$$

$$=3a^2+d^2+2(d-a)a+\frac{5}{9}(d-a)^2=(a+d)^2+\frac{5}{9}(d-a)^2$$

$$=\left\{(a+d)-2e^{\frac{1}{8}}\right\}^2+4e^{\frac{1}{8}}(a+d)-4e^{\frac{1}{4}}+\frac{5}{9}(d-a)^2\geq 4e^{\frac{1}{8}}\left[a+d-e^{\frac{1}{8}}\right]$$

215. Prove that if x,y and z are in $[-5,3]$ then

$$\sqrt{3x-5y-xy+15}+\sqrt{3y-5z-yz+15}+\sqrt{3z-5x-xz+15}\leq 12$$

When does equality occur?

Hungary NMO

Solution by Do Huu Duc Thinh-Ho Chi Minh-Vietnam

If $x,y,z\in[-5,3]$ then: $\sum\sqrt{3x-5y-xy+15}\leq 12$. We have:

$\sum\sqrt{3x-5y-xy+15}=\sum\sqrt{(3-y)(5+x)}$. Since $x,y,z\in[-5;3]$ then $3-x$; $3-y; 3-z; 5+x; 5+y; 5+z\geq 0$, so, by applying Cauchy's inequality:

$\sum\sqrt{(3-y)(5+x)}\leq\sum\left(\frac{3-y+5+x}{2}\right)=\frac{24}{2}=12\Rightarrow$ Q.E.D. The equality happens iff

$$\begin{cases} 3-y=5+x; 3-z=5+y; 3-x=5+z \\ x,y,z\in[-5;3] \end{cases}\Leftrightarrow x=y=z=-1$$

216. If $a,b,c,d>0, abcd=1$ then:

$$a\left(\frac{b}{b+a}+\frac{d}{d+a}\right)+c\left(\frac{b}{b+c}+\frac{d}{d+c}\right)\le\frac{1}{2}(ab+bc+cd+da)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Boris Colakovic-Belgrade-Serbia

$a,b,c,d>0; abcd=1$ **then** $a\left(\frac{b}{b+a}+\frac{d}{d+a}\right)+c\left(\frac{b}{b+c}+\frac{d}{d+c}\right)\le\frac{1}{2}(ab+bc+cd+da)$

$\frac{ab}{b+a}=\frac{1}{cd(b+a)}=\frac{1}{bcd+acd}=\frac{1}{\frac{1}{a}+\frac{1}{b}}\overset{HM-GM}{\le}\sqrt{ab}$. ***Similarly*** $\frac{ad}{d+a}\le\sqrt{ad}, \frac{bc}{b+c}\le\sqrt{bc}, \frac{cd}{d+c}\le\sqrt{cd}$

Adding above inequalities $\Rightarrow LHS\le\sqrt{ab}+\sqrt{bc}+\sqrt{cd}+\sqrt{da}\le\frac{1}{2}(ab+bc+cd+da)$

Prove that $\sqrt{cb}+\sqrt{bc}+\sqrt{cd}+\sqrt{da}\le\frac{1}{2}(ab+bc+cd+da)$. ***We will observe***

$f(x)=\frac{1}{2}x-\sqrt{x}$. ***How is*** $f(x)\ge-\frac{1}{2}\ \forall x>0\Rightarrow\frac{1}{2}ab-\sqrt{ab}\ge-\frac{1}{2}$. ***Similarly***

$\frac{1}{2}bc-\sqrt{bc}\ge-\frac{1}{2}, \frac{1}{2}cd-\sqrt{cd}\ge-\frac{1}{2}, \frac{1}{2}da-\sqrt{da}\ge-\frac{1}{2}$. ***Now is***

$$\frac{1}{2}\left(\sqrt{ab}-1\right)^2+\frac{1}{2}\left(\sqrt{bc}-1\right)^2+\frac{1}{2}\left(\sqrt{cd}-1\right)^2+\frac{1}{2}\left(\sqrt{da}-1\right)^2\ge 0$$

Sign "=" is valid for $a=b=c=d=1$.

Solution 2 by Le Minh Cuong-Ho Chi Minh-Vietnam

We have $LHS=\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{cd}{c+d}+\frac{da}{d+a}\le\frac{ab}{2\sqrt{ab}}+\frac{bc}{2\sqrt{bc}}+\frac{cd}{2\sqrt{cd}}+\frac{da}{2\sqrt{da}}\le$

$\le\frac{1}{2}\left(\sqrt{ab}+\sqrt{bc}+\sqrt{cd}+\sqrt{da}\right)$. ***It need show that:*** $\sqrt{ab}+\sqrt{bc}+\sqrt{cd}+\sqrt{da}\le$

$\le ab+bc+cd+da$. ***Indeed,*** $4(ab+bc+cd+da)\overset{BCS}{\ge}\left(\sqrt{ab}+\sqrt{bc}+\sqrt{cd}+\sqrt{da}\right)^2$

$$\overset{AM-GM}{\ge}4\sqrt[4]{\sqrt{ab}\cdot\sqrt{bc}\cdot\sqrt{dc}\cdot\sqrt{da}}\left(\sqrt{ab}+\sqrt{bc}+\sqrt{cd}+\sqrt{da}\right)\ge$$

$\ge 4\left(\sqrt{ab}+\sqrt{bc}+\sqrt{cd}+\sqrt{da}\right)$. ***The equality holds for*** $a=b=c=d=1$.

Solution 3 by Serban George Florin-Romania

$f(x)=\frac{x}{x+a}, f:(0,\infty)\to\mathbb{R}, f'(x)=\frac{a}{(x+a)^2}, f''(x)=\frac{-2a}{(x+a)^3}<0\Rightarrow f$ ***concave on*** $(0,\infty)\Rightarrow$

$$\Rightarrow f\left(\frac{b+d}{2}\right)\ge\frac{f(b)+f(d)}{2}, f(b)+f(d)\le 2f\left(\frac{b+d}{2}\right)\Rightarrow\frac{b}{b+a}+\frac{d}{d+a}\le$$

$$\le 2\cdot\frac{\frac{b+d}{2}}{\frac{b+d}{2}+a}=\frac{2(b+d)}{b+d+2a}\Big|\cdot a;\ a\left(\frac{b}{b+a}+\frac{d}{d+a}\right)\le\frac{2a(b+d)}{2a+b+d}.\ \textit{Analogous}\ g:(0,\infty)\to\mathbb{R}$$

$$g(x)=\frac{x}{x+c},\ g''(x)=\frac{-2c}{(x+c)^3}<0\Rightarrow g\ \textit{concave on}\ (0,\infty)\Rightarrow g(b)+g(d)\le 2g\left(\frac{b+d}{2}\right);$$

$$\frac{b}{b+c}+\frac{d}{d+c}\le 2\frac{\frac{b+d}{2}}{\frac{b+d}{2}+c}\Rightarrow\frac{b}{b+c}+\frac{d}{d+c}\le\frac{2(b+d)}{b+d+2c}\Big|\cdot c\Rightarrow c\left(\frac{b}{b+c}+\frac{d}{d+c}\right)\le$$

$$\le\frac{2c(b+d)}{b+d+2c}\le\frac{ab+ad+bc+cd}{2}=\frac{(b+d)(a+c)}{2}\Big|:(b+d)\Rightarrow\frac{2a}{2a+b+d}+$$

$$+\frac{2c}{2c+b+d}\le\frac{a+c}{2}.\ \textit{Let be}\ h(x)=\frac{x}{x+b+d},\ h:(0,\infty)\to\mathbb{R},\ h''(x)=\frac{-2(b+d)}{(x+b+d)}<0\Rightarrow\frac{2a}{2a+b+d}+$$

$$+\frac{2c}{2c+b+d}\le\frac{2(a+c)}{a+c+b+d}\le\frac{2\cdot(a+c)}{4\sqrt[4]{abcd}}=\frac{2(a+c)}{4\cdot 1}=\frac{a+c}{2}$$

217. In ΔABC, $a\ne b$ the following relationship holds:

$$\frac{\left(2b+2c-3\sqrt[3]{abc}\right)\left(1+\left(\sqrt{a}-\sqrt{b}\right)^2\right)}{\left(\sqrt{a}-\sqrt{b}\right)^2\left(1+a+b+c-3\sqrt[3]{abc}\right)}>1$$

Proposed by Daniel Sitaru – Romania

Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$c+\sqrt{ab}+\sqrt{ab}\ge 3\sqrt[3]{abc};\ c-3\sqrt[3]{abc}\ge -2\sqrt{ab}\Leftrightarrow a+b+c-3\sqrt[3]{abc}\ge$$

$$\ge a+b-2\sqrt{ab}=\left(\sqrt{a}-\sqrt{b}\right)^2\Leftrightarrow\frac{1}{\left(\sqrt{a}-\sqrt{b}\right)^2}+1\ge\frac{1}{a+b+c-3\sqrt[3]{abc}}+1\Leftrightarrow$$

$$\Leftrightarrow\frac{1+\left(\sqrt{a}-\sqrt{b}\right)^2}{\left(\sqrt{a}-\sqrt{b}\right)^2}\ge\frac{1+a+b+c-3\sqrt[3]{abc}}{a+b+c-3\sqrt[3]{abc}}\overset{a<b+c}{>}\frac{1+a+b+c-3\sqrt[3]{abc}}{2b+2c-3\sqrt[3]{abc}}\Leftrightarrow$$

$$\Leftrightarrow\frac{\left(2b+2c-3\sqrt[3]{abc}\right)\left(1+\left(\sqrt{a}-\sqrt{b}\right)^2\right)}{\left(\sqrt{a}-\sqrt{b}\right)^2\left(1+a+b+c-3\sqrt[3]{abc}\right)}>1$$

218. Let a,b,c be positive real numbers such that:

$$\begin{cases}ab>6\\ \frac{a}{8}+3b+\frac{2c}{3}=\frac{abc}{9}+\frac{67}{4a}\end{cases}$$

Find the minimum value of the expression:

$$P = 3a + 2b + c$$

Proposed by Do Quoc Chinh-Vietnam

Solution by proposer

From the hypothesis we have: $c\left(\frac{ab}{9}-\frac{2}{3}\right)=\frac{a}{8}+3b-\frac{67}{4a}\Leftrightarrow c=\frac{9(a^2+24ab-134)}{8a(ab-6)}$

Therefore, we have: $P=3a+2b+c=3a+2b+\frac{9(a^2+24ab-134)}{8a(ab-6)}$

Applying the AM-GM inequality, we have: $2b+\frac{9(a^2+24ab-134)}{8a(ab-6)}=2b+\frac{9[a^2+10+24(ab-6)]}{8a(ab-6)}$

$$=\frac{2(ab-6)}{a}+\frac{9(a^2+10)}{8a(ab-6)}+\frac{39}{a}\geq\frac{2}{a}\cdot\sqrt{2(ab-6)\cdot\frac{9(a^2+10)}{8(ab-6)}}+\frac{36}{a}$$

$$=\frac{3\left(13+\sqrt{a^2+10}\right)}{a}\Rightarrow P\geq 3\left(a+\frac{13+\sqrt{a^2+10}}{a}\right)$$

Applying the Cauchy – Schwarz and AM-GM inequality, we have:

$$P\geq 3\left(a+\frac{13+\sqrt{a^2+10}}{a}\right)=3\left(a+\frac{13}{a}+\frac{\sqrt{(15+10)(a^2+10)}}{5a}\right)$$

$$\geq 3\left(a+\frac{13}{a}+\frac{a\sqrt{15}+10}{5a}\right)=3\left(a+\frac{15}{a}+\frac{\sqrt{15}}{5}\right)$$

$$\geq 3\left(2\sqrt{a\cdot\frac{15}{a}}+\frac{\sqrt{15}}{5}\right)=\frac{33\sqrt{15}}{5}\Rightarrow P\geq\frac{33\sqrt{15}}{5}$$

Therefore, $P_{\min}=\frac{33\sqrt{15}}{5}$*. The equally holds for* $a=\sqrt{15}, b=\frac{13\sqrt{5}}{20}, c=\frac{23\sqrt{15}}{10}$

219. If $a,b,c\geq 1$ then:

$$\frac{(1+a+a^2)(1+b+b^2+b^3)(1+c+c^2+c^3+c^4)}{(1+a^2)(1+b^3)(1+c^4)}\leq\frac{15}{2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\text{Let } f(x)=\frac{1+x+x^2}{1+x^2}, x\geq 1;\ f'(x)=\frac{d}{dx}\left[1+\frac{x}{1+x^2}\right]=\frac{(1+x^2)-2x^2}{(1+x^2)^2}=\frac{1-x^2}{(1+x^2)^2}<0, \forall x>1$$

$$\Rightarrow f(x) \text{ decreases on } [1,\infty) \therefore f(x)\leq f(1)\ \forall x\geq 1$$

$$\Rightarrow \frac{1+a+a^2}{1+a^2}\leq\frac{3}{2}\ \forall a\geq 1 \quad (1)$$

$$\text{Let } g(x)=\frac{1+x+x^2+x^3}{1+x^3}=1+\frac{x+x^2}{1+x^3}$$

$$g'(x)=\frac{(1+x^3)(1+2x)-3x^2(x+x^2)}{(1+x^3)^2}=\frac{1+2x+x^3+2x^4-3x^3-3x^4}{(1+x^3)^2}=\frac{1+2x-2x^3-x^4}{(1+x^3)^2}.$$

$$g'(x)=\frac{(1-x^4)-2x(1-x^2)}{(1+x^3)^2}=\frac{(1-x^2)(1+x^2-2x)}{(1+x^3)^2}=\frac{(1-x)^3(1+x)}{(1+x^3)^2}<0\ \forall x>1$$

$$\Rightarrow g(x) \text{ decreases on } [1,\infty) \therefore g(x)\leq g(1)$$

$$\Rightarrow \frac{1+b+b^2+b^3}{1+b^3}\leq\frac{4}{2}=2\ \forall b\geq 1 \quad (2)$$

$$\text{Let } h(x)=\frac{1+x+x^2+x^3+x^4}{1+x^4}, x\geq 1$$

$$=1+\frac{x+x^2+x^3}{1+x^4}$$

$$h'(x)=\frac{(1+2x+3x^2)(1+x^4)-(x+x^2+x^3)(4x^3)}{(1+x^4)^2}$$

$$=\frac{1+2x+3x^2+x^4+2x^5+3x^6-4x^4-4x^5-4x^6}{(1+x^4)^2}$$

$$=\frac{1+2x+3x^2-3x^4-2x^5-x^6}{(1+x^4)^2}=\frac{(1-x^6)+2x(1-x^3)+3x^2(1-x^2)}{(1+x^4)^2}<0\ \forall x\geq 1$$

$$\Rightarrow h(x) \text{ decreases on } [1,\infty) \therefore h(x)\leq h(1)\ \ \forall x\geq 1$$

$$\Rightarrow \frac{1+c+c^2+c^3+c^4}{1+c^4}\leq\frac{5}{2}\ \ \forall c\geq 1 \quad (3)$$

Multiplyig (1), (2), (3) we get: $\frac{(1+a+a^2)(1+b+b^2+b^3)(1+c+c^2+c^4)}{(1+a^2)(1+b^3)(1+c^4)}\leq\frac{15}{2}$

Solution 2 by Rajsekhar Azaad-India

$$\frac{1+a+a^2}{1+a^2}-\frac{3}{2}=\frac{-(a-1)^2}{2(1+a^2)}\leq 0\Rightarrow\frac{1+a+a^2}{1+a^2}\leq\frac{3}{2} \quad (i)$$

Again, $\frac{1+b+b^2+b^3}{1+b^3}-2=\frac{1+b+b^2+b^3-2-2b^3}{1+b^3}=\frac{-(b^3+1)+b(b+1)}{1+b^3}=\frac{-(b+1)(b-1)^2}{1+b^3}\le 0$

$$\Rightarrow \frac{1+b+b^2+b^3}{1+b^3}\le 2 \quad (ii)$$

Again, $\frac{1+c+c^2+c^3+c^4}{1+c^4}-\frac{5}{2}=\frac{-3(c^4+1)+2c(c^2+1)+2c^2}{2(1+c^4)}=-\frac{(c-1)^2(3c^2+4c+3)}{2(1+c^4)}\le 0$

$$\Rightarrow \frac{1+c+c^2+c^3+c^4}{1+c^4}\le\frac{5}{2} \quad (iii)$$

on multiplying (i), (ii) and (iii): $\frac{(1+a+a^2)(1+b+b^2+b^3)(1+c+c^2+c^3+c^4)}{(1+a^2)(1+b^3)(1+c^4)}\le\frac{3}{2}\cdot 2\cdot\frac{5}{2}=\frac{15}{2}$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\frac{1+c+c^2+c^3+c^4}{1+c^4}=1+\frac{c+c^2+c^3}{1+c^4}\le 1+\frac{2c(1+c^2)+2c^2}{(1+c^2)^2}$$

$$\left(\because 1+c^4\overset{Chebyshev}{\ge}\frac{1}{2}(1+c^2)^2\right)$$

$$=1+\frac{2c}{1+c^2}+\frac{2c^2}{(1+c^2)^2}\overset{(1)}{\le}1+\frac{2c}{2c}+\frac{2c^2}{4c^2}\left(\because 1+c^2\overset{A-G}{\ge}2c\right)=1+1+\frac{1}{2}=\frac{5}{2}$$

Again, $\frac{1+b+b^2+b^3}{1+b^3}=\frac{(1+b)(1+b^2)}{(b+1)(b^2-b+1)}=\frac{(b^2+1-b)+b}{(b^2+1-b)}=1+\frac{b}{b^2+1-b}\overset{(2)}{\le}1+\frac{b}{b}=2$

$$\left(\because b^2+1-b\overset{A-G}{\ge}b\right)$$

Lastly, $\frac{1+a+a^2}{1+a^2}=1+\frac{a}{1+a^2}\overset{(3)}{\le}1+\frac{a}{2a}=\frac{3}{2}\left(\because 1+a^2\overset{A-G}{\ge}2a\right)$

(1) $\times$ ***(2)*** $\times$ ***(3)*** $\Rightarrow LHS\le\frac{5}{2}\cdot 2\cdot\frac{3}{2}=\frac{15}{2}$ ***(proved)***

Solution 4 by Soumitra Mandal-Chandar Nagore-India

$$\frac{(1+a+a^2)(1+b+b^2+b^3)(1+c+c^2+c^3+c^4)}{(1+a^2)(1+b^3)(1+c^4)}=$$

$$=\left(1+\frac{a}{1+a^2}\right)\left(1+\frac{b(1+b)}{1+b^3}\right)\left(1+\frac{c(1+c+c^2)}{1+c^4}\right)$$

$$\overset{AM\ge GM}{\le}\left(1+\frac{1}{2}\right)\left(1+\frac{b}{1-b+b^2}\right)\left(1+\frac{3}{2}\cdot\frac{c(1+c^2)}{1+c^4}\right)\left[\because \frac{3(1+x^2)}{2}\ge x^2+x+1\right]$$

$$\overset{AM\ge GM}{\le}\frac{3}{2}\cdot 2\cdot\left(1+\frac{3}{2}\cdot\frac{2c}{1+c^2}\right)\left[\because 2(1+x^4)\ge(1+x^2)^2\right]\overset{AM\ge GM}{\le}\frac{3}{2}\cdot 2\cdot\frac{5}{2}=\frac{15}{2}\ \textit{(proved)}$$

220. If $0 < a \le b$ and $c, d, e \ge 0$ then:

$$a^3 \le \frac{(a + c\sqrt{ab} + b)(a + d\sqrt{ab} + b)(a + e\sqrt{ab} + b)}{(c+2)(d+2)(e+2)} \le b^3$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

For $0 < a \le b, a(c+2) \le a + \sqrt{ab} + b \le b(c+2)$

$\Leftrightarrow c\sqrt{ab} + b - ac - a \ge 0$ and $bc + b - c\sqrt{ab} - a \ge 0$

$\Leftrightarrow (c\sqrt{a})(\sqrt{b} - \sqrt{a}) + (b - a) \ge 0$ and $c\sqrt{b}(\sqrt{b} - \sqrt{a}) + (b - a) \ge 0$

$\Leftrightarrow (c\sqrt{a} + \sqrt{b} + \sqrt{a})(\sqrt{b} - \sqrt{a}) \ge 0$ and $(c\sqrt{b} + \sqrt{b} + \sqrt{a})(\sqrt{b} - \sqrt{a}) \ge 0$

which is true in view of $b \ge a$. Thus $a \le \frac{a + c\sqrt{ab} + b}{c+2} \le b$. Similarly for d and e.

Multiplying three inequalities, we get: $a^3 \le \frac{(a+c\sqrt{ab}+b)(a+d\sqrt{ab}+b)(a+e\sqrt{ab}+b)}{(c+2)(d+2)(e+2)} \le b^3$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$1)\ a + c\sqrt{ab} + b \overset{Mg \le Ma}{\le} (a+b)\left(\frac{c}{2} + 1\right) = \frac{a+b}{2}(c+2)$$

$$2)\ a + c\sqrt{ab} + b \overset{Ma \ge Mg}{\ge} \sqrt{ab}(c+2).\ \text{Similarly}$$

$$\frac{\prod \sqrt{ab}(c+2)}{\prod (c+2)} \le \frac{\prod (a + c\sqrt{ab} + b)}{\prod (c+2)} \le \frac{\prod \frac{a+b}{2}(c+2)}{\prod (c+2)}$$

$$a^3 \overset{a \le b}{\le} ab\sqrt{ab} \le \frac{\prod (a + c\sqrt{ab} + b)}{\prod (c+2)} \le \left(\frac{a+b}{2}\right)^3 \overset{a \le b}{\le} b^3$$

221. If $0 < a \le b \le c$ then:

$$3a^2b \le \prod \sqrt[3]{a^3 + ab\sqrt{ab} + b^3} \le 3bc^2$$

Proposed by Daniel Sitaru – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

We know for $x, y \ge 0$ then $x^2 + xy + y^2 \ge 3xy$ and $\frac{3}{2}(x^2 + y^2) \ge x^2 + xy + y^2$

$$\prod_{cyc}\sqrt[3]{(a^3+ab\sqrt{ab}+b^3)}$$

$$\Rightarrow\sqrt[3]{\prod_{cyc}\left(3a^{\frac{3}{2}}b^{\frac{3}{2}}\right)}\le\prod_{cyc}\sqrt[3]{a^3+ab\sqrt{ab}+b^3}\le\sqrt[3]{\frac{27}{8}\prod_{cyc}(a^3+b^3)}$$

$$\Rightarrow 3abc\le\prod_{cyc}\sqrt[3]{a^3+ab\sqrt{ab}+b^3}\le\frac{3}{2}\sqrt[3]{\prod_{cyc}(a^3+b^3)}$$

$$\Rightarrow 3ba^2\le\prod_{cyc}\sqrt[3]{a^3+ab\sqrt{ab}+b^3}\le\frac{3}{2}\sqrt[3]{(2b^3)(2c^2)(2c^3)}[\because a\le b\le c]$$

$$\therefore 3a^2b\le\prod_{cyc}\sqrt[3]{a^3+ab\sqrt{ab}+b^3}\le 3bc^2$$

222. If $a,b,c\ge 0$ then:

$$2a^2+6ab+7b^2\ge 2\sqrt[8]{c}\left(5\sqrt[5]{a^2b^3}-\sqrt[8]{c}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Catinca Alexandru-Romania

$$2a^2+6ab+7b^2\ge 2\sqrt[8]{c}\left(5\sqrt[5]{a^2b^3}-\sqrt[8]{c}\right)\to GM\le AM$$

$$\Leftrightarrow 2a^2+6ab+7b^2+2\sqrt[4]{c}\ge 2\sqrt[8]{c}(2a+3b)$$

(since $5\sqrt[5]{a^2b^3}\le 5\cdot\frac{a+a+b+b+b}{5}=2a+3b$***)***

$$\Leftrightarrow 2a^2-4a\sqrt[8]{c}+2\sqrt[4]{c}+6ab+7b^2-6b\sqrt[8]{c}\ge 0\Leftrightarrow 2\left(a-\sqrt[8]{c}\right)^2+7b^2+6b\left(a-\sqrt[8]{c}\right)\ge 0$$

$$\Leftrightarrow 7b^2+6b\left(a-\sqrt[8]{c}\right)+2\left(a-\sqrt[8]{c}\right)^2\ge 0$$

$$\Delta_b=36\left(a-\sqrt[8]{c}\right)^2-4\cdot 7\cdot 2\left(a-\sqrt[8]{c}\right)^2=-\left(a-\sqrt[8]{c}\right)^2\cdot 20\le 0$$

$$\Rightarrow 7b^2+6b\left(a-\sqrt[8]{c}\right)+2\left(a-\sqrt[8]{c}\right)^2\ge 0$$

Solution 2 by Rozeta Atanasova-Macedonia

$$a,b,c\ge 0\Rightarrow 2a^2+4ab+7b^2\ge 2\sqrt[8]{c}\left(5\sqrt[5]{a^2b^3}-\sqrt[8]{c}\right)\quad \textit{(1)}$$

Let $x = \sqrt[8]{c} \Rightarrow 2a^2 + 4ab + 7b^2 - 10\sqrt[5]{a^2b^3}x + 2x^2 \ge 0 \Rightarrow$

$$D = 100\sqrt[5]{a^4b^6} - 8(2a^2 + 4ab + 7b^2) \le 0 \Rightarrow 4a^2 + 8ab + 14b^2 - 25\sqrt[5]{a^4b^4} \ge 0$$

But $4a^2 + 8ab + 14b^2 = 4a^2 + 4ab + 4ab + 7b^2 + 7b^2$

$$\overset{AM-GM}{\ge} 5\sqrt[5]{4^3 \cdot 7^2 \cdot a^4 \cdot b^6} = 10\sqrt[5]{98a^4b^6} > 10\sqrt[5]{2,5^5a^4b^6} = 25\sqrt[5]{a^5b^6} \Rightarrow \textit{is true}$$

Solution 3 by Marian Ursarescu-Romania

For $a = b = c = 0; a \ge 0$ *(true)*

$$\left.\begin{array}{c} a, b, c > 0; 2a^2 + 6ab + 7b^2 \ge 2\sqrt[8]{c}\left(5\sqrt[5]{a^2b^3} - \sqrt[8]{c}\right) \\ \textit{But } 5\sqrt[5]{a^2b^3} \le 2a + 3b \end{array}\right\} \Rightarrow$$

$$2\sqrt[8]{c}\left((2a + 3b) - \sqrt[8]{c}\right) \le 2a^2 + 6ab + 7b^2 \Leftrightarrow$$

$$-2\sqrt[8]{c^2} + 2(2a + 3b)\sqrt[8]{c} \le 2a^2 + 6ab + b^2 \quad (1)$$

$$\sqrt[8]{c} = x, x > 0 \Rightarrow -2x^2 + 2(2a + 3b)x = f(x)$$

$$\max f(x) = \frac{-\Delta}{4a} \Leftrightarrow \frac{-4(2a + 3b)^2}{-8} = \frac{(2a + 3b)^2}{2} \Rightarrow$$

$$f(x) \le \frac{(2a+3b)^2}{2} \quad (2)$$

From (1)+(2)⇒ we must show: $\frac{(2a+3b)^2}{2} \le 2a^2 + 6ab + 7b^2 \Leftrightarrow$

$$4a^2 + 12ab + 9b^2 \le 4a^2 + 12ab + 14b^2 \Leftrightarrow 9b^2 \le 14b^2 \Leftrightarrow 5b^2 \ge 0 \textit{ true.}$$

223. If $0 \le x, y, z \le a$ then:

$$\sqrt{x^2 - xz + z^2} + \sqrt{y^2 + z^2} + \sqrt{x^2 + xy + y^2} \le a\left(1 + \sqrt{2} + \sqrt{3}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Abdul Aziz-Semarang-Indonesia

$$\sqrt{x^2 - xz + z^2} = \sqrt{x^2 + z(z - x)} \le \sqrt{x^2} \le x \le a \Rightarrow \sqrt{x^2 - xz + z^2} \le a \quad (1)$$

$$\frac{(y^2)^{\frac{1}{2}} + (z^2)^{\frac{1}{2}}}{2} \ge \left(\frac{y^2 + z^2}{2}\right)^{\frac{1}{2}}$$

$$\frac{y + z}{2} \ge \frac{\sqrt{y^2 + z^2}}{\sqrt{2}} \Rightarrow \sqrt{y^2 + z^2} \le \left(\frac{y + z}{\sqrt{2}}\right) \Rightarrow \sqrt{y^2 + z^2} \le \frac{a + a}{\sqrt{2}} \le \frac{2a}{\sqrt{2}} \le \sqrt{2}a$$

$$\Rightarrow \sqrt{y^2+z^2} \le \sqrt{2}a \quad \textit{(2)}$$

$$\frac{(x^2)^{\frac{1}{2}}+(xy)^{\frac{1}{2}}+(y^2)^{\frac{1}{2}}}{3} \ge \left(\frac{x^2+xy+y^2}{3}\right)^{\frac{1}{2}} \Rightarrow \left(\frac{x+\sqrt{xy}+y}{3}\right) \ge \frac{\sqrt{x^2+xy+y^2}}{\sqrt{3}}$$

$$\Rightarrow \sqrt{x^2+xy+y^2} \le \left(\frac{x+\sqrt{xy}+y}{\sqrt{3}}\right) \le \frac{a+\sqrt{a^2}+a}{\sqrt{3}} \le \frac{3a}{\sqrt{3}} \Rightarrow \sqrt{x^2+xy+y^2} \le \sqrt{3}a \quad \textit{(3)}$$

Adding (1), (2), (3) we have:

$$\sqrt{x^2-xz+z^2}+\sqrt{y^2+z^2}+\sqrt{x^2+xy+y^2} \le a\left(1+\sqrt{2}+\sqrt{3}\right) \textit{ (proved)}$$

Solution 2 by Ravi Prakash-New Delhi-India

WLOG $x = \max\{x, z\}$

$$\sqrt{x^2-xz+z^2}+\sqrt{y^2+z^2}+\sqrt{x^2+xy+y^2} = \sqrt{x^2+z(z-x)}+\sqrt{y^2+z^2}+\sqrt{x^2+xy+y^2}$$

$$\le \sqrt{x^2}+\sqrt{y^2+z^2}+\sqrt{x^2+xy+y^2} \le \sqrt{a^2}+\sqrt{a^2+a^2}+\sqrt{a^2+a^2+a^2} = a\left(1+\sqrt{2}+\sqrt{3}\right)$$

Equality holds when $x = y = z = a$.

Solution 3 by Amit Dutta-Jamshedpur-India

Assume $x \le z$, ***then*** $x^2-xz+z^2 = x(x-z)+z^2 \le z^2 \Rightarrow \sqrt{x^2-xz+z^2} \le z \le a$

$$\sqrt{y^2+z^2} \le \sqrt{a^2+a^2} = a\sqrt{2} \text{ and } \sqrt{x^2+xy+y^2} \le \sqrt{a^2+a^2+a^2} = a\sqrt{3}$$

$$\therefore \sqrt{x^2-xz+z^2}+\sqrt{y^2+z^2}+\sqrt{x^2+xy+y^2} \le a\left(1+\sqrt{2}+\sqrt{3}\right)$$

224. If $x, y, z, t \ge 1$ then:

$$\frac{xy+2yz+2zt+2xz+ty+tx+9}{2x+2y+3z+2t} \ge 2$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$\forall x, y, z, t \ge 1, xy+2yz+2zx+ty+tx+9 \ge 4x+4y+6z+4t$. ***Let*** $x = a+1$, $y = b+1, z = c+1, t = d+1$ $(a, b, c, d \ge 0)$. ***Then, given inequality becomes:***

$$(a+1)(b+1)+2(b+1)(c+1)+2(c+1)(d+1)+2(c+1)(a+1)+$$
$$+(d+1)(b+1)+(d+1)(a+1)+9-4(a+1)-4(b+1)-6(c+1)-4(d+1) \ge 0$$

$$\Leftrightarrow ab+2ac+ad+2bc+bd+2cd \ge 0 \to \textit{true} \because a, b, c, d \ge 0 \textit{ (proved)}$$

225. If $x, y, z, t > 0$ then:

$$\frac{(x^6+y^6)^2(x^4+y^4+z^4)^3(x^3+y^3+z^3+t^3)^4}{(x^{12}+y^{12})(x^{12}+y^{12}+z^{12})(x^{12}+y^{12}+z^{12}+t^{12})} \le 1152$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Amit Dutta-Jamshedpur-India

AM of m^{th} power $\ge$ m^{th} power of AM $\forall m \in R \setminus (0,1) \Rightarrow \frac{x_1^m+x_2^m+\dots+x_n^m}{n} \ge \left(\frac{x_1+x_2+\dots+x_n}{n}\right)^m$.

Using this, we have: $\frac{(x^6)^2+(y^6)^2}{2} \ge \left(\frac{x^6+y^6}{2}\right)^2 \Rightarrow \frac{(x^6+y^6)^2}{x^{12}+y^{12}} \le \frac{4}{2} \le 2$ ***(1)***

Similarly, $\frac{(x^4)^3+(y^4)^3+(z^4)^3}{3} \ge \left(\frac{x^4+y^4+z^4}{3}\right)^3 \Rightarrow \frac{(x^4+y^4+z^4)^3}{x^{12}+y^{12}+z^{12}} \le \frac{3^3}{3} \le 9$ ***(2)***

And, $\frac{(x^3)^4+(y^3)^4+(z^3)^4+(t^3)^4}{4} \ge \left(\frac{x^3+y^3+z^3+t^3}{4}\right)^4 \Rightarrow \frac{(x^3+y^3+z^3+t^3)^4}{x^{12}+y^{12}+z^{12}+t^{12}} \le \frac{4^4}{4} \le 64$ ***(3)***

Multiplying (1), (2), (3), we get:

$$\frac{(x^6+y^6)^2(x^4+y^4+z^4)^3(x^3+y^3+z^3+t^3)^4}{(x^{12}+y^{12})(x^{12}+y^{12}+z^{12})(x^{12}+y^{12}+z^{12}+t^{12})} \le (2)(9)(64) \le 1152$$

Solution 2 by Marian Ursarescu-Romania

From Hölder's inequality, we have: $a_1^p + a_2^p + \dots + a_n^p \ge \frac{(a_1+a_2+\dots+a_n)^p}{n^{p-1}}, p \in \mathbb{N}^*$

$$x^{12}+y^{12} = (x^6)^2+(y^6)^2 \ge \frac{(x^6+y^6)^2}{2} \Rightarrow \frac{(x^6+y^6)^2}{x^{12}+y^{12}} \le 2 \quad \textbf{(1)}$$

$$x^{12}+y^{12}+z^{12} = (x^4)^3+(y^4)^3+(z^4)^3 \ge \frac{(x^4+y^4+z^4)^3}{9} \Rightarrow$$

$$\frac{x^4+y^4+z^4}{x^{12}+y^{12}+z^{12}} \le 9 \quad \textbf{(2)}$$

$$x^{12}+y^{12}+z^{12}+t^{12} = (x^3)^4+(y^3)^4+(z^3)^4+(t^3)^4 \ge \frac{(x^3+y^3+z^3+t^3)^4}{64}$$

$$\Rightarrow \frac{(x^3+y^3+z^3+t^3)^4}{x^{12}+y^{12}+z^{12}+t^{12}} \le 64 \quad \textbf{(3)}$$

From (1)+(2)+(3)$\Rightarrow \frac{(x^6+y^6)^2(x^4+y^4+z^4)^3(x^3+y^3+z^3+t^3)^4}{(x^{12}+y^{12})(x^{12}+y^{12}+z^{12})(x^{12}+y^{12}+z^{12}+t^{12})} \le 1152$

226. If $a,b,c,d>1$ and $abcd=e^4$. Prove that:

$$\frac{\ln d}{\log_d(ab^2c^3)}+\frac{\ln c}{\log_c(da^2b^3)}+\frac{\ln b}{\log_b(cd^2a^3)}+\frac{\ln a}{\log_a(bc^2d^3)}\geq\frac{2}{3}$$

Proposed by Lazaros Zachariadis-Thessaloniki-Greece

Solution by Daniel Sitaru – Romania

$$abcd=e^4\to\sum\ln a=4\quad(1)$$

$$\sum\frac{\ln d}{\log_d(ab^2c^3)}=\sum\frac{\ln d}{\frac{\ln a+2\ln b+3\ln c}{\ln d}}=\sum\frac{\ln^2 d}{\ln a+2\ln b+3\ln c}\geq$$

$$\overset{BERSGTROM}{\geq}\frac{(\sum\ln a)^2}{6\sum\ln a}=\frac{\sum\ln a}{6}\overset{(1)}{=}\frac{4}{6}=\frac{2}{3}$$

227. Let $x,y,z\in(0,+\infty)$. Prove that:

$$x^2+y^2+z^2+xy+yz+zx\geq2\sqrt{3(x+y+z)xyz}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Daniel Sitaru – Romania

$$a=y+z,b=z+x,c=x+y,s=x+y+z,S=\sqrt{xyz(x+y+z)}$$

$$\sum a^2\overset{IONESCU-WEITZENBOCK}{\geq}4\sqrt{3}S\leftrightarrow s^2-r^2-4Rr\geq2\sqrt{3}S\leftrightarrow$$

$$-2s^2+4s^2-\sum bc\geq2\sqrt{3}S\leftrightarrow s^2-3s^2+\sum s(b+c)-\sum bc\geq2\sqrt{3}S\leftrightarrow$$

$$s^2-\sum(s-b)(s-c)\geq2\sqrt{3}S\leftrightarrow\left(\sum x\right)^2-\sum xy\geq2\sqrt{3xyz(x+y+z)}\leftrightarrow$$

$$x^2+y^2+z^2+xy+yz+zx\geq2\sqrt{3xyz(x+y+z)}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$(1)\Leftrightarrow\sum x^2+\sum xy\geq2\sum xy\overset{?}{\geq}2\sqrt{3xyz\left(\sum x\right)}\Leftrightarrow\left(\sum xy\right)^2\geq3xyz\left(\sum x\right)\Leftrightarrow$$

$$\Leftrightarrow\sum x^2y^2\geq xyz(\sum x)\to\textit{true}\because\sum a^2\geq\sum ab\ \forall\ a,b\quad\textit{(Proved)}$$

228. If $a,b,c>0,a+b+c=3$ then:

$$\frac{\sqrt{2}}{2}(\sqrt{a}+\sqrt{b}+\sqrt{c}+3)\geq\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution by Daniel Sitaru – Romania

$$f(x)=x^{\frac{1}{2}}, f''(x)=\frac{1}{4}x^{-\frac{3}{4}}>0, f:(0,\infty)\to\mathbb{R}, f-\textit{convexe}$$

$$\frac{1}{3}\sum f(a)+f\left(\frac{a+b+c}{3}\right)\geq\frac{2}{3}\sum f\left(\frac{a+b}{2}\right)\to\frac{1}{3}\sum\sqrt{a}+f\left(\frac{3}{3}\right)\geq\frac{2}{3}\sum\sqrt{\frac{a+b}{2}}\to$$

$$\sum\sqrt{a}+3f(1)\geq\frac{2}{\sqrt{2}}\sum\sqrt{a+b}\to\frac{\sqrt{2}}{2}(\sqrt{a}+\sqrt{b}+\sqrt{c}+3)\geq\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}$$

229. Let $x,y,z\in(0;\infty)\wedge x+y+z=3$. Prove:

$$\frac{1}{(x+y)^3}+\frac{1}{(y+z)^3}+\frac{1}{(z+x)^3}+\frac{3}{8}\geq 16\left(\frac{1}{(2x+y+z)^3}+\frac{1}{(2y+x+z)^3}+\frac{1}{(2z+x+y)^3}\right)$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution by Daniel Sitaru – Romania

$$f:(0,\infty)\to(0,\infty), f(x)=x^{-3}, f'(x)=-3x^{-4}, f''(x)=12x^{-5}>0, f-\textit{convexe}$$

By Popoviciu's inequality:

$$\frac{1}{3}\big(f(a)+f(b)+f(c)\big)+f\left(\frac{a+b+c}{3}\right)\geq\frac{2}{3}\left(f\left(\frac{a+b}{2}\right)+f\left(\frac{b+c}{2}\right)+f\left(\frac{c+a}{2}\right)\right)$$

For $a=x+y, b=y+z, c=z+x$:

$$\frac{1}{3}\sum f(x+y)+f\left(\frac{2(x+y+z)}{3}\right)\geq\frac{2}{3}\sum f\left(\frac{x+y+y+z}{2}\right)$$

$$\frac{1}{3}\sum\frac{1}{(x+y)^3}+\frac{1}{2^3}\geq\frac{2}{3}\sum\frac{1}{\left(\frac{x+2y+z}{2}\right)^3}$$

$$\frac{1}{(x+y)^3}+\frac{1}{(y+z)^3}+\frac{1}{(z+x)^3}+\frac{3}{8}\geq 16\left(\frac{1}{(2x+y+z)^2}+\frac{1}{(2y+z+x)^2}+\frac{1}{(2z+x+y)^2}\right)$$

230. If $x,y,z\in\left(0,\frac{\pi}{2}\right)$ then:

$$\prod \ln(1+\tan^2 x)\cdot\prod \ln(1+\cot^2 y)\le \prod \ln^2\left(\frac{2}{\sin 2z}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

For $0<x<\frac{\pi}{2}$; $\ln(1+\tan^2 x)\ln(1+\cot^2 x)\le$

$$\le\left\{\frac{\ln(1+\tan^2 x)+\ln(1+\cot^2 x)}{2}\right\}^2=\left\{\frac{1}{2}\ln(\sec^2 x\csc^2 x)\right\}^2=\left(\ln\left(\frac{2}{\sin 2x}\right)\right)^2$$

Now, $0<x,y,z<\frac{\pi}{2}$; $\prod \ln(1+\tan^2 x)\prod \ln(1+\cot^2 y)=$

$$=\prod \ln(1+\tan^2 x)\,(1+\cot^2 x)\le\prod\left[\ln\left(\frac{2}{\sin 2x}\right)\right]^2$$

231. If $x,y\ge 0, n\ge 1, n\in\mathbb{Q}$, $AM=\frac{x+y}{2}$, $GM=\sqrt{xy}$ then :

$$\left(\frac{x^n+y^n}{\sqrt{2}}\right)^2\ge AM^{2n}+GM^{2n}$$

Proposed by Uche Eliezer Okeke-Anambra-Nigeria

Solution 1 by Henry Ricardo-Tapan-New York

The power means inequality gives us:

$$\sqrt[n]{\frac{x^n+y^n}{2}}\ge AM\ge GM\leftrightarrow\left(\frac{x^n+y^n}{2}\right)^2\ge AM^{2n}\ge GM^{2n}\to$$

$$\to 2\left(\frac{x^n+y^n}{2}\right)^2\ge AM^{2n}+GM^{2n}\to\left(\frac{x^n+y^n}{\sqrt{2}}\right)^2\ge AM^{2n}+GM^{2n}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\left(\frac{x^n+y^n}{\sqrt{2}}\right)^2=\frac{(x^n+y^n)^2}{4}+\frac{(x^n+y^n)^2}{4}\ge$$

$$\ge\frac{1}{4}\left(\frac{(x+y)^n}{2^{n-1}}\right)^2+(xy)^n=AM^{2n}+GM^{2n}$$

232. If $x,y\in\mathbb{R},\Omega=\begin{vmatrix} sinxsiny & sinxcosy & cosx \\ cosx & sinxsiny & sinxcosy \\ sinxcosy & cosx & sinxsiny \end{vmatrix}$ then:

$$|\Omega|\le 1$$

Proposed by Daniel Sitaru-Romania

Solution by Ravi Prakash-New Delhi-India

$$\overrightarrow{OA}=sinxsiny\vec{i}+sinxcosy\vec{j}+cosx\vec{k}$$

$$\overrightarrow{OB}=cosx\vec{i}+sinxsiny\vec{j}+sinxcosy\vec{k}$$

$$\overrightarrow{OC}=sinxcosy\vec{i}+cosx\vec{j}+sinxsiny\vec{k}$$

$$\left|\overrightarrow{OA}\right|^2=\left|\overrightarrow{OB}\right|^2=\left|\overrightarrow{OC}\right|^2=sin^2xsin^2y+sin^2xcos^2y+cos^2x=$$

$$=sin^2x(sin^2y+cos^2y)+cos^2x=1\to\left|\overrightarrow{OA}\right|=\left|\overrightarrow{OB}\right|=\left|\overrightarrow{OC}\right|=1$$

$$|\Omega|=\left|\overrightarrow{OA}\cdot\left(\overrightarrow{OB}\mathrm{X}\overrightarrow{OC}\right)\right|\overset{HADAMARD}{\leqq}\left|\overrightarrow{OA}\right|\cdot\left|\overrightarrow{OB}\right|\cdot\left|\overrightarrow{OC}\right|=1$$

233. If $0<a<b$ then:

$$e^{\frac{1}{b}}<\left(\frac{a+b}{2\sqrt{ab}}\right)^{\frac{2}{(\sqrt{b}-\sqrt{a})^2}}<e^{\frac{1}{a}}$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

Suppose $0<a<b$*, then* $a<\sqrt{ab}<\frac{a+b}{2}<b$*. Let* $f(x)=\ln x, x\in\left[\sqrt{ab},\frac{a+b}{2}\right]$

By the first mean value theorem, there exists $c\in\left(\sqrt{ab},\frac{a+b}{2}\right)$ *such that*

$$\frac{\ln\left(\frac{a+b}{2}\right)-\ln\sqrt{ab}}{\frac{a+b}{2}-\sqrt{ab}}=\frac{1}{c}\Rightarrow\frac{2}{\left(\sqrt{b}-\sqrt{a}\right)^2}\ln\left(\frac{a+b}{2\sqrt{ab}}\right)=\frac{1}{c}$$

$$\Rightarrow\left(\frac{a+b}{2\sqrt{ab}}\right)^{\frac{2}{(\sqrt{b}-\sqrt{a})^2}}=e^{\frac{1}{c}}\quad (1)$$

But $a<\sqrt{ab}<c<\frac{a+b}{2}<b\Rightarrow\frac{1}{b}<\frac{1}{c}<\frac{1}{a}$ *(2)*

From (1), (2), we get: $e^{\frac{1}{b}} < \left(\frac{a+b}{2\sqrt{ab}}\right)^{\frac{2}{(\sqrt{b}-\sqrt{a})^2}} < e^{\frac{1}{a}}$

234. If $P \in \mathbb{R}[x]$ with distinct roots $x_1, x_2, \dots, x_n \in \mathbb{R}, n \in \mathbb{N}^*$ then:

$$\frac{P''(x)}{P(x)} < \left(\frac{P'(x)}{P(x)}\right)^2 + \sum_{k=1}^{n} \frac{P''(x_k)}{P'(x_k)}, \forall x \in \mathbb{R} - \{x_1, x_2, \dots, x_n\}$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

$$\text{Let } P(x) = A(x-x_1)(x-x_2)\dots(x-x_n)$$

$$P'(x) = A(x-x_2)(x-x_3)\dots(x-x_n) + A(x-x_1)(x-x_3)\dots(x-x_n) + \dots + A(x-x_1)(x-x_2)\dots(x-x_{n-1})$$

$$P''(x) = \begin{bmatrix} A(x-x_3)(x-x_4)\dots(x-x_n) \\ +A(x-x_2)(x-x_4)\dots(x-x_n) \\ +\dots+A(x-x_2)\dots(x-x_{n-1}) \end{bmatrix} + \begin{bmatrix} A(x-x_3)(x-x_4)\dots(x-x_n) \\ +A(x-x_1)(x-x_4)\dots(x-x_n) \\ +\dots+A(x-x_1)\dots(x-x_{n-1}) \end{bmatrix}$$

$$+\dots+\begin{bmatrix} A(x-x_2)\dots(x-x_{n-1}) \\ +A(x-x_1)\dots(x-x_{n-1}) \\ +\dots+A(x-x_1)\dots(x-x_{n-2}) \end{bmatrix}$$

$\frac{P''(x_1)}{P'(x_1)} = \frac{2}{x_1-x_2} + \frac{2}{x_1-x_3} + \dots + \frac{2}{x_1-x_n}$. *Similarly,*

$$\frac{P''(x_r)}{P'(x_r)} = 2\sum_{\substack{j=1 \\ j\neq r}}^{n} \frac{1}{x_r - x_j} \Rightarrow \sum_{r=1}^{n} \frac{P''(x_r)}{P'(x_r)} = 0$$

Also, $\frac{P''(x)}{P(x)} - \left(\frac{P'(x)}{P(x)}\right)^2 = \frac{d}{dx}\left[\frac{P'(x)}{P(x)}\right] = \frac{d}{dx}\left[\frac{d}{dx}(\ln(P(x)))\right] = \frac{d^2}{dx^2}[\ln|A| + \ln|x-x_1| + \dots +$

$$\ln x{-}xn$$

$$= \frac{d}{dx}\left[\frac{1}{x-x_1} + \frac{1}{x-x_2} + \dots + \frac{1}{x-x_n}\right] = -\left[\frac{1}{(x-x_1)^2} + \frac{1}{(x-x_2)^2} + \dots + \frac{1}{(x-x_n)^2}\right] < 0$$

Hence, $\frac{P''(x)}{P(x)} < \left(\frac{P'(x)}{P(x)}\right)^2 + \sum_{r=1}^{n} \frac{P''(x_r)}{P'(x_r)}$

235. If $a,b,c>0, a+b+c=3, x\in\mathbb{R}$ then:

$$\left(\sqrt[3]{a\sin^2 x}+\sqrt[3]{b\cos^2 x}\right)\left(\sqrt[3]{b\sin^2 x}+\sqrt[3]{c\cos^2 c}\right)\left(\sqrt[3]{c\sin^2 x}+\sqrt[3]{a\cos^2 x}\right)\le 4$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Do Huu Duc Thinh-Ho Chi Minh-Vietnam

Using the inequality $m+n\le\sqrt[3]{4(m^3+n^3)}\ \forall m,n>0$:

$$\prod\left(\sqrt[3]{a\sin^2 x}+\sqrt[3]{b\cos^2 x}\right)\le 4\sqrt[3]{\prod(a\sin^2 x+b\cos^2 x)}$$

Also by AM-GM inequality we get: $\sqrt[3]{\prod(a\sin^2 x+b\cos^2 x)}\le\frac{\sum(a\sin^2 x+b\cos^2 x)}{3}=$

$=\frac{(a+b+c)(\sin^2 x+\cos^2 x)}{3}=\frac{3\cdot 1}{3}=1$. *Hence:* $\prod\left(\sqrt[3]{a\sin^2 x}+\sqrt[3]{b\cos^2 x}\right)\le 4$ *(Q.E.D.)*

Solution 2 by Serban George Florin-Romania

$$\sqrt[3]{a\cdot\sin^2 x\cdot 1}+\sqrt[3]{b\cdot\cos^2 x\cdot 1}\le\left(\sqrt[3]{a}^3+\sqrt[3]{b}^3\right)^{\frac{1}{3}}\left(\sqrt[3]{\sin^2 x}^3+\sqrt[3]{\cos^2 x}^3\right)^{\frac{1}{3}}\left(\sqrt[3]{1}^3+\sqrt[3]{1}^3\right)^{\frac{1}{3}}, \text{(Holder)}$$

$$\sqrt[3]{a\cdot\sin^2 x}+\sqrt[3]{b\cdot\cos^2 x}\le\sqrt[3]{2(a+b)}, \sqrt[3]{a\cdot\sin^2 x}+\sqrt[3]{b\cdot\cos^2 x}$$

$$\left(\sqrt[3]{a\cdot\sin^2 x}+\sqrt[3]{b\cdot\cos^2 x}\right)\left(\sqrt[3]{b\cdot\sin^2 x}+\sqrt[3]{c\cdot\cos^2 x}\right)\left(\sqrt[3]{c\cdot\sin^2 x}+\sqrt[3]{a\cdot\cos^2 x}\right)\le$$

$$\le\sqrt[3]{2(a+b)2(b+c)2(a+c)},$$

$$\left(\sqrt[3]{a\cdot\sin^2 x}+\sqrt[3]{b\cdot\cos^2 x}\right)\left(\sqrt[3]{b\cdot\sin^2 x}+\sqrt[3]{c\cdot\cos^2 x}\right)\left(\sqrt[3]{c\cdot\sin^2 x}+\sqrt[3]{a\cdot\cos^2 x}\right)\le$$

$$\le 2\sqrt[3]{(a+b)(b+c)(a+c)}$$

$$2\sqrt[3]{(a+b)(b+c)(a+c)}\le 2\frac{a+b+b+c+c+a}{3}=2\cdot\frac{6}{3}=4, (M_a\ge M_g)$$

$$\left(\sqrt[3]{a\cdot\sin^2 x}+\sqrt[3]{b\cdot\cos^2 x}\right)\left(\sqrt[3]{b\cdot\sin^2 x}+\sqrt[3]{c\cdot\cos^2 x}\right)\left(\sqrt[3]{c\cdot\sin^2 x}+\sqrt[3]{a\cdot\cos^2 x}\right)\le 4.$$

236. If $a,b,c,d\in\mathbb{R}$ then:

$$a+b+c+d\le\frac{1}{2}+(a+b)(c+d)+a^2+b^2+c^2+d^2$$

Proposed by Uche Eliezer Okeke-Anambra-Nigeria

Solution 1 by Ravi Prakash-New Delhi-India

$$(a+b+c+d-1)^2+(a-b)^2+(c-d)^2\ge 0 \Rightarrow a^2+b^2+c^2+d^2-2(a+b+c+d)+1$$
$$+2(ab+bc+cd+ad+ac+bd)+a^2+b^2-2ab+c^2+d^2-2cd\ge 0$$
$$\Rightarrow 2(a^2+b^2+c^2+d^2)-2(a+b+c+d)+2(a+b)(c+d)+1\ge 0$$
$$\Rightarrow a+b+c+d\le \frac{1}{2}+(a+b)(c+d)+a^2+b^2+c^2+d^2$$

Solution 2 by Nho Nguyen Van-Nghe An-Vietnam

We have:

$a^2+b^2 \overset{?}{\ge} \frac{1}{2}(a+b)^2;\ c^2+d^2 \overset{?}{\ge} \frac{1}{2}(c+d)^2$. **So:** $\frac{1}{2}+(a+b)(c+d)+a^2+b^2+c^2+d^2$

$$\ge \frac{1}{2}\left[(a+b)^2+2(a+b)(c+d)+(c+d)^2+1\right]=\frac{1}{2}(a+b+c+d)^2+\frac{1}{2}$$
$$\overset{?}{\ge} \frac{1}{2}\left[2(a+b+c+d)-1\right]+\frac{1}{2}=a+b+c+d \text{ (done) } (x^2\ge 2x-1)$$

237. If $0<a\le b<\frac{\pi}{2}$ then:

$$\frac{\tan b}{\tan a}\ge e^{2(b-a)}$$

Proposed by Nho Nguyen Van-Nghe An-Vietnam

Solution by Daniel Sitaru-Romania

$$f:[a,b]\to\mathbb{R}, f(x)=\ln(\tan x)$$
$$f(b)-f(a)\overset{LAGRANGE}{\triangleq} f'(c)(b-a), c\in(a,b)\to \ln(\tan b)-\ln(\tan a)=\frac{1}{\sin c\cos c}(b-a)$$
$$\ln\left(\frac{\tan a}{\tan b}\right)=\frac{2(b-a)}{\sin 2c}\ge 2(b-a)\to \ln\left(\frac{\tan a}{\tan b}\right)\ge \ln e^{2(b-a)}\to \frac{\tan a}{\tan b}\ge e^{2(b-a)}$$

238. Prove that:

$$2^x+3^x+4^x\ge x\ln 24+3, \forall x\in\mathbb{R}$$

Proposed by Nho Nguyen Van-Nghe An-Vietnam

Solution 1 by Daniel Sitaru-Romania

$$\begin{cases} f(x) = 2^x - xln2 \\ g(x) = 3^x - xln3 \\ h(x) = 4^x - xln4 \end{cases} \to \begin{cases} f'(x) = (2^x - 1)ln2 \\ g'(x) = (3^x - 1)ln3 \\ h'(x) = (4^x - 1)ln4 \end{cases} \to \begin{cases} f(x) \ge f(0) = 1 \\ g(x) \ge g(0) = 1 \\ h(x) \ge h(0) = 1 \end{cases} \to$$

$$f(x) + g(x) + h(x) \ge 3 \to 2^x - xln2 + 3^x - xln3 + 4^x - xln4 \ge 3$$

$$2^x + 3^x + 4^x \ge xln24 + 3, \forall x \in \mathbb{R}$$

Solution 2 by Michel Rebeiz-Lebanon

$$x - 1 > lnx\ , for\ x > 0$$

$$\begin{cases} 2^x - 1 > ln2^x \\ 3^x - 1 > ln3^x \\ 4^x - 1 > ln4^x \end{cases} \to 2^x - 1 + 3^x - 1 + 4^x - 1 > x(ln2 + ln3 + ln4)$$

$$2^x + 3^x + 4^x \ge xln24 + 3, \forall x \in \mathbb{R}$$

239. If $a, b \in \mathbb{R}, A, B \in M_n(\mathbb{R}), AB = BA$ then:

$$det(I_n + 2(a^2 + b^2)(A^2 + B^2) + 2(a + b)(A + B) + 8abAB) \ge 0$$

Proposed by Marian Ursarescu-Romania

Solution by Ravi Prakash-New Delhi-India

We first show that if $x, y \in M_n(\mathbb{R})$ *and* $xy = yx$, *then* $\det(x^2 + y^2) \ge 0$.

Note that $x^2 + y^2 = (x + iy)(x - iy)$ $[\because xy = yx]$

Now, $\det(x^2 + y^2) = \det\big((x + iy)(x - iy)\big) = \det(x + iy)\det(x - iy)$

$$= \det(x + iy)\det(\overline{x + iy}) = \det(x + iy)\overline{\det(x + iy)} = |\det(x + iy)|^2 \ge 0$$

Now, for $a, b \in R, A, B \in M_n(\mathbb{R}), AB = BA$, *we have*

$$I_n + 2(a^2 + b^2)(A^2 + B^2) + 2(a + b)(A + B) + 8abAB =$$

$$= I_n + \left[(a + b)^2 + (a - b)^2\right](A^2 + B^2) + 2(a + b)(A + B) + \left[(a + b)^2 - (a - b)^2\right](2AB)$$

$$= I_n + (a + b)^2(A^2 + B^2 + 2AB) + 2(a + b)(A + B) + (a - b)^2(A^2 + B^2 - 2AB)$$

$$= I_n + (a + b)^2(A + B)^2 + 2(a + b)(A + B) + (a - b)^2(A - B)^2$$

$$= [I_n + (a + b)(A + B)]^2 + \big((a - b)(A - B)\big)^2 = x^2 + y^2$$

where $x = I_n + (a + b)(A + B) \in M_n(\mathbb{R})$ *and* $y = (a - b)(A - B) \in M_n(\mathbb{R})$

Thus, $\det\{I_n + 2(a^2 + b^2)(A^2 + B^2) + 2(a + b)(A + B) + 8abAB\} =$

$$= \det(x^2 + y^2) \geq 0$$

240. **For** $0 < a < b \wedge x_1, x_2, \dots, x_n \in [a; b] \wedge \alpha > 0$. **Prove:**

$$\prod_{k=1}^{n} x_k^{\frac{\alpha}{n}} + \frac{(ab)^\alpha}{\prod_{k=1}^{n} x_k^{\frac{\alpha}{n}}} \leq a^\alpha + b^\alpha$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Daniel Sitaru-Romania

$$a \leq x_1, x_2, \dots, x_n \leq b \to a^n \leq \prod_{k=1}^{n} x_k \leq b^n \to a^\alpha \leq \prod_{k=1}^{n} x_k^{\frac{\alpha}{n}} \leq b^\alpha \to$$

$$\left(a^\alpha - \prod_{k=1}^{n} x_k^{\frac{\alpha}{n}}\right)\left(b^\alpha - \prod_{k=1}^{n} x_k^{\frac{\alpha}{n}}\right) \leq 0 \to (ab)^\alpha - (a^\alpha + b^\alpha)\prod_{k=1}^{n} x_k^{\frac{\alpha}{n}} + \left(\prod_{k=1}^{n} x_k^{\frac{\alpha}{n}}\right)^2 \leq 0 \to$$

$$\left(\prod_{k=1}^{n} x_k^{\frac{\alpha}{n}}\right)^2 + (ab)^\alpha \leq (a^\alpha + b^\alpha)\prod_{k=1}^{n} x_k^{\frac{\alpha}{n}} \to \prod_{k=1}^{n} x_k^{\frac{\alpha}{n}} + \frac{(ab)^\alpha}{\prod_{k=1}^{n} x_k^{\frac{\alpha}{n}}} \leq a^\alpha + b^\alpha$$

Solution 2 by Marian Ursarescu-Romania

We have $\left(\sqrt[n]{x_1 x_2 \dots x_n}\right)^\alpha + \frac{(ab)^\alpha}{\sqrt[n]{(x_1 x_2 \dots x_n)^\alpha}} \leq a^\alpha + b^\alpha$ ***(1)***

Let $\left(\sqrt[n]{x_1 x_2 \dots x_n}\right)^\alpha = y$, ***becaue*** $x_1, x_2, \dots, x_n \in [a, b] \Rightarrow y \in [a^\alpha, b^\alpha]$ ***(2)***

From (1) $\Rightarrow y + \frac{(ab)^\alpha}{y} \leq a^\alpha + b^\alpha \Rightarrow$

$$y^2 - (a^\alpha + b^\alpha)y + (ab)^\alpha \leq 0 \quad \textit{(3)}$$

$$\Delta = (a^\alpha + b^\alpha)^2 - 4(ab)^\alpha = (a^\alpha - b^\alpha)^2 \geq 0 \Rightarrow$$

$$y_{1,2} = \frac{a^\alpha + b^\alpha \pm (a^\alpha - b^\alpha)}{2} \Rightarrow y_1 = a^\alpha, y_2 = b^\alpha \Rightarrow$$

Relations (3) is true because $y \in [a^\alpha, b^\alpha]$, ***true from (2).***

241. $P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 \in \mathbb{R}[X], n \geq 2$

If $a_0, a_1, \dots, a_n > 0$ then: $P\left(1+\frac{1}{n}\right) \ge P(1) + \frac{1}{n}P'(1)$

Proposed by Marian Ursarescu-Romania

Solution 1 by Ravi Prakash-New Delhi-India

For $k \in \mathbb{N}, n \in \mathbb{N}$; $\left(1+\frac{1}{n}\right)^k \ge 1 + \frac{k}{n}$

$$\therefore P\left(1+\frac{1}{n}\right) = \sum_{k=0}^{n} a_k\left(1+\frac{1}{n}\right)^k \ge \sum_{k=0}^{n} a_k\left(1+\frac{k}{n}\right) \quad [\because a_k > 0]$$

$$= \sum_{k=0}^{n} a_k + \frac{1}{n}\sum_{k=1}^{n} k a_k = P(1) + \frac{1}{n}P'(1)$$

Solution 2 by Abdallah El Farissi-Bechar-Algerie

If $a_0, a_1, \dots, a_n > 0$, then $P''(x) \ge 0$ for all $x \ge 0$, then P' is increasing function for $x \ge 0$. By Lagrange mean value theorem on $\left[1, 1+\frac{1}{n}\right]$

$P\left(1+\frac{1}{n}\right) - P(1) = \frac{1}{n}P'(c)$ and $c \in \left]1, 1+\frac{1}{n}\right[$ then $P\left(1+\frac{1}{n}\right) - P(1) \ge \frac{1}{n}P'(1)$

242. If $a, b, c > 0$ then:

$$a^a \cdot b^b \cdot c^c \ge \left(\frac{a+b}{2}\right)^{\frac{a+b}{2}}\left(\frac{b+c}{2}\right)^{\frac{b+c}{2}}\left(\frac{c+a}{2}\right)^{\frac{c+a}{2}} \ge (abc)^{\frac{a+b+c}{3}}$$

USA-TST(Radu Zaciu-Romania)

Solution by Soumitra Mandal-Chandar Nagore-India

Applying Weighted AM≥GM: $\sqrt[a+b]{a^a b^b} \ge \frac{a+b}{2}$, $\sqrt[b+c]{b^b c^c} \ge \frac{b+c}{2}$ and $\sqrt[c+a]{c^c a^a} \ge \frac{c+a}{2}$

$$\Rightarrow \prod_{cyc} a^{2a} \ge \prod_{cyc}\left(\frac{a+b}{2}\right)^{a+b} \Rightarrow \prod_{cyc} a^a \ge \prod_{cyc}\left(\frac{a+b}{2}\right)^{\frac{a+b}{2}}$$

Again applying Weighted $AM \ge GM$;

$$\prod_{cyc}\left(\frac{a+b}{2}\right)^{\frac{a+b}{2}} \geq \left(\frac{\sum_{cyc}\left(\frac{a+b}{2}\right)}{\frac{(a+b)/2}{(a+b)/2}+\frac{(b+c)/2}{(b+c)/2}+\frac{(c+a)/2}{(c+a)/2}}\right)^{a+b+c} = \left(\frac{a+b+c}{3}\right)^{a+b+c} \geq (abc)^{\frac{a+b+c}{3}}$$

243. If $0 < x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n$ is an arithmetical progression with common difference d then:

$$\tan^{-1}\frac{d}{1+x_1x_2} + \tan^{-1}\frac{d}{1+x_2x_3} + \cdots + \tan^{-1}\frac{d}{1+x_{n-1}x_n} \leq \ln\sqrt{\frac{x_n}{x_1}}$$

Proposed by Mihaly Bencze-Romania

Solution by Daniel Sitaru-Romania

$$f(x) = \tan^{-1}x - \frac{\ln x}{2} \to f'(x) = \frac{1}{1+x^2} - \frac{1}{2x} = -\frac{(x-1)^2}{2x(1+x^2)} \leq 0 \to f - decreasing$$

$$x_1 \leq x_n \to f(x_1) \geq f(x_n)$$

$$\sum \tan^{-1}\frac{d}{1+x_{k-1}x_k} = \sum \tan^{-1}\frac{x_k - x_{k-1}}{1+x_{k-1}x_k} = \sum(\tan^{-1}x_k - \tan^{-1}x_{k-1}) =$$

$$= \tan^{-1}x_n - \tan^{-1}x_1 \leq \ln\sqrt{\frac{x_n}{x_1}} \leftrightarrow \tan^{-1}x_1 - \frac{1}{2}\ln x_1 \geq \tan^{-1}x_n - \frac{1}{2}\ln x_n \leftrightarrow$$

$$\leftrightarrow f(x_1) \geq f(x_n)$$

244. For $a, b \in (0; +\infty) \land 0 \leq \theta \leq \pi$. Prove:

$$\frac{(a^3+b^3)(a^6+b^6)(a^8+b^8)}{(a+b)(a^5+b^5)(a^{11}+b^{11})} \leq 1 + \sin\theta$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Ravi Prakash-New Delhi-India

Consider $(a^3+b^3)(a^6+b^6)(a^8+b^8) - (a+b)(a^5+b^5)(a^{11}+b^{11})$

$$= (a^3+b^3)(a^{14}+a^8b^6+a^6b^8+b^{14}) - (a+b)(a^{16}+a^5b^{11}+a^{11}b^5+b^{16})$$

$$= a^{17} + a^{11}b^6 + a^9b^8 + a^3b^{14} + b^{17} + a^6b^{11} + a^8b^9 + a^{14}b^3 -$$

$$-\left[a^{17} + a^6b^{11} + a^{12}b^5 + ab^{16} + b^{17} + a^{11}b^6 + b^{12}a^5 + a^{16}b\right]$$

$$= a^9b^8 + a^8b^9 + a^3b^{14} + a^{14}b^3 - a^{12}b^5 - a^5b^{12} - ab^{16} - a^{16}b$$

$$= a^9b^5(b^3 - a^3) + a^5b^9(a^3 - b^3) + ab^{14}(a^2 - b^2) + a^{14}b(b^2 - a^2)$$

$$= a^5b^5(a^3 - b^3)(b^3 - a^3) + ab(b^{13} - a^{13})(a^2 - b^2) \le 0$$

$$\Rightarrow (a^3 + b^3)(a^6 + b^6)(a^8 + b^8) \le (a + b)\left(a^5 + b^5\right)(a^{11} + b^{11})$$

$$\Rightarrow \frac{(a^3 + b^3)(a^6 + b^6)(a^8 + b^8)}{(a + b)(a^5 + b^5)(a^{11} + b^{11})} \le 1 \le 1 + \sin\theta\,;\ (0 \le \theta \le \pi)$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\forall\theta \in [0,\pi], \sin\theta \ge 0 \Rightarrow 1 + \sin\theta \ge 1$$

$\therefore$ ***it suffices to prove:*** $1 \ge \frac{(a^3+b^3)(a^6+b^6)(a^8+b^8)}{(a+b)(a^5+b^5)(a^{11}+b^{11})}$

$$\Leftrightarrow a^{17} + a^{11}b^6 + a^{12}b^5 + a^{16}b + a^6b^{11} + b^{17} + ab^{16} + a^5b^{12} \ge$$

$$\ge a^{17} + a^{11}b^6 + a^{14}b^3 + a^8b^9 + a^9b^8 + a^3b^{14} + a^6b^{11} + b^{17}$$

$$\Leftrightarrow a^{16}b + ab^{16} + a^{12}b^5 + a^5b^{12} \overset{(1)}{\ge} a^{14}b^3 + a^3b^{14} + a^9b^8 + a^8b^9$$

Now, $a^{16}b + ab^{16} = ab\left(a^{15} + b^{15}\right) \overset{Chebyshev}{\ge} \frac{1}{2}ab(a^4 + b^4)(a^{11} + b^{11}) \ge$

$$\underset{(a)}{\overset{A-G}{\ge}} ab \cdot a^2b^2(a^{11} + b^{11}) = a^{14}b^3 + a^3b^{14}$$

Also, $a^{12}b^5 + a^5b^{12} = a^5b^5(a^7 + b^7) \overset{Chebyshev}{\ge} \frac{1}{2}a^5b^5(a + b)(a^6 + b^6) \ge$

$$\underset{(b)}{\overset{A-G}{\ge}} a^5b^5 \cdot a^3b^3(a + b) = a^9b^8 + a^8b^9$$

(a)+(b)⇒ (1) is true (proved)

245. If $a, b > 0, a \ne b$ **then:**

$$0 < \frac{\frac{a - b}{\ln a - \ln b} - \sqrt{ab}}{\frac{a + b}{2} - \sqrt{ab}} < \frac{1}{3}$$

Proposed by B.G.Carlson-USA

Solution 1 by Omran Kouba-Damascus-Syria

First note that: $\cosh t + 2 - 3\frac{\sinh t}{t} = \sum_{n=2}^{\infty}\frac{2(n-1)}{(2n+1)!}t^{2n} \geq 0$

With equality if and only if $t = 0$*. This is equivalent to:* $\frac{\frac{\sinh t}{t}-1}{\cosh t-1} < \frac{1}{3}$ *for* $t \neq 0$

Now, setting $t = \ln\sqrt{\frac{a}{b}}$*, yields the upper inequality. The lower inequality is trivial since it follows in the same way from* $\frac{\sinh t}{t} > 1$ *for* $t \neq 0$*.*

Solution 2 by Khanh Hung Vu-Ho Chi Minh-Vietnam

Put $A = \frac{\frac{a-b}{\ln a-\ln b}-\sqrt{ab}}{\frac{a+b}{2}-\sqrt{ab}}$*. We need to prove that* $0 < A < \frac{1}{3}$

1) LEMMA: $\frac{a-b}{\ln a-\ln b} > \sqrt{ab}$ *when* $a, b > 0$ *and* $a \neq b$

We have $\frac{a-b}{\ln a-\ln b} > \sqrt{ab} \Rightarrow \frac{\ln a-\ln b}{a-b} < \frac{1}{\sqrt{ab}}$

$$\Rightarrow \frac{\ln\left(\frac{a}{b}\right)}{\frac{a}{b}-1} < \sqrt{\frac{b}{a}} \quad (1)$$

Put $\frac{a}{b} = t\ (t > 0, t \neq 1)$*, we have (1)* $\Rightarrow \frac{\ln t}{t-1} < \frac{1}{\sqrt{t}}$ *(2)*

Put $f(t) = \ln t - \frac{t-1}{\sqrt{t}}$

$f'(t) = \frac{-(\sqrt{t}-1)^2}{2\sqrt{t^3}} < 0 \Rightarrow f(t)$ *is decreasing function* $\Rightarrow f(t) < f(1)$ *when* $t > 1$ *and* $f(t) > f(1)$ *when* $t < 1 \Rightarrow f(t) < 0$ *when* $t > 1$ *and* $f(t) > 0$ *when* $t < 1$*.*

1.1.) If $t > 1$*. We have (2)* $\Rightarrow \ln t < \frac{t-1}{\sqrt{t}}$ *(True)*

1.2) If $t < 1$*. We have (2)* $\Rightarrow \ln t > \frac{t-1}{\sqrt{t}}$ *(True)*

$\Rightarrow$ *(1) true* $\Rightarrow \frac{\ln a-\ln b}{a-b} < \frac{1}{\sqrt{ab}}$

Applying the lemma $\Rightarrow \frac{a-b}{\ln a-\ln b} > \sqrt{ab}$ *(since* $0 < \frac{\ln a-\ln b}{a-b} < \frac{1}{\sqrt{ab}}$*)*

On the other hand, by AM-GM inequality, we have $\frac{a+b}{2} - \sqrt{ab} > 0$ *(since* $a \neq b$*)*

2) We need to prove that $A < \frac{1}{3} \Rightarrow$

$$\Rightarrow \frac{3(a-b)}{\ln a - \ln b} - 3\sqrt{ab} < \frac{a+b}{2} - \sqrt{ab} \Rightarrow \frac{3(a-b)}{\ln a - \ln b} < \frac{a+b}{2} + 2\sqrt{ab}$$

$$\Rightarrow \frac{3\left(\frac{a}{b}-1\right)}{\ln\left(\frac{a}{b}\right)} < \frac{\frac{a}{b}+1}{2} + 2\sqrt{\frac{a}{b}} \quad \textit{(3)}$$

Put $\frac{a}{b} = t \ (t > 0, t \neq 1)$***, we have (3)*** $\Rightarrow \frac{3(t-1)}{\ln t} < \frac{t+1}{2} + 2\sqrt{t}$ ***(4)***

Put $g(t) = \frac{t+1}{2} + 2\sqrt{t} - \frac{3(t-1)}{\ln t}$

$$g'(t) = \frac{1}{\sqrt{t}} + \frac{1}{2} + \frac{3(t-1) - 3t\cdot\ln t}{t\cdot\ln^2 t} = \frac{2\sqrt{t}\cdot\ln^2 t + t\cdot\ln^2 t + 6(t-1) - 6t\cdot\ln t}{2t\cdot\ln^2 t}$$

Put $h(t) = 2\sqrt{t}\cdot\ln^2 t + t\cdot\ln^2 t + 6(t-1) - 6t\cdot\ln t$

$$h'(t) = \frac{\ln t\cdot\left(-4\sqrt{t} + 4 + \left(\sqrt{t}+1\right)\cdot\ln t\right)}{\sqrt{t}}$$

$h'(t) = 0 \Rightarrow \ln t = 0$ ***(5) or*** $-4\sqrt{t} + 4 + \left(\sqrt{t}+1\right)\cdot\ln t = 0$ ***(6)***

(5): $\ln t = 0 \Rightarrow t = 1$

(6): $-4\sqrt{t} + 4 + \left(\sqrt{t}+1\right)\cdot\ln t = 0 \Rightarrow \ln t = \frac{4(\sqrt{t}-1)}{\sqrt{t}+1}$

Put $y(t) = \ln t - \frac{4(\sqrt{t}-1)}{\sqrt{t}+1}$

$y'(t) = \frac{(\sqrt{t}-1)^2}{t(\sqrt{t}+1)^2} > 0 \Rightarrow y(x)$ ***is increasing function***$\Rightarrow y(x) = 0$ ***has at most 1 root***

On the other hand, we have $y(1) = 0 \Rightarrow t = 1$ ***is the root of (6).***

So $h'(t) = 0 \Rightarrow t = 1$***. So we have2.1)*** $g'(t) < 0$ ***when*** $t < 1$

So when $t < 1 \Rightarrow g(t)$ ***is decreasing function*** $\Rightarrow g(t) > \lim_{t\to 1^+} g(t) \Rightarrow g(t) > 0$

2.2) $g'(t) > 0$ ***when*** $t > 1$

So when $t < 1 \Rightarrow g(t)$ ***is an increasing function*** $\Rightarrow g(t) > \lim_{t\to 1^+} g(t)$

So, $g(t) > 0 \ \forall t > 0 \Rightarrow$ ***(4) true*** $\Rightarrow$ ***(3)*** $\Rightarrow A < \frac{1}{3} \Rightarrow$ ***Q.E.D***

t	0		1		$+\infty$
$g(t)$	-6	↗	0	↗	$+\infty$

246. If $x, y, z > 0$ then:

$$x+y+z \geq ln\left(\frac{z+2}{(x-1)^2-2x+5}\right)+ln\left(\frac{y+2}{(z-1)^2-2z+5}\right)+ln\left(\frac{x+2}{(y-1)^2-2y+5}\right)+3$$

Proposed by Lazaros Zachariadis-Thessaloniki-Greece

Solution 1 by Soumava Chakraborty-Kolkata-India

$$RHS = \ln\left(\frac{z+2}{(x-1)^2-2x+5}\right)\left(\frac{y+2}{(z-1)^2-2z+5}\right)\left(\frac{x+2}{(y-1)^2-2y+5}\right)+3=$$

$$=\ln\left(\frac{x+2}{(x-1)^2-2x+5}\right)\left(\frac{y+2}{(y-1)^2-2y+5}\right)\left(\frac{z+2}{(z-1)^2-2z+5}\right)+3$$

$$=\ln\left(\frac{x+2}{(x-1)^2-2x+5}\right)+\ln\left(\frac{y+2}{(y-1)^2-2y+5}\right)+\ln\left(\frac{z+2}{(z-1)^2-2z+5}\right)+3$$

Let $f(x)=1-x+\ln\left(\frac{x+2}{(x-1)^2-2x+5}\right)\ \forall x \geq 0;\ f(0)=1-\ln 3<0$

$$f'(x)=\frac{(1-x)(x^2+2)}{(x+2)(x^2-4x+6)} \because x^2-4x+6=(x-2)^2+2>0,$$

$\therefore f'(x)>0\ \forall x \in (0,1);\ f'(1)=0$ *and* $f''(1)=\left(\frac{x^4+8x^3-48x^2+32x-20}{(x+2)^2(x-4x+6)^2}\right)\Big|_{x=1}<0$

$\therefore f(x)$ ***attains a maxima at*** $x=1$ ***and*** $f(1)=0$ ***and*** $f'(x)<0\ \forall x \in (1,\infty)$

$\therefore f(0)<0$ ***and then*** $f(x)$ ***increases and at*** $x=1$***, it reaches a maxima with*** $f(1)=0$ ***and then*** $f(x)$ ***decreases***

$\therefore x \in [0,\infty), f(x) \leq 0 \Rightarrow \forall x \in (0,\infty), f(x) \leq 0$ ***with equality at*** $x=1$

$\Rightarrow \forall x>0, 1-x+\ln\left(\frac{x+2}{(x-1)^2-2x+5}\right) \leq 0$ ***with equality at*** $x=1 \rightarrow$ ***(1)***

Similarly, $\forall y>0, 1-y+\ln\left(\frac{y+2}{(y-1)^2-2y+5}\right) \overset{(2)}{\leq} 0$ ***with equality at*** $y=1$

and, $\forall z>0, 1-z+\ln\left(\frac{z+2}{(z-1)^2-2z+5}\right) \overset{(3)}{\leq} 0$ ***with equality at*** $z=1$

(1)+(2)+(3)$\Rightarrow 3-\sum x+\ln\left(\frac{x+2}{(x-1)^2-2x+5}\right)+\ln\left(\frac{y+2}{(y-1)^2-2y+5}\right)+\ln\left(\frac{z+2}{(z-1)^2-2z+5}\right) \leq 0$

$\forall x, y, z > 0$

$$\Rightarrow \forall x, y, z>0, x+y+z \geq \ln\left(\frac{z+2}{(x-1)^2-2x+5}\right)+\ln\left(\frac{y+2}{(z-1)^2-2z+5}\right)+\ln\left(\frac{x+2}{(y-1)^2-2y+5}\right)$$

(proved)

Solution 2 by proposer

$$f(x)=e^{x-1}(x^2-4x+6) \text{ is convex because } f''(x)=e^{x-1}x^2>0, \forall x>0$$

$$y=f'(1)(x-1)+f(1)\Leftrightarrow y=x+2 \text{ is the tangent line at } (1,f(1))$$

$$\text{so we have: } e^{x-1}(x^2-4x+6)\ge x+2 \overset{x^2-4x+6>0}{\Leftrightarrow} e^{x-1}\ge \frac{x+2}{(x-1)^2-2x+5} \quad (1)$$

$$\text{Likewise we have } e^{y-1}\ge \frac{y+2}{(y-1)^2-2y+5} \ (2) \text{ and } e^{z-1}\ge \frac{z+2}{(z-1)^2-2z+5} \ (3)$$

$$\overset{(1).(2).(3)}{\Rightarrow} e^{x-1}y^{y-1}e^{z-1}\ge \frac{(x+2)}{(y-1)^2-2y+5}\cdot\frac{(y+2)}{(z-1)^2-2z+5}\cdot\frac{(z+2)}{(x-1)^2-2x+5} \overset{\ln x\nearrow}{\Leftrightarrow}$$

$$(x-1)+(y-1)+(z-1)\ge \sum_{cyc}\ln\left(\frac{x+2}{(y-1)^2-2y+5}\right)\Leftrightarrow$$

$$x+y+z\ge \sum_{cyc}\ln\left(\frac{x+2}{(y-1)^2-2y+5}\right)+3 \ \text{(proved)}$$

equality holds when $x=y=z=1$.

247. If $0<a\le b\le c$ then:

$$\frac{1}{1+e^{a-b+c}}+\frac{1}{1+e^b}\le\frac{1}{1+e^a}+\frac{1}{1+e^c}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Chris Kyriazis-Athens-Greece

Let's consider the function $f(x)=\frac{1}{1+e^x}, x>0$. *Easily:* $f'(x)=-\frac{e^x}{(1+e^x)^2}<0, \forall x>0$ *(f strictly decreasing) and* $f''(x)=-e^x\frac{(1-e^x)}{(1+e^x)^3}>0, \forall x>0$. *So,* f *is convex for every* $x>0$.

Working with the fundamental definition of convexity, I have that:

$\frac{c-b}{c-a}a+\left(1-\frac{c-b}{c-a}\right)\cdot c=\frac{c-b}{c-a}\cdot a+\frac{b-a}{c-a}\cdot c=\frac{ca-ab+bc-ac}{c-a}=b$. *And* $\frac{c-b}{c-a}+1-\frac{c-b}{c-a}=1$. *So,*

$$f(b)=f\left(\frac{c-b}{c-a}\cdot a+\left(1-\frac{c-b}{c-a}\right)c\right)\le\frac{c-b}{c-a}f(a)+\left(1-\frac{c-b}{c-a}\right)f(c)=\frac{c-b}{c-a}f(a)+\frac{b-a}{c-a}f(c) \quad (1)$$

Also: $a-b+c=a-\left(\frac{c-b}{c-a}a+\frac{b-a}{c-a}\cdot c\right)+c=\frac{b-a}{c-a}a+\frac{c-b}{c-a}c$. *So,*

$$f(a-b+c)=f\left(\frac{b-a}{c-a}a+\frac{c-b}{c-a}c\right)\le\frac{b-a}{c-a}f(a)+\frac{c-b}{c-a}f(c) \quad (2)$$

Adding (1) + (2): $f(b)+f(a-b+c)\le f(a)+f(c)$ *as we desire!*

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $0 < a \le b \le c$, *we give* $b \le a + m, c = b + p = a + m + p,\ m, p \ge 0$. *Consider*

$$(e^m - 1) = (e^m - 1) \Rightarrow (e^m - 1) \le (e^m - 1)(e^{a+a+m+p}) \Rightarrow$$

$$\Rightarrow e^m - 1 \le e^{a+a+m+m+p} - e^{a+a+m+p} \Rightarrow e^m + e^{a+a+m+p} \le 1 + e^{a+a+m+m+p} \Rightarrow$$

$$\Rightarrow e^m + e^{a+a+m+p} \le 1 + e^{a+a+m+m+p} \Rightarrow e^m + e^{a+m+p} + e^{a+m} + e^{a+a+m+p} \le$$

$$\le 1 + e^{a+m+p} + e^{a+m} + e^{a+a+m+m+p} \Rightarrow e^m(1 + e^a)(1 + e^{a+p}) \le$$

$$\le (1 + e^{a+m})(1 + e^{a+p+m}) \Rightarrow e^m(e^p - 1)(1 + e^a)(1 + e^{a+p}) \le$$

$$\le (e^p - 1)(1 + e^{a+m})(1 + e^{a+p+m}) \Rightarrow \frac{e^m(e^p - 1)}{(1 + e^{a+m})(1 + e^{a+p+m})} \le \frac{(e^p - 1)}{(1 + e^a)(1 + e^{a+p})}$$

$$\Rightarrow \frac{e^{a+p+m} - e^{a+m}}{(1 + e^{a+m})(1 + e^{a+p+m})} \le \frac{e^{a+p} - e^a}{(1 + e^a)(1 + e^{a+p})} \Rightarrow \frac{1}{1 + e^{a+m}} - \frac{1}{1 + e^{a+p+m}} \le$$

$$\le \frac{1}{1 + e^a} - \frac{1}{1 + e^{a+p}} \Rightarrow \frac{1}{1 + e^{a+p}} + \frac{1}{1 + e^{a+m}} \le \frac{1}{1 + e^a} + \frac{1}{1 + e^{a+p+m}} \Rightarrow$$

$$\Rightarrow \frac{1}{1 + e^{a-(a+m)+(a+m+p)}} + \frac{1}{1 + e^{a+m}} \le \frac{1}{1 + e^a} + \frac{1}{1 + e^{a+p+m}} \Rightarrow \frac{1}{1 + e^{a-b+c}} + \frac{1}{1 + e^b} \le$$

$$\le \frac{1}{1+e^a} + \frac{1}{1+e^a}.$$ *Therefore it is to be true.*

248. **For $0 < a < b$. Prove: $\frac{e^{b^2} - e^{a^2}}{b-a} \ge (a + b)(ab + 1)$.**

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Soumitra Mandal-Chandar Nagore-India

Let $f(x) = 2xe^{x^2}$ *for all* $x \ge 0$

$$f'(x) = 2e^{x^2} + 4x^2e^{x^2}, f''(x) = 4xe^{x^2} + 8xe^{x^2} + 8x^3e^{x^2} \ge 0 \text{ for all } x \ge 0$$

Hence f is convex $\therefore$ applying Hermite – Hadamard Inequality.

$$\frac{f(a) + f(b)}{2} \ge \frac{1}{b - a}\int_a^b f(x)\,dx \ge f\left(\frac{a + b}{2}\right) \Rightarrow \frac{1}{b - a}\int_a^b 2xe^{x^2}\,dx \ge 2\left(\frac{a + b}{2}\right)e^{\left(\frac{a+b}{2}\right)^2}$$

$$\Rightarrow \frac{e^{b^2} - e^{a^2}}{b-a} \ge (a + b)\left(1 + \left(\frac{a+b}{2}\right)^2\right) \because e^x \ge 1 + x \therefore \frac{e^{b^2} - e^{a^2}}{b-a} \ge (a + b)(1 + ab)$$ ***(proved)***

Solution 2 by Soumava Chakraborty-Kolkata-India

$$ab+1\overset{GM\le AM}{<}\frac{a^2+b^2}{2}+1=\frac{a^2+b^2+2}{2}$$

$$\Rightarrow (b^2-a^2)(ab+1)\overset{(1)}{<}\frac{(b^2-a^2)(a^2+b^2+2)}{2}=\frac{b^4-a^4+2b^2-2a^2}{2}$$

Given inequality $\Leftrightarrow (b^2-a^2)(ab+1)\overset{(2)}{\le} e^{b^2}-e^{a^2}$

(1), (2) $\Rightarrow$ *it suffices to prove:* $\frac{b^4-a^4+2b^2-2a^2}{2}\le e^{b^2}-e^{a^2}$

$$\Leftrightarrow 2e^{b^2}-b^4-2b^2\ge 2e^{a^2}-a^4-2a^2 \quad (3)$$

Let $f(x)=2e^{x^2}-x^4-2x^2\ \forall x>0$

$$f'(x)=4x\left(e^{x^2}-x^2-1\right)>4x(1+x^2-x^2-1)\ \left(\because e^{x^2}>1+x^2\forall x>0\right)$$

$=0\Rightarrow f'(x)>0\Rightarrow f(x)$ *is an increasing* f^n *on* $(0,\infty)\ \therefore$ *as* $b>a, f(b)>f(a)$

$\Rightarrow 2e^{b^2}-b^4-2b^2>2e^{a^2}-a^4-2a^2\Rightarrow$ *(3) is true (proved)*

249. For $a\ge 1\wedge b\ge 1$. Prove:

$$\frac{\sum_{k=0}^{8}b^{8-k}a^k}{\sum_{k=0}^{7}a^{7-k}b^k}\ge\frac{9}{8}.$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution by Ravi Prakash-New Delhi-India

Let $c>1$*. By the Cauchy's mean value theorem, there exists* $\alpha\in(1,c)$ *such that*

$$\frac{c^9-1}{c^8-1}=\frac{9\alpha^8}{8\alpha^7}=\frac{9}{8}\alpha>\frac{9}{8}\ (1)$$

Case 1 $a=b=1$, *then* $\frac{\sum_{k=0}^{8}b^{8-k}a^k}{\sum_{k=0}^{7}b^ka^{7-k}}=\frac{9}{8}.$

Case 2 $a\ne b$. *Let* $a>b\ge 1$. *Put* $\frac{a}{b}=c>1$. *Now,*

$\sum_{k=0}^{8}b^{8-k}a^k=\frac{b^8(c^9-1)}{c-1}$ *and* $\sum_{k=0}^{7}a^{7-k}b^k=\frac{b^7(c^8-1)}{c-1}$. *Thus,*

$$\frac{\sum_{k=0}^{8}b^{8-k}a^k}{\sum_{k=0}^{7}b^ka^{7-k}}=\frac{b(c^9-1)}{c^8-1}>\frac{9}{8}b\ge\frac{9}{8}\ [\because b\ge 1]$$

250. For $a,b,c\in(0;+\infty)$. Prove:

$$\frac{e^{a^b+b^c+c^a+a^c+c^b+b^a}}{a^{b+c}b^{a+c}c^{a+b}} \ge e^6$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

From the graphs of $y = e^x$ ***and*** $y = x + 1$, ***it is clear that:*** $\forall x, e^x \ge x + 1 \to$ ***(1)***

Choosing $x = a^b - 1$ ***in (1), we get:*** $e^{a^b} - 1 \ge a^b \Rightarrow \frac{e^{a^b}}{a^b} \overset{(a)}{\ge} e$

Similarly, $\frac{e^{b^c}}{b^c} \overset{(b)}{\ge} e, \frac{e^{c^a}}{c^a} \overset{(c)}{\ge} e, \frac{e^{a^c}}{a^c} \overset{(d)}{\ge} e, \frac{e^{c^b}}{c^b} \overset{(e)}{\ge} e, \frac{e^{b^a}}{b^a} \overset{(f)}{\ge} e$

(a) · (b) · (c) · (d) · (e) · (f) $\Rightarrow \frac{e^{a^b+b^c+c^a+a^c+c^b+b^a}}{a^{b+c}b^{a+c}c^{a+b}} \ge e^6$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

Because $d^{(a^b-1)}e^{(a^c-1)}e^{(b^a-1)}e^{(b^c-1)}e^{(c^a-1)}e^{(c^b-1)} \ge a^b a^c b^a b^c c^a c^b$

$$\Rightarrow \frac{e^{a^b}e^{a^c}e^{b^a}e^{b^c}e^{e^a}e^{c^b}}{e^6} \ge a^{(b+c)}b^{(c+a)}c^{(a+b)} \Rightarrow \frac{e^{(a^b+b^c+c^a+a^c+c^b+b^a)}}{a^{(b+c)} \cdot b^{(c+a)} \cdot c^{(a+b)}} \ge e^6$$

Therefore it is to be true.

251. If $a, b, c > 0$ then:

$$\frac{e^a + e^b + e^c}{\sqrt{a} + \sqrt{b} + \sqrt{c}} > 2$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution by Daniel Sitaru-Romania

$$f(x) = e^x - 2\sqrt{x}, f'(x) = e^x - \frac{1}{\sqrt{x}}, f''(x) = e^x + \frac{1}{2x\sqrt{x}} > 0$$

$$f(a) + f(b) + f(c) \overset{Jensen}{\ge} 3f\left(\frac{a+b+c}{3}\right) \leftrightarrow$$

$$\leftrightarrow e^a - 2\sqrt{a} + e^b - 2\sqrt{b} + e^c - 2\sqrt{c} \ge 3e^{\frac{a+b+c}{3}} - 6\sqrt{\frac{a+b+c}{3}} >$$

$$> 3\left(\frac{a+b+c}{3}+1\right)-6\sqrt{\frac{a+b+c}{3}} = 3\left(\sqrt{\frac{a+b+c}{3}}-1\right)^2 \geq 0 \leftrightarrow$$

$$e^a - 2\sqrt{a} + e^b - 2\sqrt{b} + e^c - 2\sqrt{c} > 0 \to \frac{e^a+e^b+e^c}{\sqrt{a}+\sqrt{b}+\sqrt{c}} > 2$$

252. For $\Delta\, ABC$ have $\widehat{BAC} = \frac{\pi}{2}$, put $\widehat{ABC} = \alpha$, $\widehat{ACB} = \beta$ and $\theta \geq 2$

Prove: $\frac{2}{(\sqrt{2})^\theta} \leq sin^\theta\, \alpha + sin^\theta\, \beta \leq 1$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{2}{(\sqrt{2})^\theta} \overset{(i)}{\leq} \sin^\theta\alpha + \sin^\theta\beta \overset{(ii)}{\leq} 1$$

$$A = \frac{\pi}{2} \Rightarrow B + C = \frac{\pi}{2} \Rightarrow \alpha + \beta = \frac{\pi}{2} \Rightarrow \sin\beta = \cos\alpha \quad (1)$$

$$\because \alpha + \beta = \frac{\pi}{2}, \therefore 0 < \alpha, \beta < \frac{\pi}{2} \Rightarrow 0 < \sin\alpha, \sin\beta < 1 \because \theta \geq 2$$

$$\therefore \sin^\theta\alpha \overset{(a)}{\leq} \sin^2\alpha \;\;\&\;\; \sin^\theta\beta \overset{(b)}{\leq} \sin^2\beta = \cos^2\alpha$$

(a)+(b)$\Rightarrow \sin^\alpha\alpha + \sin^\theta\beta \leq \sin^2\alpha + \cos^2\alpha = 1 \Rightarrow$ *(ii) is true (*)*

Let $\alpha = \frac{\pi}{4} + x \;\;\&\;\; \beta = \frac{\pi}{4} - x;\; -\frac{\pi}{4} < x < \frac{\pi}{4}$

$$\therefore \sin\alpha \overset{(2)}{=} \sin\left(\frac{\pi}{4}+x\right) = \frac{\cos x + \sin x}{\sqrt{2}} \;\;\&\;\; \sin\beta \overset{(3)}{=} \cos\left(\frac{\pi}{4}+x\right) = \frac{\cos x - \sin x}{\sqrt{2}}$$

(2), (3) $\Rightarrow \sin^\theta\alpha + \sin^\theta\beta = \frac{1}{(\sqrt{2})^\theta}\left[\{(\cos x+\sin x)^2\}^{\frac{\theta}{2}} + \{(\cos x - \sin x)^2\}^{\frac{\theta}{2}}\right]$

$\overset{(4)}{=} \frac{1}{(\sqrt{2})^\theta}\left[(1+\sin 2x)^{\frac{\theta}{2}} + (1+(-\sin 2x))^{\frac{\theta}{2}}\right]$. *From Bernoulli's inequality, we have,*

$$\forall r \geq 1 \;\&\; \forall t > -1, (1+t)^r \geq 1 + rt \quad (5)$$

$\because -\frac{\pi}{2} < 2x < \frac{\pi}{2}, \therefore -1 < \sin 2x < 1$. *So,* $\because \sin 2x > -1 \;\&\; \frac{\theta}{2} \geq 1$,

$$\therefore (1+\sin 2x)^{\frac{\theta}{2}} \geq 1 + \frac{\theta}{2}\cdot\sin 2x \quad (5)$$

Again, $\because -\sin 2x > -1 \ \& \ \frac{\theta}{2} \geq 1$,

$$\therefore \left(1 + (-\sin 2x)\right)^{\frac{\theta}{2}} \geq 1 + \frac{\theta}{2}(-\sin 2x) \quad \textit{(6)}$$

(5) + (6) along with (4) $\Rightarrow \sin^{\theta}\alpha + \sin^{\theta}\beta \geq \frac{2+\frac{\theta}{2}\sin 2x - \frac{\theta}{2}\sin 2x}{(\sqrt{2})^{\theta}} = \frac{2}{(\sqrt{2})^{\theta}} \Rightarrow$ *(i) is true (*)*

(Proved)

253. For $0 < a < b < 1$. Prove: $\frac{b\sqrt[3]{b}-a\sqrt[3]{a}}{b\sqrt{b}-a\sqrt{a}} \geq \frac{8}{9}$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Ravi Prakash-New Delhi-India

Let $f(x) = x^{\frac{4}{3}}; g(x) = x^{\frac{3}{2}}, a \leq x \leq b.$

By the Cauchy's mean value theorem $\exists c \in (a,b)$*, s.t*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)} \Rightarrow \frac{b^{\frac{4}{3}} - a^{\frac{4}{3}}}{b^{\frac{3}{2}} - a^{\frac{3}{2}}} = \frac{4}{3}\cdot\frac{2}{3}\cdot\frac{c^{\frac{1}{3}}}{c^{\frac{1}{2}}} = \frac{8}{9}\left(\frac{1}{c^{\frac{1}{6}}}\right) > \frac{8}{9} \quad \left[\because c^{\frac{1}{6}} < b^{\frac{1}{6}} < 1\right]$$

Solution 2 by Soumava Chakraborty-Kolkata-India

Given inequality $\Leftrightarrow 9b\sqrt[3]{b} - 8b\sqrt{b} \geq 9a\sqrt[3]{a} - 8a\sqrt{a}.$

Let $f(x) = 9x\sqrt[3]{x} - 8x\sqrt{x} \ \forall x \in (0,1)$

$$f'(x) = 12\left(\sqrt[3]{x} - \sqrt{x}\right) \because 1 > x \therefore x^2 > x^3 \Rightarrow \sqrt[3]{x} > \sqrt{x} \Rightarrow 12\left(\sqrt[3]{x} - \sqrt{x}\right) > 0 \Rightarrow f'(x) > 0$$

$\Rightarrow$ *in* $(0,1), f(x)$ *is an increasing function and* $\because b > a$

$\therefore f(b) > f(a) \Rightarrow 9b\sqrt[3]{b} - 8b\sqrt{b} \geq 9a\sqrt[3]{a} - 8a\sqrt{a}$ *(proved)*

254. If $a, b, c > 0, x, y, z > 1$ then:

$$log_{y^b z^c} x^a + log_{z^b x^c} y^a + log_{x^b y^c} z^a \geq \frac{3a}{b+c}$$

Proposed by D.M.Batinetu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by Marian Ursarescu-Romania

Inequality $\Leftrightarrow a\left(\log_{y^b z^c} x + \log_{z^b x^c} y + \log_{x^b y^c} z\right) \ge \frac{3a}{b+c} \Leftrightarrow$

$$\frac{1}{\log_x y^b z^c} + \frac{1}{\log_y z^b x^c} + \frac{1}{\log_z x^b y^c} \ge \frac{3}{b+c} \Leftrightarrow \frac{1}{b\log_x y + c\log_x z} + \frac{1}{b\log_y z + c\log_y x} +$$

$$+\frac{1}{b\log_z x + c\log_z y} \ge \frac{3}{b+c} \Leftrightarrow$$

$$\frac{\ln x}{b\ln y + c\ln z} + \frac{\ln y}{b\ln z + c\ln x} + \frac{\ln z}{b\ln x + c\ln y} \ge \frac{3}{b+c} \quad (1)$$

Let $\ln x = m, \ln y = n, \ln z = p, m, n, p > 0$

$$(1) \Leftrightarrow \frac{m}{bn+cp} + \frac{n}{bp+cm} + \frac{p}{bm+cn} \ge \frac{3}{b+c} \quad (2)$$

Inequality (2) is a generalization of Nesbitt inequality (to prove let $bn + cp = x_1$, $bp + cm = x_2$ *and* $bm + cn = x_3$ *and use* $x + \frac{1}{\alpha} \ge 2, \forall \alpha > 0$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc} \frac{\log x^a}{\log(y^b z^c)} = a\sum_{cyc} \frac{\log z}{b\log y + c\log z} = a\sum_{cyc} \frac{(\log x)^2}{b\log y\log x + c\log z\log x} \ge$$

$$\ge a\frac{(\log x + \log y + \log z)^2}{(b+c)(\log x\log y + \log y\log z + \log z\log x)} \ge \frac{3a}{b+c} \quad \textit{(proved)}$$

255. **Prove without computer:**

$$e^e\left(1 - e^{\tan e}\right) > e^\pi - \pi^\pi$$

Proposed by Rovsen Pirguliyev-Sumgait-Azerbaidian

Solution by Abdallah Almalih-Damascus-Syria

Put $f(x) = (1 + \tan^2 x)e^{\tan x + e} + \pi x^{\pi-1}$ *where* $x \in [e, \pi]$. *Clearly, we have* $f(x) > 0$.

So, $\int_e^\pi f(x)\,dx > 0$. *But*

$$\int_e^\pi (1 + \tan^2 x)\,e^{\tan x + e} + \pi x^{\pi-1} dx = \left[e^{\tan x + e} + x^\pi\right]_e^\pi = e^{\tan\pi + e} + \pi^\pi - \left(e^{\tan e + e} + e^\pi\right)$$

$$= e^e\left[e^{\tan\pi} - e^{\tan e}\right] - (e^\pi - \pi^\pi) = e^e\left(1 - e^{\tan e}\right) - (e^\pi - \pi^\pi) > 0$$

Hence $e^e\left(1 - e^{\tan e}\right) > e^\pi - \pi^\pi$

256. If $a, b, c \geq 0$ then:

$$3(\sinh a+\sinh b+\sinh c)\geq(a+b+c)\left(\sqrt[3]{\cosh a}+\sqrt[3]{\cosh b}+\sqrt[3]{\cosh c}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

Let $f(x)=\cosh x\ \forall x\geq 0;\ f'(x)=\sinh x=\frac{e^x-e^{-x}}{2}=\frac{e^{2x}-1}{2e^x}$

$$=\frac{(e^x+1)(e^x-1)}{2e^x}\geq 0\ (\because e^x\geq 1\ as\ x\geq 0)$$

$\therefore f(x)$ *is an increasing* f^n*, WLOG, we may assume* $a\geq b\geq c$

Then, as $\cosh x$ *is an increasing* $f^n, \forall x\geq 0$*,* $\therefore \cosh a\geq\cosh b\geq\cosh c$

$$\Rightarrow \sqrt[3]{\cosh a}\geq\sqrt[3]{\cosh b}\geq\sqrt[3]{\cosh c}\ \therefore \sum a\sqrt[3]{\cosh a}\overset{Chebyshev}{\geq}\frac{1}{3}\left(\sum a\right)\left(\sum\sqrt[3]{\cosh a}\right)$$

$$\Rightarrow\left(\sum a\right)\left(\sum\sqrt[3]{\cosh a}\right)\overset{(1)}{\leq}3\sum\left(a\sqrt[3]{\cosh a}\right)$$

(1) $\Rightarrow$ *it suffices to show:* $\sum\sinh a\geq\sum\left(a\sqrt[3]{\cosh a}\right)$ *(i)*

For 2 positive $m\ \&\ n$*, let*

$$A=A(m,n)=\frac{m+n}{2}, G=G(m,n)=\sqrt{mn}\ \ \&\ L=L(m,n)=\frac{m-n}{\ln m-\ln n}$$

We have, $\sqrt[3]{G^2A}\overset{(a)}{<}L$ *(E.B. Leach & M.C. Scholander)*

Now, $A(e^x,e^{-x})=\cosh x, G(e^x,e^{-x})=1, L(e^x,e^{-x})=\frac{e^x-e^{-x}}{2x}=\frac{\sinh x}{x}$

$\therefore$ *applying (a), we get,* $\sqrt[3]{\cosh x}<\frac{\sinh x}{x}, \forall x>0\ \therefore a,b,c>0, \sinh a>a\sqrt[3]{\cosh a}$ *etc*

$$\Rightarrow\sum\sinh h>\sum a\sqrt[3]{\cosh a}\quad (2)$$

For $a=0, \sinh a=0\ \&\ a\sqrt[3]{\cosh a}=0\Rightarrow\sinh a=a\sqrt[3]{\cosh a}$

Similarly, for $b\ \&\ c=0, \sinh b=b\sqrt[3]{\cosh b}\ \&\sinh c=c\sqrt[3]{\cosh c}$

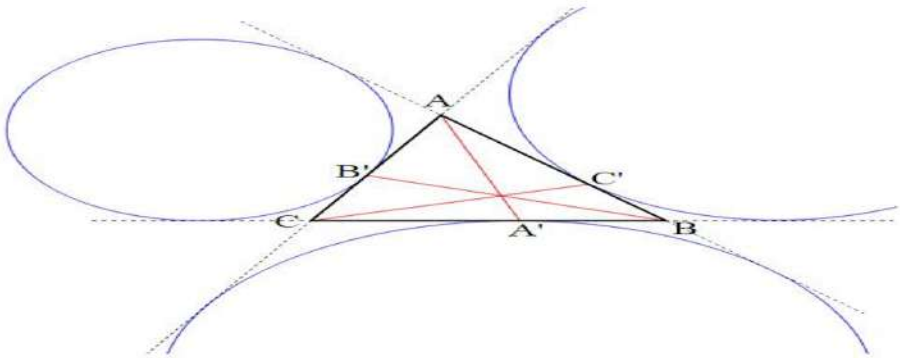
$\therefore$ *when* $a=b=c=0$*,*

$$\sum\sinh a=\sum a\sqrt[3]{\cosh a}\ (=0)\quad (3)$$

Combining (2) & (3), (i) is true (Proved)

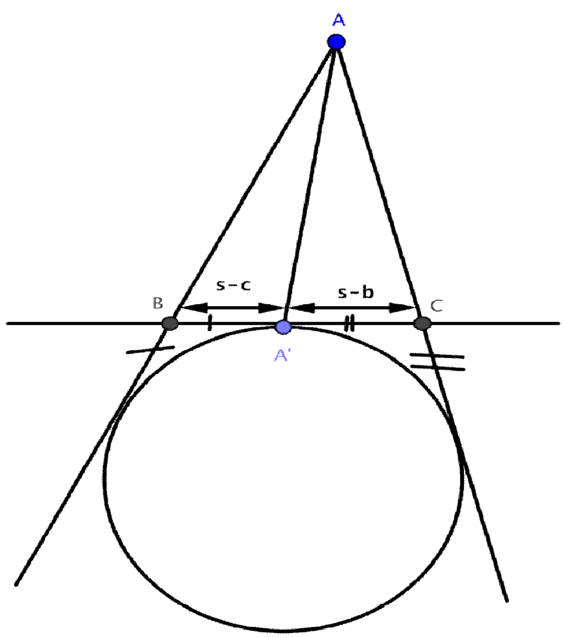
257. Let a, b, c, R, r, s be usual notations for side lengths, circumradius, inradius, semiperimeter of ABC. A', B', C' be tangency points of sides BC, AC and AB with excircles. (Their intersection is Nagel point of ABC). Prove the following inequality holds:

$$a^3[AA']^2 + b^3[BB']^2 + c^3[CC']^2 \geq 4s^2Rr$$



Proposed by Abdilkadir Altintas-Afyonkarashisar-Turkey

Solution by Rajsekhar Azaad-India



Lemma; $\sum a^3 = S[2S^2 - 6r^2 - 12Rr]$ ***(i)***

Now, applying Stewart's theorem: $(s-b)c^2 + (s-c)b^2 = a[(AA')^2 + (s-c)(s-b)]$

$$\Rightarrow a[AA']^2 = s(b-c)^2 + as(s-a) \Rightarrow a^3[AA']^2 = s[a^2(b-c)^2 + a^3(s-a)] =$$

$$= s[a^2(b^2+c^2) - 2a^2bc + sa^3 - a^4]$$

$$a^3[AA']^2 + b^3[BB']^2 + c^3[CC']^2 = s\left[2\sum a^2b^2 - \sum a^4 - 2abc \cdot \sum a + s\sum a^3\right]$$

$$= s\left[16\Delta^2 - 2 \cdot 4Rrs \cdot 2s + s\sum a^3\right]$$

$$= s[16r^2s^2 - 16Rrs^2 + s^2(2s^2 - 6r^2 - 12Rr)];\text{ from (i)}$$

$$= s^3[16r^2 - 16Rr + 2s^2 - 6r^2 - 12Rr] = s^3[2s^2 + 10r^2 - 28Rr]$$

$$\geq s^3[2(16Rr - 5r^2) + 10r^2 - 28Rr];\text{ (Gerretsen)}$$

$$= s^3[32Rr - 10r^2 + 10r^2 - 28Rr] = s^3 \cdot 4Rr = 4s^2Rr\ \text{ (proved)}$$

258. If $a, b, c \in (0,1), 2(a+b+c) = 3$ then:

$$\sum(3 + (log_a c)^4)\left(3 + \frac{1}{(a+b)^4}\right) \geq 48$$

Proposed by Daniel Sitaru – Romania

Solution by Serban George Florin-Romania

$$3 + \frac{1}{(a+b)^4} = 1 + 1 + 1 + \frac{1}{(a+b)^4} \overset{(Ma\geq Mg)}{\geq} 4\sqrt[4]{1 \cdot 1 \cdot 1 \cdot \frac{1}{(a+b)^4}} = \frac{4}{a+b}$$

$$3 + (\log_a c)^4 = 1 + 1 + 1 + (\log_a c)^4 \overset{(Ma\geq Mg)}{\geq} 4\sqrt[4]{1 \cdot 1 \cdot 1(\log_a c)^4} = 4\log_a c$$

$$\sum(3 + (\log_a c)^4)\left(1 + \frac{1}{(a+b)^4}\right) \geq \sum 16 \cdot \frac{\log_a c}{a+b} = 16\sum\frac{\log_a c}{a+b}$$

$$16\sum\frac{\log_a c}{a+b} \overset{(Ma\geq Mg)}{\geq} 16 \cdot 3\sqrt[3]{\frac{\prod \log_a c}{\prod(a+b)}} =$$

$$= \frac{48}{\sqrt[3]{(a+b)(b+c)(a+c)}} \overset{(Ma\geq Mg)}{\geq} \frac{48}{\frac{a+b+b+c+a+c}{3}} = \frac{48}{\frac{2(a+b+c)}{3}} = \frac{48}{\frac{3}{3}} = 48$$

$$a, b, c \in (0,1) \Rightarrow \log_a c, \log_b c, \log_a b > 0$$

259. If $x, y, z, t \geq 1$ then:

$$(\ln xy)(\ln^2 x + \ln^2 y - \ln x \ln y - \ln z \ln t) \geq (\ln zt)(\ln x \ln y + \ln z \ln t - \ln^2 z - \ln^2 t)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

Let $a = \ln x, b = \ln y, c = \ln z, d = \ln t \ (a, b, c, d \geq 0)$. *Using this substitution, given inequality* $\Leftrightarrow (a+b)(a^2+b^2-ab-cd) \geq (c+d)(ab+cd-c^2-d^2)$

$$\Leftrightarrow a^3+b^3+c^3+d^3 \overset{(1)}{\geq} abc+bcd+cda+dab$$

Now, $a^3+b^3+c^3 = 3abc+(a+b+c)(a^2+b^2+c^2-ab-bc-ca) \overset{(2)}{\geq} 3abc$, $\forall a, b, c \geq 0$

Similarly, $b^3+c^3+d^3 \overset{(3)}{\geq} 3bcd, \forall b, c, d \geq 0, c^3+d^3+a^3 \overset{(4)}{\geq} 3cda, \forall c, d, a \geq 0$ &

$$d^3+a^3+b^3 \overset{(5)}{\geq} 3dab, \forall d, a, b \geq 0$$

(2)+(3)+(4)+(5) $\Rightarrow a^3+b^3+c^3+d^3 \geq abc+bcd+cda+dab \Rightarrow$ *(1) is true*

(Proved)

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

Because $\ln^3 x + \ln^3 y + \ln^3 z + \ln^3 t \geq \ln x \ln y \ln z + \ln x \ln y \ln t + \ln x \ln z \ln t +$
$+ \ln y \ln z \ln t \Rightarrow \ln^3 x + \ln^3 y + \ln^3 z + \ln^3 t + \ln y \ln^2 x + \ln x \ln^2 y + \ln t \ln^2 z +$
$+ \ln z \ln^2 t \geq \ln x \ln y \ln z + \ln x \ln y \ln t + \ln x \ln z \ln t + \ln y \ln z \ln t +$
$+ \ln y \ln^2 x + \ln x \ln^2 y + \ln t \ln^2 z + \ln z \ln^2 t \Rightarrow \ln xy \ln^2 x + \ln xy \ln^2 y + \ln zt \ln^2 z +$
$+ \ln zt \ln^2 t \geq \ln zt \ln x \ln y \ln zt \ln z \ln t + \ln xy \ln x \ln xy + \ln xy \ln z \ln t \Rightarrow$
$\Rightarrow \ln xy (\ln^2 x + \ln^2 y - \ln x \ln y - \ln z \ln t) \geq \ln zt (\ln x \ln y + \ln z \ln t - \ln^2 z - \ln^2 t)$

Therefore it is to be true.

260. If $1 \leq x < y$ then:

$$\frac{(y^5-x^5)(y^7-x^7)(y^9-x^9)}{(y^6-x^6)(y^8-x^8)(y^{10}-x^{10})} < \frac{21}{32}$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India,

Let $1 \leq x < y, n, m \in \mathbb{N}, n < m$. By the Cauchy's mean value theorem:

$$\frac{y^n - x^n}{y^m - x^m} = \frac{n\alpha^{n-1}}{m\alpha^{m-1}} \text{ for some } \alpha \in (x, y)$$

$$= \frac{n}{m}\alpha^{n-m} = \frac{n}{m} \cdot \frac{1}{\alpha^{m-n}}$$

$$< \frac{n}{m} \; [\because \alpha > x \geq 1 \Rightarrow \alpha > 1]$$

$$\therefore \frac{y^5 - x^5}{y^6 - x^6} \cdot \frac{y^7 - x^7}{y^8 - x^8} \cdot \frac{y^9 - x^9}{y^{10} - x^{10}} < \left(\frac{5}{6}\right)\left(\frac{7}{8}\right)\left(\frac{9}{10}\right) = \frac{21}{32}$$

Generalization by Sagar Kumar-India

$$\Psi = \prod_{r=0}^{n}\left(\frac{y^{2r+1} - x^{2r+1}}{y^{2r+2} - x^{2r+2}}\right) < \frac{1}{4^{n+1}}\binom{n+1}{2n+2}, 1 \leq x < y$$

$$\lim_{n\to\infty}(n+1)\Psi \leq \frac{1}{\sqrt{\pi}}$$

261. If $0 \leq a, b, c \leq 1$ then:

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ e^{a^2} & e^{b^2} & e^{c^2} \end{vmatrix} < e - 1$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$LHS = be^{c^2} - ce^{b^2} + ce^{a^2} - ae^{c^2} + ae^{b^2} - be^{a^2}$$

Case 1: Exactly one variable among $a, b, c = 0$. WLOG we may assume $a = 0$.

$$\therefore LHS = be^{c^2} - ce^{b^2} + c - b = b\left(e^{c^2} - 1\right) + c\left(1 - e^{b^2}\right)$$

$$\leq e^{c^2} - 1 + c\left(1 - e^{b^2}\right)\left(\because b \leq 1 (\& b \geq 0) \& e^{c^2} - 1 \geq 0 \text{ as } c \geq 0\right)$$

$$\overset{(1)}{\leq} e - 1 + c(1 - b^2) \; (\because c \leq 1 \;\&\; e^{c^2} \text{ is increasing on } [0, \infty))$$

Now, $e^{b^2} > 1 + b^2 \Rightarrow -e^{b^2} < -1 - b^2 \Rightarrow 1 - e^{b^2} < -b^2 \leq 0 \;\&\; \because c > 0$

$\therefore c(1-e^{b^2})<0$ ***(2); (1), (2)*** $\Rightarrow LHS < e-1$. ***Analogous proof is evident if we assume*** $b=0$ ***or*** $c=0$.

Case 2: Exactly 2 variables among $a,b,c=0$. ***WLOG we may assume*** $a=b=0$. ***Then, LHS***$=-c+c=0<e-1$. ***Analogous proof is evident if we assume*** $b=c=0$ ***or*** $c=a=0$.

Case 3: $a,b,c>0$

$$LHS = be^{c^2}-bc\left(\frac{e^{b^2}}{b}\right)+ce^{a^2}-ca\left(\frac{e^{c^2}}{c}\right)+ae^{b^2}-ab\left(\frac{e^{a^2}}{a}\right)$$

$$\le (be+ce+ae)-\left(ab\cdot\frac{e^{a^2}}{a}+bc\cdot\frac{e^{b^2}}{b}+ca\cdot\frac{e^{c^2}}{c}\right)$$

($\because c\le 1$ ***etc, &*** e^{c^2} ***etc is increasing on*** $[0,\infty)$ ***&*** $b\ge 0$ ***etc)***

$$\overset{(3)}{\le} 3e-\left(ab\cdot\frac{e^{a^2}}{a}+bc\cdot\frac{e^{b^2}}{b}+ca\cdot\frac{e^{c^2}}{c}\right)\ (\because a,b,c\le 1)$$

Let $f(x)=\frac{e^{x^2}}{x}, \forall x\in(0,1]; f'(x)=\frac{(2x^2-1)e^{x^2}}{x^2}$ & $f''(x)=\frac{(4x^4-2x^2+2)e^{x^2}}{x^3}$

$f'(x)=0$ ***iff*** $x=\frac{1}{\sqrt{2}}$ ***&*** $f''\left(\frac{1}{\sqrt{2}}\right)>0$

$\therefore f(x)$ ***attains a minima at*** $x=\frac{1}{\sqrt{2}}>0$ ***&*** $\because f(x)$ ***never attains a maxima in*** $(0,1]$,

$$\therefore f_{\min}=f\left(\frac{1}{\sqrt{2}}\right)\Rightarrow \frac{e^{x^2}}{x}\ge\frac{\sqrt{e}}{\frac{1}{\sqrt{2}}}=\sqrt{2e}\Rightarrow -\frac{e^{x^2}}{x}\le-\sqrt{2e}\ \ (4)$$

Now, $ab\le 1\Rightarrow ab-1\le 0\Rightarrow\left(-\frac{e^{a^2}}{a}\right)(ab-1)\le 0\Rightarrow -ab\cdot\frac{e^{a^2}}{a}\le-\frac{e^{a^2}}{a}\underset{(a)}{\overset{by\,(4)}{\le}}-\sqrt{2e}$

Similarly, $-bc\cdot\frac{e^{b^2}}{b}\overset{(b)}{\le}-\sqrt{2e}\ \&-ca\cdot\frac{e^{c^2}}{c}\overset{(c)}{\le}-\sqrt{2e}$

(a)+(b)+(c) along with (3) $\Rightarrow LHS\le 3e-3\sqrt{2e}\overset{?}{<}e-1$

$$\Leftrightarrow (2e+1)^2\overset{?}{<}18e\Leftrightarrow 4e^2+1-14e\overset{?}{<}0\ \ (5)$$

Now, $e\le\frac{11}{4}\Rightarrow 4e^2<\frac{121}{4}\to$ ***(i). Also,*** $e>\frac{5}{2}\Rightarrow 14e>35\Rightarrow -14e<-35$ ***(ii)***

(i)+(ii)$\Rightarrow 4e^2+1-14e<\frac{121}{4}+1-35=\frac{121+4-140}{4}=\frac{-15}{4}<0\Rightarrow$ ***(5) is true***

$\therefore LHS < e - 1$

Case 4: $a = b = c = 0$

$LHS = 0 < e - 1 \therefore$ ***combining all the cases,*** $LHS < e - 1$ ***(proved)***

262. If $0 \leq a, b, c, d \leq 2$ then:

$$\frac{9a}{1 + bcd} + \frac{9b}{1 + cda} + \frac{9c}{1 + dab} + \frac{9d}{1 + abc} + 9e^{abcd} \leq 8 + 9e^{16}$$

Proposed by Daniel Sitaru – Romania

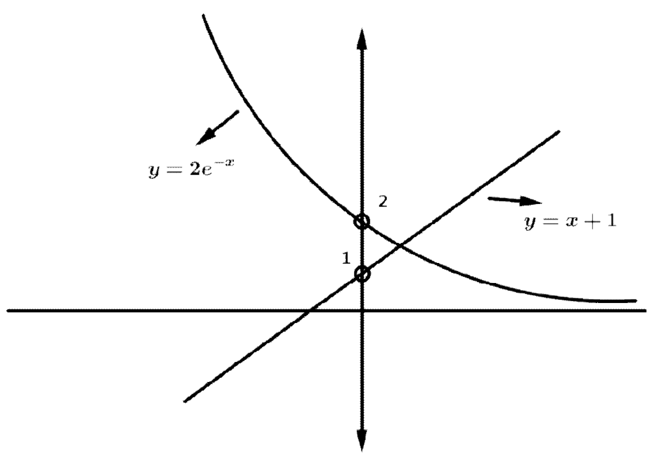
Solution by Soumava Chakraborty-Kolkata-India

We shall show that: $\frac{9a}{1+bcd} + \frac{9}{4}e^{abcd} \overset{(1)}{\leq} 2 + \frac{9}{4}e^{16}$

LHS of (1) $\overset{(2)}{\leq} \frac{18}{1+bcd} + \frac{9}{4}e^{2bcd}$ $(\because a \leq 2 \,\&\, a \geq 0)$

Let $f(x) = \frac{18}{1+x} + \frac{9}{4}e^{2x}; f'(x) = \frac{9e^{2x}(x+1)^2 - 36}{2(x+1)^2}$ & $f''(x) = 9e^{2x} + \frac{36}{(x+1)^3}$

Now, $f'(x) = 0 \Rightarrow e^x(x+1) = 2 \Rightarrow x + 1 = 2e^{-x}$ ***(3)***

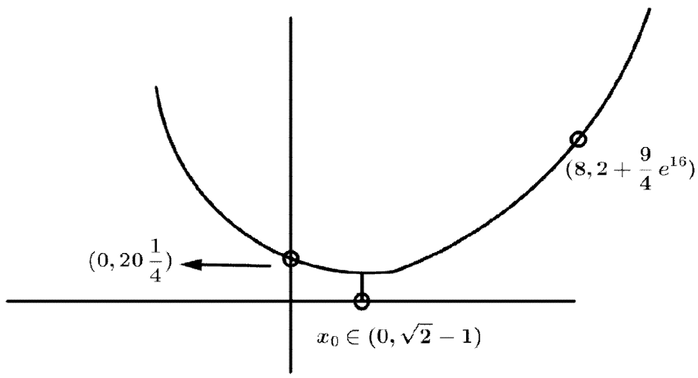


Also, $e^x = \frac{2}{x+1} \geq x + 1 \Rightarrow x \leq \sqrt{2} - 1 \therefore$ ***(3) has only root & it*** $\in (0, \sqrt{2} - 1) \Rightarrow$

$\Rightarrow f'(x) = 0$ ***at one & only one value*** $x_0 \in (0, \sqrt{2} - 1)$

$\& \because f''(x) > 0, \forall x \geq 0, \therefore f''(x_0) > 0 \Rightarrow f(x)$ ***attains a minima at*** $x_0 \in (0, \sqrt{2} - 1)$

Also, $f(0)=18+\frac{9}{4}=20\frac{1}{4}$ & $f(8)=\frac{18}{1+8}+\frac{9}{4}e^{16}=2+\frac{9}{4}e^{16}>f(0)$ & $\because f(x)$ *never attains a maxima in* $[0,8]$, $\therefore$ *the graph of* $f(x)$ *in* $[0,8]$ *should be like below:*



Hence, it is clear that in $[0,8]$, $f(x)_{\max}=f(8)=2+\frac{9}{4}e^{16}\Rightarrow\frac{18}{1+x}+\frac{9}{4}e^{2x}\le 2+\frac{9}{4}e^{16}$

$\Rightarrow\frac{18}{1+bcd}+\frac{9}{4}e^{2bcd}\le 2+\frac{9}{4}e^{16}$ *(putting* $x=bcd$ & $bcd\le 8$*)*

$$\left.\begin{array}{r}\Rightarrow\frac{9a}{1+bcd}+\frac{9}{4}e^{abcd}\overset{by\,(2)}{\le}2+\frac{9}{4}e^{16}\\ Similarly,\frac{9b}{1+cda}+\frac{9}{4}e^{abcd}\le 2+\frac{9}{4}e^{16}\\ \frac{9c}{1+dab}+\frac{9}{4}e^{abcd}\le 2+\frac{9}{4}e^{16}\\ \frac{9d}{1+abc}+\frac{9}{4}e^{abcd}\le 2+\frac{9}{4}e^{16}\end{array}\right\}$$

Adding the last 4, we obtain the desired inequality (proved)

263. For $b>a\ge 1\wedge n\in\mathbb{N}\wedge n\ge 2$. Prove:

$$\prod_{k=1}^{n}\frac{b^{2k+1}-a^{2k+1}}{b^{2k}-a^{2k}}\ge\frac{(2n+1)!}{4^n(n!)^2}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution by Soumitra Mandal-Chandar Nagore-India

We know $x^{2k}\ge x^{2k-1}$ *for all* $x\ge 1$

$$\int_a^b x^{2k} \ge \int_a^b x^{2k-1}\,dx \Rightarrow \frac{b^{2k+1}-a^{2k+1}}{b^{2k}-a^{2k}} \ge \frac{2k+1}{2k}$$

$$\Rightarrow \prod_{k=1}^{n}\left(\frac{b^{2k+1}-a^{2k+1}}{b^{2k}-a^{2k}}\right) \ge \prod_{k=1}^{n}\left(\frac{2k+1}{2k}\right) = \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n+1)}{2^n\cdot n!}$$

$$= \frac{1\cdot 2\cdot 3\cdot \ldots \cdot 2n\cdot (2n+1)}{2^n\cdot n!(2\cdot 4\cdot 6\cdot \ldots \cdot 2n)} = \frac{(2n+1)!}{4^n(n!)^2}$$ ***(proved)***

264. In acute ΔABC the following relationship holds:

$$\frac{1}{\pi}(A\tan^{\alpha}A + B\tan^{\alpha}B + C\tan^{\alpha}C) \ge \sqrt{3^{\alpha}}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution by Daniel Sitaru-Romania

WLOG: $A \le B \le C \to \tan A \le \tan B \le \tan C \to \tan^{\alpha}A \le \tan^{\alpha}B \le \tan^{\alpha}C$

$$\sum A\tan^{\alpha}A \overset{CEBYSHEV}{\ge} \frac{1}{3}\sum A \sum \tan^{\alpha}A = \frac{\pi}{3}\sum \tan^{\alpha}A \leftrightarrow$$

$$\leftrightarrow \frac{1}{\pi}\sum A\tan^{\alpha}A \ge \frac{1}{3}\sum \tan^{\alpha}A \overset{JENSEN}{\ge} \frac{1}{3}\cdot 3\tan^{\alpha}\left(\frac{A+B+C}{3}\right) = \tan^{\alpha}\frac{\pi}{3} = 3^{\frac{\alpha}{2}}$$

265. If $a,b,c \in (0,1], x,y > 0$ then:

$$\frac{3}{2}\log(x^2+y^2) > (a+b+c)\log x + (3-a-b-c)\log y$$

Proposed by Daniel Sitaru – Romania

Solution by Khanh Hung Vu-Ho Chi Minh-Vietnam

If $a,b,c \in (0;1], x,y > 0$ ***then*** $\frac{3}{2}\log(x^2+y^2) > (a+b+c)\log x + (3-a-b-c)\log y$ ***(1)***

Case 1. $\log\left(\frac{x}{y}\right) > 0$

We have (1) $\Rightarrow (a+b+c-3)\cdot(\log x - \log y) + 3\log x < \frac{3}{2}\log(x^2+y^2) \Rightarrow$

$$\Rightarrow (a+b+c-3)\cdot\log\left(\frac{x}{y}\right)+3\log x<\frac{3}{2}\log(x^2+y^2)$$

We have $\log\left(\frac{x}{y}\right)>0$ ***and*** $a+b+c-3\le 0$ ***so*** $(a+b+c-3)\cdot\log\left(\frac{x}{y}\right)\le 0$

$$\Rightarrow (a+b+c-3)\cdot\log\left(\frac{x}{y}\right)+3\log x\le 3\log x$$

On the other hand, we have $\frac{3}{2}\log(x^2+y^2)>\frac{3}{2}\log(x^2)=3\log x$***. So,***

$(a+b+c-3)\cdot\log\left(\frac{x}{y}\right)+3\log x<\frac{3}{2}\log(x^2+y^2)\Rightarrow$ ***(1) true***

Case 2. $\log\left(\frac{x}{y}\right)<0$

We have (1) $\Rightarrow (a+b+c)\cdot(\log x-\log y)+3\log y<\frac{3}{2}\log(x^2+y^2)\Rightarrow$

$$\Rightarrow (a+b+c)\cdot\log\left(\frac{x}{y}\right)+3\log y<\frac{3}{2}\log(x^2+y^2)$$

We have $\log\left(\frac{x}{y}\right)<0$ ***and*** $a+b+c>0$ ***so,*** $(a+b+c)\cdot\log\left(\frac{x}{y}\right)<0\Rightarrow$

$$\Rightarrow (a+b+c)\cdot\log\left(\frac{x}{y}\right)+3\log y<3\log y$$

On the other hand, we have $\frac{3}{2}\log(x^2+y^2)>\frac{3}{2}\log(y^2)=3\log y$

So $(a+b+c)\cdot\log\left(\frac{x}{y}\right)+3\log y<\frac{3}{2}\log(x^2+y^2)\Rightarrow$ ***(1) true***

Therefore, we have QED.

266. If $\alpha\ge 2$ then $\sum_{k=1}^{\infty}(\xi(\alpha k)-1)\le\frac{3}{4}$ where ξ denote the Riemann function.

Proposed by Mihály Bencze – Romania

Solution by Omran Kouba-Damascus-Syria

For $\alpha\ge 2$ ***prove that*** $\sum_{k\ge 1}(\zeta(\alpha k)-1)\le\frac{3}{4}$ ***where*** ζ ***is the Riemann zeta function.***

Clearly the function $\alpha\to\sum_{k\ge 1}(\zeta(\alpha k)-1)$ ***is decreasing on*** $[2,\infty)$ ***so***

$$\sum_{k\ge1}(\zeta(\alpha k)-1)\le\sum_{k\ge1}(\zeta(2k)-1)=\sum_{k\ge1}\left(\sum_{j\ge2}\frac{1}{j^{2k}}\right)=\sum_{j\ge2}\left(\sum_{k\ge1}\frac{1}{j^{2k}}\right)=\sum_{j\ge2}\frac{1}{j^2-1}$$

$$=\frac{1}{2}\sum_{j\ge2}\left(\frac{2j-1}{j(j-1)}-\frac{2j+1}{(j+1)j}\right)=\frac{3}{4}$$

267. **If $n\in\mathbb{N}^*, m\in\mathbb{R}^*, x_k>0, k\in\overline{1,n}$ then:**

$$\sum_{k=1}^{n}\left((\tan^{-1}x_k)^{m+1}+\left(\tan^{-1}\frac{1}{x_k}\right)^{m+1}\right)\ge\frac{n\cdot\pi^{m+1}}{2^{2m+1}}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution by Marian Ursărescu – Romania

$a^{m+1}+b^{m+1}\ge\frac{(a+b)^{m+1}}{2^m}, \forall a,b>0, n\in\mathbb{N}^*$ ***(demonstration by induction)***

$$\Rightarrow(\arctan x_k)^{m+1}+\left(\arctan\frac{1}{x_k}\right)^{m+1}\ge\frac{\left(\arctan x_k+\arctan\frac{1}{x_k}\right)^{m+1}}{2^m}\quad (1)$$

But $\arctan x+\arctan\frac{1}{x}=\frac{\pi}{2}$ ***(2)***, $\forall x>0$, ***because function*** $f:(0,+\infty)\to\mathbb{R}$,

$f(x)=\arctan x+\arctan\frac{1}{x}, f'(x)=0\Rightarrow f(x)=const=k$, ***but*** $f(1)=\frac{\pi}{2}\Rightarrow k=\frac{\pi}{2}$

From (1)+(2) $\Rightarrow(\arctan x_k)^{m+1}+\left(\arctan\frac{1}{x_k}\right)^{m+1}\ge\frac{\left(\frac{\pi}{2}\right)^{m+1}}{2^m}\Rightarrow$

$$(\arctan x_k)^{m+1}+\left(\arctan\frac{1}{x_k}\right)^{m+1}\ge\frac{\pi^{m+1}}{2^{2m+1}}\quad (3)$$

From (3) $\Rightarrow\sum_{k=1}^{n}\left[(\arctan x_k)^{m+1}+\left(\arctan\frac{1}{x_k}\right)^{m+1}\right]\ge\frac{n\pi^{m+1}}{2^{2m+1}}$

268. **For $a,b\in[1;+\infty)\wedge m,n\in\mathbb{N}^*\wedge m\ge n\ge2$. Prove:**

$$\frac{\sum_{k=0}^{m}a^{m-k}b^k}{\sum_{l=0}^{n}a^{n-l}b^l}\ge\frac{m+1}{n+1}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Ravi Prakash-New Delhi-India

If $a = b$: $\sum_{k=0}^{m} a^{m-k}b^k = (m+1)a^m$ *and*

$$\sum_{k=0}^{n} a^{n-k}b^k = (n+1)a^n \therefore \frac{\sum_{k=0}^{m} a^{m-k}b^k}{\sum_{k=0}^{n} a^{n-k}b^k} = \frac{m+1}{n+1}a^{m-n} \geq \frac{m+1}{n+1}$$

$[\because a \geq 1, m-n \geq 0]$. *If* $a \neq b$,

$$\sum_{k=0}^{m} a^{m-k}b^k = \frac{a^m\left(\left(\frac{b}{a}\right)^{m+1}-1\right)}{\frac{b}{a}-1} = \frac{b^{m+1}-a^{m+1}}{b-a}$$

and

$$\sum_{k=0}^{n} a^{n-k}b^k = \frac{b^{n+1}-a^{n+1}}{b-a} \therefore S = \frac{\sum_{k=0}^{m} a^{m-k}b^k}{\sum_{k=0}^{n} a^{n-k}b^k} = \frac{b^{m+1}-a^{m+1}}{b^{n+1}-a^{n+1}} = \frac{(m+1)c^m}{(n+1)c^n}$$

[By Cauchy's Mean Value Theorem] for some $c$ycling between a and b.

$\Rightarrow S = \frac{m+1}{n+1}c^{m-n} \geq \frac{m+1}{n+1}$ *as* $1 \leq a < c < b$ *or* $1 \leq b < c < a$.

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\frac{\sum_{k=0}^{m} a^{m-k}b^k}{\sum_{l=0}^{n} a^{n-l}b^l} \geq \frac{m+1}{n+1} \Leftrightarrow \frac{\left(\sum_{k=0}^{m} a^{m-k}b^k\right)}{(b-a)\left(\sum_{l=0}^{n} a^{n-l}b^l\right)} \geq \frac{m+1}{n+1}$$

$$\Leftrightarrow \frac{\frac{b^{m+1}-a^{m+1}}{m+1}}{\frac{b^{n+1}-a^{n+1}}{n+1}} \geq 1 \Leftrightarrow \int_a^b x^m dx \geq \int_a^b x^n\, dx \Leftrightarrow x^{m-n} \geq 1 \Leftrightarrow m \geq n, \text{ which is true}$$

269. If $a, b, c, d, e, f > 0$ then:

$$\frac{a+b+c}{\sqrt[3]{abc}\left(\frac{d}{e}+\frac{e}{f}+\frac{f}{d}\right)} \leq \frac{\sqrt[3]{def}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}{d+e+f}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Michael Sterghiou-Greece

For every triad of positive real numbers x, y, z we have:

$x^2y + x^2y + z^2x \geq 3x\sqrt[3]{(xyz)^2}$. *Cyclic application and addition gives:*

$$x^2y + y^2z + z^2x \geq (x + y + z) \cdot \sqrt[3]{(xyz)^2} \text{ or}$$

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq (x + y + z) \cdot (xyz)^{-\frac{1}{3}} \text{ or } \frac{x+y+z}{(xyz)^{\frac{1}{3}}\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)} \leq 1.$$

The reverse fraction is obviously ≥ 1. *For the triads* a, b, c *and* f, d, e *we have*

$$\frac{a+b+c}{\sqrt[3]{abc}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)} \leq 1 \leq \frac{\sqrt[3]{def} \cdot \left(\frac{d}{e}+\frac{e}{f}+\frac{f}{d}\right)}{d+e+f}.$$ *We are done!*

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For all $a, b, c, d, e, f > 0$, *we let* $a = m^3, b = n^3, c = p^3, d = x^3, e = y^3, f = z^3$.

Consider $\frac{(a+b+c)}{\sqrt[3]{abc}\left(\frac{d}{e}+\frac{e}{f}+\frac{f}{d}\right)} \leq \frac{\sqrt[3]{def}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}{(d+e+f)}$. *Iff* $\frac{(a+b+c)(d+e+f)}{\sqrt[3]{abc}\sqrt[3]{def}} \leq \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{d}{e}+\frac{e}{f}+\frac{f}{d}\right)$.

Iff $\left(\sqrt[3]{\frac{a^2}{bc}}+\sqrt[3]{\frac{b^2}{ca}}+\sqrt[3]{\frac{c^2}{ab}}\right)\left(\sqrt[3]{\frac{d^2}{ef}}+\sqrt[3]{\frac{e^2}{fd}}+\sqrt[3]{\frac{f^2}{de}}\right) \leq \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{d}{e}+\frac{e}{f}+\frac{f}{d}\right)$.

Iff $\left(\frac{m^2}{np}+\frac{n^2}{pm}+\frac{p^2}{mn}\right)\left(\frac{x^2}{yz}+\frac{y^2}{zx}+\frac{z^2}{xy}\right) \leq \left(\frac{m^3}{n^3}+\frac{n^3}{p^3}+\frac{p^3}{m^3}\right)\left(\frac{x^3}{y^3}+\frac{y^2}{z^2}+\frac{z^2}{x^3}\right)$ *and it is to be true*

because $\frac{x^3}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^3} \geq \frac{x^2}{yz}+\frac{y^2}{xz}+\frac{z^2}{xy}$ *and* $\frac{m^2}{n^3}+\frac{n^3}{p^3}+\frac{p^3}{m^3} \geq \frac{m^2}{np}+\frac{n^2}{pm}+\frac{p^2}{mn}$. *Therefore it is to be true.*

270. In ΔABC the following relationship holds:

$$\frac{\left((a+1)(b+1)(c+1)\right)^{\frac{1}{2}}}{e^{a+b+c}} < 1$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution by Daniel Sitaru-Romania

$$\begin{cases} e^{b+c} > e^a > a+1 \\ e^{c+a} > e^b > b+1 \\ e^{a+b} > e^c > c+1 \end{cases} \rightarrow \prod e^{b+c} > \prod (a+1) \rightarrow e^{2a+2b+2c} > \prod (a+1) \rightarrow$$

$$\rightarrow e^{a+b+c} > \sqrt{(a+1)(b+1)(c+1)} \rightarrow \frac{\left((a+1)(b+1)(c+1)\right)^{\frac{1}{2}}}{e^{a+b+c}} < 1$$

271.

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right),$$

$$R_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log\left(n+\frac{1}{2}\right)$$

Prove that:

$$\frac{1}{23(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}, n\in\mathbb{N}^*$$

Proposed by D.W.de Temple-AMM

Solution by Omran Kouba-Damascus-Syria

First, let us define $a_n = \ln\left(n+\frac{1}{2}\right)-\ln\left(n-\frac{1}{2}\right)-\frac{1}{n}$*. Note that*

$$a_n = \int_{-\frac{1}{2}}^{-\frac{1}{2}}\left(\frac{1}{n+t}-\frac{1}{n}\right)dt = -\frac{1}{n}\int_{-\frac{1}{2}}^{-\frac{1}{2}}\frac{t}{t+n}dt = -\frac{1}{n}\left(\int_0^{\frac{1}{2}}\frac{-t}{-t+n}dt+\int_0^{\frac{1}{2}}\frac{t}{t+n}dt\right) =$$

$$= \frac{1}{n}\int_0^{\frac{1}{2}}\frac{2t^2}{n^2-t^2}dt. \text{ So,}$$

$$\frac{1}{n}\int_0^{\frac{1}{2}}\frac{2t^2}{n^2-0}dt < a_n < \frac{1}{n}\int_0^{\frac{1}{2}}\frac{2t^2}{n^2-\frac{1}{4}}dt$$

Equivalently $\frac{1}{12n^3} < a_n < \frac{1}{12n\left(n^2-\frac{1}{4}\right)}$*. Using the trivial inequalities:*

$$\frac{1}{n^2}-\frac{1}{(n+1)^2} < \frac{2}{n^3}, \frac{2}{n\left(n^2-\frac{1}{4}\right)} < \frac{1}{\left(n-\frac{1}{2}\right)^2}-\frac{1}{\left(n+\frac{1}{2}\right)^2}$$

We conclude that $\frac{1}{24}\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right) < a_n < \frac{1}{24}\left(\frac{1}{\left(n-\frac{1}{2}\right)^2}-\frac{1}{\left(n+\frac{1}{2}\right)^2}\right)$*. Consequently*

$$\frac{1}{24(n+1)^2} < \sum_{k=n+1}^{\infty} a_k < \frac{1}{24\left(n+\frac{1}{2}\right)^2} \quad (1). \text{ Now,}$$

$$\sum_{k=1}^{n} a_k = \ln\left(n+\frac{1}{2}\right)+\ln 2-\sum_{k=1}^{n}\frac{1}{k}$$

So, $\sum_{k=1}^{n} a_k = \ln 2 - \gamma$***. Thus,*** $\sum_{k=n+1}^{n} a_k \sum_{k=1}^{n} \frac{1}{k} - \ln\left(n+\frac{1}{2}\right) - \gamma$***. Combining this with (1) we get:***

$$\frac{1}{24(n+1)^2} < \sum_{k=1}^{n} \frac{1}{k} - \ln\left(n+\frac{1}{2}\right) - \gamma < \frac{1}{24\left(n+\frac{1}{2}\right)^2} \quad (1)$$

Which is stronger than the proposed inequality.

Observation by editor:

The inequality (1) was also discovered by the romanian mathematician ***TANASE NEGOI*** *few years ago and it was published as a Math Note in GMA.*

272. $\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n\right)$

Find an increasing order for:

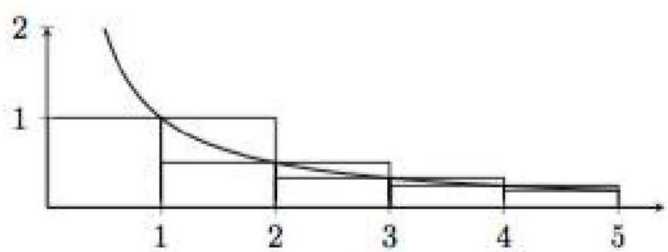
$$\Omega_1 = \gamma^{\sqrt{\pi e}}, \Omega_2 = \pi^{\sqrt{e\gamma}}, \Omega_3 = e^{\sqrt{\gamma\pi}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Emre Tuvay-Turkey

From Riemann sum of the area of curve $y = \frac{1}{x}$ ***we have the followings for lower bound.***

$$\sum_{k=1}^{n} \frac{1}{k} > \int_1^{n+1} \frac{1}{x}dx > \int_1^{n} \frac{1}{x}dx = \ln n > 0$$



As for upper bound again from Riemann sum keeping $y = \frac{1}{x}$ ***function's values above the rectangles and adding the area of 1st rectangle we have***

$1 + \int_1^n \frac{1}{x}dx > \sum_{k=1}^{n} \frac{1}{k}$***, hence,*** $0 < \gamma < 1$***. For convergence, showing the sequence***

$U_n = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right)$ ***monotonic decreasing should suffice.***

$U_{n+1} - U_n = \frac{1}{n+1} - \ln(n+1) + \ln n$ ***again, by checking the area under*** $y = \frac{1}{x}$ ***curve***

for $x = n$ ***and*** $x = n+1$ ***we see that***

$$\int_n^{n+1} \frac{1}{x} dx > \frac{1}{n+1} \Rightarrow \ln(n+1) - \ln n > \frac{1}{n+1}$$

hence, $U_{n+1} - U_n < 0 \Rightarrow U_n = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right)$ ***is monotonic decreasing.***

Therefore,

$\lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right)$ ***converges to*** γ ***where*** $0 < \gamma < 1$***. So,*** $0 < \gamma < 1 < e < \pi$***.***

Now, for ordering of $\Omega_1 = \gamma^{\sqrt{\pi e}}, \Omega_2 = \pi^{\sqrt{e\gamma}}, \Omega_3 = e^{\sqrt{\gamma\pi}}$***. Considering a generic case,***

$b^{\sqrt{a}}$ ***and*** $a^{\sqrt{b}}$ ***(where*** $a, b \in \Re_{\geq 0}$ ***and*** $b > a$***) which can be written as*** $\left(b^{\frac{1}{\sqrt{b}}}\right)^{\sqrt{a}\sqrt{b}}$ ***and***

$\left(a^{\frac{1}{\sqrt{a}}}\right)^{\sqrt{b}\sqrt{a}}$ ***respectively. From checking function,*** $f(x) = \left(x^{\frac{1}{\sqrt{x}}}\right)$,

$f'(x) = x^{\frac{1}{\sqrt{x}}-1}\left(\frac{1}{\sqrt{x}} - \frac{\ln x}{2\sqrt{x}}\right)$***. Critical point*** $f'(x) = \frac{x^{\frac{1}{\sqrt{x}}}}{x^{\frac{3}{2}}}\left(1 - \frac{\ln x}{2}\right) = 0 \Rightarrow x = e^2$***.***

$$f'(x) = \begin{cases} > 0, when\ x < e^2; \\ = 0, when\ x = e^2; \\ < 0, when\ x > e^2; \end{cases} \quad \text{so, } f(x) = \begin{cases} increasing, when\ x < e^2; \\ maxvalue, when\ x = e^2; \\ decreasing, when\ x > e^2; \end{cases} \text{ since,}$$

$\gamma < e < \pi < e^2 \Rightarrow f(\gamma) < f(e) < f(\pi)$ ***hence*** $\Omega_1 < \Omega_3 < \Omega_2$

Solution 2 by Marian Ursarescu-Romania

Let $f: (0, +\infty) \to \mathbb{R}, f(x) = \frac{\ln x}{x}; f'(x) = \frac{\frac{1}{x}\cdot\sqrt{x} - \ln x \cdot \frac{1}{2\sqrt{x}}}{x^2} = \frac{2 - \ln x}{2x^2\sqrt{x}}, f'(x) = 0 \Rightarrow x = e^2$

x	0 $\qquad\qquad$ e^2 $\qquad\qquad$ $+\infty$
$f'(x)$	+ + + + + + + + + 0 − − − − − − − − −
$f(x)$	↗ $f(e^2)$ ↘

We have $\gamma < e < \pi < e^2 \Rightarrow f(\gamma) < f(e) < f(\pi)$

From $f(\gamma) < f(e) \Rightarrow \frac{\ln\gamma}{\sqrt{\gamma}} < \frac{\ln e}{\sqrt{e}} \Rightarrow \sqrt{e} < \ln\gamma < \sqrt{\gamma}\ln e \Rightarrow \sqrt{e\pi}\ln\gamma < \sqrt{\gamma\pi}\cdot\ln e^{\sqrt{\gamma}} \Rightarrow$

$$\Rightarrow \ln\gamma^{\sqrt{e\pi}} < \ln e^{\sqrt{\gamma\pi}} \Rightarrow \gamma^{\sqrt{e\pi}} < e^{\sqrt{\gamma\pi}} \Rightarrow \Omega_1 < \Omega_3 \quad (1)$$

From $f(e) < f(\pi) \Rightarrow \frac{\ln e}{\sqrt{e}} < \frac{\ln\pi}{\sqrt{\pi}} \Rightarrow \sqrt{\pi}\ln e < \sqrt{e}\ln\pi$

$$\Rightarrow \sqrt{\pi\gamma}\ln e < \sqrt{e\gamma}\ln\pi \Rightarrow \ln e^{\sqrt{\pi\gamma}} < \ln\pi^{\sqrt{e\gamma}} \Rightarrow\Rightarrow e^{\sqrt{\pi\gamma}} < \pi^{\sqrt{e\gamma}} \Rightarrow \Omega_3 < \Omega_2 \quad (2)$$

From (1)+(2)$\Rightarrow \Omega_1 < \Omega_3 < \Omega_2$

273. If $a,b,c > 1, n\in\mathbb{N}, n\geq 2$ then:

$$\sum\frac{\sqrt[n]{a^n+1}}{a^n+1}+\sum\frac{\sqrt[n]{a^n-1}}{a^n-1} > \frac{6}{\sqrt[3]{a^{n-1}b^{n-1}c^{n-1}}}$$

Proposed by Daniel Sitaru – Romania

Solution by Le Van-Ho Chi Minh-Vietnam

With $x > 1$ *and* $n\geq 2$*, building the function:* $f(x) = \frac{\sqrt[n]{x+1}}{x+1} - \frac{\sqrt[n]{x}}{x} = (x+1)^{\frac{1}{n}-1} - x^{\frac{1}{n}-1} \Rightarrow$

$$\Rightarrow f'(x) = \left(\frac{1}{n}-1\right)\left[(x+1)^{\frac{1}{n}-2} - x^{\frac{1}{n}-2}\right] = \left(\frac{1-n}{n}\right)\left[\frac{1}{(x+1)^{\frac{2n-1}{n}}} - \frac{1}{x^{\frac{2n-1}{n}}}\right] > 0$$

Accordingly, $f(x)$ *is a positive function which gives:*

$$f(x) > f(x-1) \Leftrightarrow \frac{\sqrt[n]{x+1}}{x+1} + \frac{\sqrt[n]{x-1}}{x-1} > \frac{2\sqrt[n]{x}}{x} = \frac{2}{\sqrt[n]{x^{n-1}}}$$

Therefore, QED is obtained by AM-GM inequality as:

$$\sum\left(\frac{\sqrt[n]{a^n+1}}{a^n+1} + \frac{\sqrt[n]{a^n-1}}{a^n-1}\right) > \frac{2}{a^{n-1}} + \frac{2}{b^{n-1}} + \frac{2}{c^{n-1}} \geq \frac{6}{\sqrt[3]{a^{n-1}b^{n-1}c^{n-1}}}$$

274. If $a,b\in\mathbb{R}, |3a+4b+2| = 5$ then:

$$a^2+b^2+4b+7\geq 4a$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Chris Kyriazis-Athens-Greece

The distance of $M(a,b)$ *from the line:* $3x+4y+2=0$ *is* 1

$$\left(d(M,\varepsilon) = \frac{|3a+4b+2|}{\sqrt{3^2+4^2}} = 1\right)$$

I have to prove that: $a^2+b^2+4b+7 \geq 4a$. *It suffices to prove that:*

$$(a-2)^2+(b+2)^2 \geq 1 \quad \textit{(1)}$$

But its easy to prove that the point $N(2,-2)$ *belong to the straight line* $3x+4y+2=0$.

So, (1) holds becomes: $d(M,\varepsilon) \leq d(M,N)$

Solution 2 by Serban George Florin-Romania

$$|3a+4b+2| = 5 \Rightarrow 3a+4b+2 \in \{5,-5\}$$

I. If $3a+4b+2=5 \Rightarrow 3a+4b=3, a=\frac{3-4b}{3}, a^2+b^2+4b+7 \geq 4a,$

$$\frac{(3-4b)^2}{9}+b^2+4b+7 \geq \frac{12-16b}{3}; 25b^2+60b+36 \geq 0, (5b+6)^2 \geq 0 \textit{ (true)}$$

II. If $3a+4b+2=-5, 3a+4b=-7, a=\frac{-7-4b}{3}, a^2+b^2+4b+7 \geq 4a,$

$$\frac{(-7-4b)^2}{9}+b^2+4b+7 \geq \frac{-28-16b}{3},$$

$$49+56b+16b^2+9b^2+36b+63+84+48b \geq 0$$

$$25b^2+140b+196 \geq 0, (5b+14)^2 \geq 0 \textit{ (true)}$$

275. If $x,y,z,t \in \left(0,\frac{\pi}{2}\right)$ then:

$$64 \cdot \cos x \cdot \cos z \cdot \sin y \cdot \sin t \cdot \sin(x-y) \cdot \sin(z-t) \leq 1$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Michael Sterghiou-Greece

$$64 \cos x \cos z \sin y \sin t \sin(x-y) \sin(z-t) \leq \quad \textit{(1)}$$

By AM-GM: LHS of (1) $\leq 64\left[\frac{\overbrace{\cos x+\cos z+\sin y+\sin t+\sin(x-y)+\sin(z-t)}^{A\,(nominator)}}{6}\right]^6$

It suffices to prove that $64\left[\frac{A}{6}\right]^6 \leq 1$ *or* $A \leq \left[\frac{1}{2^6}\cdot 2^6 \cdot 3^6\right]^{\frac{1}{6}} = 3$

As $\cos x = \sin\left(\frac{\pi}{2}-x\right)$, $\cos z = \sin\left(\frac{\pi}{2}-2\right)$ *and the function*

$f(\vartheta) = \sin\vartheta$ *is concave over*

$\left(0,\frac{\pi}{2}\right)$ $[f''(\vartheta) = \sin\vartheta < 0]$ *we can apply the Jensen inequality for the variables*

$\frac{\pi}{2}-x, \frac{\pi}{2}-2, y, t, x-y, z-t$ *of the function* $f(\vartheta) = \sin\vartheta$*. We get:*

$$A \le 6\sin\left(\frac{\frac{\pi}{2}-x+\frac{\pi}{2}-z+y+t+x-y+z-t}{6}\right) = 6\cdot\sin\frac{\pi}{6} = 6\cdot\frac{1}{2} = 3$$

We are done!

Solution 2 by Marian Ursarescu-Romania

We must show this:

$$\cos x\cos z\cdot\sin y\cdot\sin t\,(\sin x\cos y-\cos x\sin y)(\sin z\cot t-\cos z\sin t) \le \frac{1}{64} \quad (1)$$

We show this: $\cos x\sin y\,(\sin x\cos y-\cos x\sin y) \le \frac{1}{8}$ *(2)*

$$\left.\begin{array}{c}\cos x = a, \sin y = b\ (2) \Leftrightarrow ab\left(\sqrt{(1-a^2)(1-b^2)}-ab\right) \le \frac{1}{8}\\ But\ \sqrt{(1-a^2)(1-b^2)} \le \frac{2-a^2-b^2}{2}\end{array}\right\} \Rightarrow$$

$$ab\left(\frac{2-a^2-b^2}{2}-ab\right) \le \frac{1}{8} \Leftrightarrow ab(2-a^2-b^2-2ab) \le \frac{1}{4} \Leftrightarrow$$

$$4ab(2-(a+b)^2) \le 1 \quad (3)$$

But $(a+b)^2 \ge 4ab \Rightarrow -(a+b)^2 \le -4ab$ *(4)*

From (3)+(4)$\Rightarrow 4ab(2-4ab) \le 1 \Leftrightarrow 8ab-16a^2b^2 \le 1 \Leftrightarrow$

$16a^2b^2-8ab+1 \ge 0 \Leftrightarrow (4ab-1)^2 \ge 0$ *true (equality for* $a = b = \frac{1}{2}$*).*

Similarly: $\cos z\sin t\sin(z-t) \le \frac{1}{8}$ *(5)*

From (2)+(5)$\Rightarrow \cos x\cos z\cdot\sin y\cdot\sin t\cdot\sin(x-y)\sin(z-t) \le 1$*, with equality for* $x = z = \frac{\pi}{3}$ *and* $y = t = \frac{\pi}{6}$*.*

276. Let $n \in \mathbb{N} \wedge n \ge 2$ and $\theta \ge 1$. Prove:

$$\sum_{k=0}^{n}\left(C_n^k\right)^{\theta}>(n+1)\left(\frac{2^n}{n+1}\right)^{\theta}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution by Soumitra Mandal-Chandar Nagore-India

We know, $(1+x)^n=1+C_1^n x+C_2^n x^2+\dots+x^n\Rightarrow 2^n=\sum_{k=0}^{n}C_k^n$

$$\frac{1}{n+1}\sum_{k=1}^{n}(C_k^n)^{\theta}\ge\left(\frac{1}{n+1}\sum_{k=0}^{n}C_k^n\right)^{\theta}=\left(\frac{2^n}{n+1}\right)^{\theta}\Bigg|\Rightarrow\sum_{k=0}^{n}(C_k^n)^{\theta}\ge(n+1)\left(\frac{2^n}{n+1}\right)^{\theta}$$

(proved)

277. If $a_1,a_2,\dots,a_n>0, n\in\mathbb{N}^*$ then:

$$\frac{1}{2}\left(a_1^{2018}+a_2^{2018}+\dots+a_n^{2018}+n\right)\ge a_1^9+a_2^9+\dots+a_n^9$$

Proposed by Asude Ebrar Kiziloglu

Solution by proposer

WLOG: $a_1a_2\cdot\ldots\cdot a_n=1\to\ln(a_1a_2\cdot\ldots\cdot a_n)=0$

Let be $f:(0,\infty)\to\mathbb{R}, f(x)=x^{2018}-2x^9+1-2000\ln x$,

$$f'(x)=2018x^{2017}-18x^8-\frac{2000}{x}$$

$$f'(x)=\frac{1}{x}(2018x^{2018}-18x^9-2000)=\frac{2018x^{2018}-2018x^9+2000x^9-2000}{x}$$

$$f'(x)=\frac{2018x^9(x^{2009}-1)+2000(x^9-1)}{x}=\frac{x-1}{x}\left(2018x^9\sum_{k=0}^{2008}x^k+2000\sum_{k=0}^{8}x^k\right)$$

$$\min f=f(1)=0\to x^{2018}+1\ge 2x^9+2000\ln x$$

$$\sum_{k=1}^{n}a_i^{2018}+n\ge 2\sum_{k=1}^{n}a_i^9+2000\left(\sum_{i=1}^{n}\ln a_i\right)==2\sum_{k=1}^{n}a_i^9+2000\ln(a_1a_2\cdot\ldots\cdot a_n)=2\sum_{k=1}^{n}a_i^9$$

278. If $x,y,z\in\left(0,\frac{\pi}{2}\right)$ then:

$$\frac{x(\cos x+\cos z)+y(\cos y+\cos x)+z(\cos z+\cos y)}{x(\cos x+\cos y)+y(\cos y+\cos z)+z(\cos z+\cos x)}\geq 1$$

Proposed by Daniel Sitaru – Romania

Solution by Serban George Florin – Romania

$$x\cos x+x\cos z+y\cos y+y\cos x+z\cos z+z\cos y\geq x\cos x+x\cos y+y\cos y+$$
$$+y\cos z+z\cos z+z\cos x$$
$$x\cos z+y\cos x+z\cos y\geq x\cos y+y\cos z+z\cos x$$
$$x(\cos z-\cos y)+y(\cos x-\cos z)-z(\cos x-\cos y)\geq 0$$
$$x(\cos z-\cos y)+y(\cos x-\cos z)-z(\cos x-\cos z+\cos z-\cos y)\geq 0$$
$$x(\cos z-\cos b)+b(\cos x-\cos z)-z(\cos x-\cos z)-z(\cos z-\cos y)\geq 0$$
$$(y-z)(\cos x-\cos z)+(x-z)(\cos z-\cos y)\geq 0$$
$$(x-z)(\cos z-\cos y)\geq(z-y)(\cos x-\cos z)$$

If $x=z\Rightarrow 0\geq 0$ ***true. If*** $z=y\Rightarrow 0\geq 0$ ***true.***

If $x\neq z, z\neq y, x-z>0$ ***and*** $z-y>0\Rightarrow y<z<x\Rightarrow\cos z<\cos y,\cos x<\cos z$

$$\Rightarrow(x-z)(\cos y-\cos z)\leq(z-y)(\cos z-\cos x)$$
$$\frac{\cos y-\cos z}{z-y}\leq\frac{\cos z-\cos x}{x-z}\Big|\cdot(-1)$$
$$\frac{\cos y-\cos z}{y-z}\geq\frac{\cos z-\cos x}{z-x};\quad f(x)=\cos x$$

T. Lagrange $[x,z],[y,z],f'(x)=-\sin x$

$$-\sin c_1\geq-\sin c_2,\ \sin c_1\leq\sin c_2$$

$(\exists)c_1\in(y,z),(\exists)c_2\in(z,x), y<z<x\Rightarrow c_1<c_2\Rightarrow\sin c_1<\sin c_2$ ***true.***

279. For $0<a<b<1\wedge m,n\in\mathbb{N}\wedge m\geq n\geq 2$. Prove:

$$\frac{b\sqrt[m]{b}-a\sqrt[m]{b}}{b\sqrt[n]{b}-a\sqrt[n]{a}}\geq\frac{mn+n}{mn+m}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

By MVT, $\frac{b^{1+\frac{1}{m}}-a^{1+\frac{1}{m}}}{b^{1+\frac{1}{n}}-a^{1+\frac{1}{n}}}=\frac{(b-a)\left(a+\frac{1}{m}\right)c^{\frac{1}{m}}}{(b-a)\left(1+\frac{1}{n}\right)c^{\frac{1}{n}}}$ *for some* $c\in(a,b)$

$$=\frac{n(m+1)c^{\frac{1}{m}}}{m(n+1)c^{\frac{1}{n}}}\geq\frac{n(m+1)}{m(n+1)}\Leftrightarrow c^{\frac{1}{m}}\geq c^{\frac{1}{n}}\Leftrightarrow c^n\geq c^m\Leftrightarrow c^{m-n}\leq 1\quad \textit{(true)}$$

$\because 0<a<c<b<1\ \&\ m-n\geq 0$ *(proved). Equality when* $m=n$.

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\frac{b\sqrt[m]{b}-a\sqrt[m]{a}}{b\sqrt[n]{b}-a\sqrt[n]{a}}\geq\frac{mn+n}{mn+m}\Leftrightarrow\frac{m}{m+1}\left(b\sqrt[m]{b}-a\sqrt[m]{a}\right)\geq\frac{n}{n+1}\left(b\sqrt[n]{b}-a\sqrt[n]{a}\right)$$

$\Leftrightarrow\int_a^b\sqrt[m]{x}\,dx\geq\int_a^b\sqrt[n]{x}\,dx\Leftrightarrow x^n\geq x^m\Leftrightarrow\left(\frac{1}{x}\right)^{m-n}\geq 1$*, which is true* $\because 1\geq x>0$

Hence Proved

280. If $x\in\left(0,\frac{\pi}{2}\right), n\in\mathbb{N}, n\geq 3$ then:

$$\prod_{k=3}^{n}\sqrt[k]{\sin^k x+\cos^k x}\geq 2^{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\frac{n+1}{2}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

For $k\geq 3$*. Let* $f_k(x)=(\sin^k x+\cos^k x)^{\frac{1}{k}}, 0<x<\frac{\pi}{2};\ \ln f_k(x)=\frac{1}{k}\ln(\sin^k x+\cos^k x)$

$$\frac{1}{f_k(x)}f_k'(x)=\frac{1}{k}\cdot\frac{k[\sin^{k-1}x\cos x-\cos^{k-1}x\sin x]}{\sin^k x+\cos^k x}\Rightarrow$$

$$\Rightarrow f_k'(x)=\frac{(\sin x\cos x)(\sin^{k-2}x-\cos^{k-2}x)}{\sin^k x+\cos^k x}f_k(x)$$

$$f_k'(x)<0 \text{ for } 0<x<\frac{\pi}{4}$$

$$=0 \text{ for } x=\frac{\pi}{4}, >0 \text{ for } \frac{\pi}{4}<x<\frac{\pi}{2}$$

$\therefore f_k(x)$ *attains its minimum value at* $x=\frac{\pi}{4}\Rightarrow f_k(x)\geq\left(\frac{2}{2^{\frac{k}{2}}}\right)^{\frac{1}{k}}=2^{\frac{1}{k}-\frac{1}{2}}\Rightarrow$

$\Rightarrow\prod_{k=3}^{n}f_k(x)\geq 2^{a_n}$ *where* $a_n=\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}-\frac{n-2}{2}=1+\frac{1}{2}+\frac{1}{3}+\cdots-\frac{1}{n}-\frac{n+1}{2}$

Thus $\prod_{k=3}^{n}(\sin^k x+\cos^k x)^{\frac{1}{k}}\geq 2^{1+\frac{1}{2}+\cdots+\frac{1}{n}-\frac{n+1}{2}}$

Solution 2 by Michael Sterghiou-Greece

$$\prod_{k=3}^{n}\sqrt[k]{\sin^k x+\cos^k x}\geq 2^{1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n}-\frac{n+1}{2}}\quad (1)$$

We will show that $\sin^k x+\cos^k x\geq 2\left(\frac{\sqrt{2}}{2}\right)^k$*. With this LHS of (1) becomes*

$$\geq\prod_{3}^{n}\left[2-\left(\frac{\sqrt{2}}{2}\right)^k\right]^{\frac{1}{k}}=\prod_{3}^{n}\left(2^{1-\frac{k}{2}}\right)^{\frac{1}{k}}=\prod_{3}^{n}\frac{2^{\frac{1}{k}}}{2^{\frac{1}{2}}}=\frac{2\sum_3^n\frac{1}{k}}{2^{\frac{n-2}{2}}}=\frac{2^{1+\frac{1}{2}}\cdot 2^{\sum_3^n\frac{1}{k}}}{2^{\frac{3}{2}}\cdot 2^{\frac{n}{2}-\frac{3}{2}}}=$$

$$=\frac{2^{1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n}}}{2^{\frac{n+1}{2}}}=2^{1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n}-\frac{n+1}{2}}.$$ *We are done.*

Lemma: Consider the function $f(x)=\sin^k x+\cos^k x$; $[k\geq 3]$ *over* $\left(0,\frac{\pi}{2}\right)$

$f'(x)=k\cos x\sin x\left[\sin^{k-2}x-\cos^{k-2}x\right]$*. But we know that for* $0<x\leq\frac{\pi}{4}$

$\sin x<\cos x\to\sin^{k-2}x-\cos^{k-2}x<0$ *hence* $f'(x)\leq 0$ *and* $f(x)\downarrow$*. Likewise in*

$\left[\frac{\pi}{4},\frac{\pi}{2}\right]$ $f(x)\uparrow$ *and because* $f'\left(\frac{\pi}{4}\right)=0$; $\lim_{x\to 0_+}f(x)=\lim_{x\to\frac{\pi}{2}-}f(x)=1$ *we have*

$$\min f(x)=f\left(\frac{\pi}{4}\right)=2\left(\frac{\sqrt{2}}{2}\right)^k$$

281. If $x,y,z\in\mathbb{R}$ then:

$$\frac{1}{e^{\sin^2 x}}+\frac{1}{e^{\sin^2 y}}+\frac{1}{e^{\sin^2 z}}+\frac{1}{e^{\cos^2 x}}+\frac{1}{e^{\cos^2 y}}+\frac{1}{e^{\cos^2 z}}>3\left(\frac{1}{2}+\frac{\sqrt{e}}{e}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Michael Sterghiou-Greece

$\sum_{\substack{cyc\\x,y,z}}\frac{1}{e^{\sin^2 x}}+\frac{1}{e^{\cos^2 y}}>3\left(\frac{1}{2}+\frac{\sqrt{e}}{e}\right)$ *(1) LHS of (1) can be written* $\frac{1}{e}\cdot\sum_{cyc}e^{\sin^2 x}+e^{\cos^2 y}$ *by*

using $\sin^2 x+\cos^2 y=1$*. Consider the function* $f(t)=e^{\sin^2 t}+e^{\cos^2 t}$; $f(t)$ *has*

period π *so it is enough to examine the domain* $[0,\pi]$ *for extremes.*

$f'(t) = 2\sin t\cos t\left(e^{\sin^2 t} - e^{\cos^2 t}\right)$ ***with roots (critical points)*** $0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$***. Examinig those as well as the ends of the domain*** $0, \pi$ ***we can easily see that*** $\min f(t)$ ***in*** $[0, \pi] = 2\sqrt{e}$ ***at the points*** $\frac{\pi}{4}, \frac{3\pi}{4}$***. This is global minimum.***

Therefore LHS ***of (1)*** $\geq 3 \cdot 2\sqrt{e}$ ***(*) and as***

$6\sqrt{e} - 3e\left(\frac{1}{2} + \frac{\sqrt{e}}{e}\right) = \frac{3}{2}(2 - \sqrt{e})\sqrt{e} > 0$ ***we are done. (*) we apply***

$\frac{1}{e^{\sin^2 t}} + \frac{1}{e^{\cos^2 t}} = \frac{1}{e}\left(e^{\sin^2 t} + e^{\cos^2 t}\right) \geq \frac{1}{e} \cdot 2\sqrt{e}$ ***for*** $t = x, y, z$ ***and add.***

Solution 2 by Marian Ursarescu-Romania

$\frac{1}{e^{\sin^2 x}} + \frac{1}{e^{\cos^2 x}} \geq 2\sqrt{\frac{1}{e^{\sin^2 x + \cos^2 x}}} = \frac{2}{\sqrt{e}} \Rightarrow \frac{1}{e^{\sin^2 x}} + \frac{1}{e^{\cos^2 x}} \geq \frac{2}{\sqrt{e}} > \frac{1}{2} + \frac{1}{\sqrt{e}}$ ***(because*** $\frac{1}{\sqrt{e}} > \frac{1}{2} \Leftrightarrow 2 > \sqrt{e} \Leftrightarrow 4 > e$***);*** $\frac{1}{e^{\sin^2 y}} + \frac{1}{e^{\cos^2 y}} > \frac{1}{2} + \frac{1}{\sqrt{e}}$; $\frac{1}{e^{\sin^2 t}} + \frac{1}{e^{\cos^2 z}} > \frac{1}{2} + \frac{1}{\sqrt{e}} \Rightarrow$

$$\sum\left(\frac{1}{e^{\sin^2 x}} + \frac{1}{e^{\cos^2 x}}\right) > 3\left(\frac{1}{2} + \frac{1}{\sqrt{e}}\right)$$

Solution 3 by Amit Dutta-Jamshedpur-India

$$AM \geq GM$$

$$\frac{1}{e^{\sin^2 x}} + \frac{1}{e^{\cos^2 x}} \geq 2\sqrt{\frac{1}{e^{\sin^2 x + \cos^2 x}}} \geq \frac{2}{\sqrt{e}} \quad (1)$$

Similarly, $\frac{1}{e^{\sin^2 y}} + \frac{1}{e^{\cos^2 y}} \geq \frac{2}{\sqrt{e}}$ ***(2)***

$$\frac{1}{e^{\sin^2 z}} + \frac{1}{e^{\cos^2 z}} \geq \frac{2}{\sqrt{e}} \quad (3)$$

Adding (1), (2), (3):

$$LHS = \frac{1}{e^{\sin^2 x}} + \frac{1}{e^{\sin^2 y}} + \frac{1}{e^{\sin^2 z}} + \frac{1}{e^{\cos^2 x}} + \frac{1}{e^{\cos^2 y}} + \frac{1}{e^{\cos^2 z}} \geq \frac{6}{\sqrt{e}} \geq \frac{3}{\sqrt{e}} + \frac{3}{\sqrt{e}} > \frac{3}{\sqrt{e}} + \frac{3}{2} >$$

$$> 3\left(\frac{1}{2} + \frac{3}{\sqrt{e}}\right)$$

Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

Since $e \approx 2.71828, \ldots < 4$***. Hence*** $4e > e^2 \Rightarrow e > \frac{e^2}{4} \Rightarrow \sqrt{e} > \frac{e}{2} \Rightarrow 2\sqrt{e} > \frac{e}{2} + \sqrt{e} \Rightarrow$

$$\Rightarrow b\sqrt{e} = 6\sqrt[6]{e^3} > 3\left(\frac{e}{2} + \sqrt{e}\right) \Rightarrow$$

$$\Rightarrow 6\sqrt[6]{e^{\sin^2 x+\cos^2 x}\cdot e^{\sin^2 y+\cos^2 y}\cdot e^{\sin^2 z+\cos^2 z}} > 3\left(\frac{e}{2}+\sqrt{e}\right)\Rightarrow$$

$$\Rightarrow e^{\sin^2 x}+e^{\cos^2 x}+e^{\sin^2 y}+e^{\cos^2 y}+e^{\sin^2 z}+e^{\cos^2 z} > 3\left(\frac{e}{2}+\sqrt{e}\right)\Rightarrow$$

$$\Rightarrow \frac{e}{e^{\cos^2 x}}+\frac{e}{e^{\sin^2 x}}+\frac{e}{e^{\cos^2 y}}+\frac{e}{e^{\sin^2 y}}+\frac{e}{e^{\cos^2 z}}+\frac{e}{e^{\sin^2 z}} > 3\left(\frac{e}{2}+\sqrt{e}\right)\Rightarrow$$

$$\Rightarrow \frac{1}{e^{\sin^2 x}}+\frac{1}{e^{\sin^2 y}}+\frac{1}{e^{\sin^2 z}}+\frac{1}{e^{\cos^2 x}}+\frac{1}{e^{\cos^2 y}}+\frac{1}{e^{\cos^2 z}} > 3\left(\frac{1}{2}+\frac{\sqrt{e}}{e}\right)$$

Therefore it is to be true.

282. If $a, b \in \mathbb{N}, a, b \geq 2$ then:

$$(2a-1)(3a-1)\cdot\ldots\cdot(a^2-1)+(2b-1)\cdot(3b-1)\cdot\ldots\cdot(b^2-1) > 2\sqrt{\frac{a!\cdot b!\cdot a^2\cdot b^b}{ab\cdot\sqrt[ab]{a^b\cdot b^a}}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Michael Sterghiou-Greece

$$(2a-1)(3a-1)\cdot\ldots\cdot(a^2-1)+(2b-1)(3b-1)\cdot\ldots\cdot(b^2-1)\geq 2\sqrt{\frac{a!b!a^ab^b}{ab\sqrt[ab]{a^bb^a}}}\quad (1)$$

Applying AM-GM in (1) we obtain:

LHS of (1) $\geq 2\cdot\{[\prod_2^a(ka-1][\prod_2^a(\alpha b-1)]\}^{\frac{1}{2}}$ *so it suffices to show that:*

$$\prod_{k=2}^{a}(ka-1)\cdot\prod_{\alpha-2}^{a}(\alpha b-1)\geq a!\,\alpha^{\alpha-1-\frac{1}{\alpha}}\cdot b!\,b^{b-1-\frac{1}{b}}$$

From symmetry we only need to prove that: $\prod_2^\alpha(ka-1)\geq a!\,a^{\alpha-1-\frac{1}{\alpha}}$ *or*

$\prod_{k=2}^{a}\left(a-\frac{1}{k}\right)\geq a^{a-1}\cdot a^{-\frac{1}{\alpha}}$*. The last inequality can be written as*

$\prod_2^a\left[a\left(1-\frac{1}{ka}\right)\right]\geq a^{\alpha-1}\cdot\alpha^{-\frac{1}{\alpha}}$*. Taking logharithms we get:*

$\sum_a^{2a}\ln a+\sum_2^a\ln\left(1-\frac{1}{ka}\right)\geq(\alpha-1)\ln a-\frac{1}{a}\ln a$ *and as* $\sum_2^\alpha\ln a=(\alpha-1)\ln a$

It suffices to prove: $\sum_{k=2}^{\alpha}\ln\left(1-\frac{1}{ka}\right)\geq-\frac{1}{a}\ln a$*. Using the well known inequaity*

$\ln(1+x) \geq \frac{x}{1+x}$. *If* $x > -1$ *and for* $x = -\frac{1}{ka}$ *we get:* $\sum_{k=2}^{a} \ln\left(1-\frac{1}{ka}\right) \geq \sum_{2}^{a} \frac{1}{1-ka} \overset{?}{\geq} -\frac{1}{a}\ln a$. *Now, we need to prove that* $\sum_{2}^{a} \frac{1}{ka-1} \leq \frac{1}{a}\ln a$ *or* $\sum_{2}^{a} \frac{a}{ka-1} \leq \ln a$ *(2).*

We observe that:

$\frac{a}{ka-1} < \frac{1}{k-1}$ *so (2) becomes* $\sum_{2}^{a} \frac{1}{k-1} \leq \ln a$. *But* $\sum_{2}^{a} \frac{1}{k-1} < \sum_{1}^{a} \frac{1}{k} = \left(\sum_{2}^{a} \frac{1}{k-1}\right) + \frac{1}{a}$ *(3). It is easy to show that for any* $n \in \mathbb{N}$ $\ln n < \sum_{k=1}^{n} \frac{1}{k} < (\ln n) + 1$ *(using the funciton* $\frac{1}{x}$ *(*)).*

So,

(3) $\rightarrow \sum_{2}^{a} \frac{1}{k-1} < \ln a + \left(1-\frac{1}{a}\right) < \ln a$ $[a > 2]$. *We are done!*

(*) Harmonic series $\sum_{1}^{n} \frac{1}{k} < 1 + \int_{2}^{n+1} \frac{dx}{x-1} = 1 + \int_{1}^{n} \frac{dx}{x}$

Solution 2 by Omran Kouba-Damascus-Syria

Consider $f(x) = \ln(1-x) + x$. *Clearly* $f''(x) = -\frac{1}{(x-1)^2}$ *so* f *is concave. Thus the function* $x \rightarrow \frac{(f(x)-f(0))}{(x-0)}$ *is decreasing on* $(0,1)$. *Thus, for* $x \in (0,1)$ *and* $n \geq 2$ *we have:*

$\frac{f\left(\frac{x}{n}\right)}{\frac{x}{n}} > \frac{f(x)}{x}$. *Consequently* $f\left(\frac{x}{n}\right) - \frac{f(x)}{n} > 0$. *Applying this to* $x = \frac{1}{k}$ *and adding we get:*

$$0 < \sum_{k=2}^{m} f\left(\frac{1}{kn}\right) - \frac{1}{n}\sum_{k=2}^{m} f\left(\frac{1}{k}\right) = \sum_{k=2}^{m} \ln\left(1-\frac{1}{kn}\right) + \frac{1}{n}\sum_{k=2}^{m} \frac{1}{k} - \frac{1}{n}\sum_{k=2}^{m} \ln\frac{k-1}{k} - \frac{1}{n}\sum_{k=2}^{m} \frac{1}{k} =$$

$$= \frac{\ln m}{n} + \ln\frac{\prod_{k=2}^{m}(kn-1)}{n^{m-1}m!} = \ln\left(\frac{m^{\frac{1}{n}}}{n^{m-1}m!}\prod_{k=2}^{m}(kn-1)\right)$$

So, we have proved that for integers $n, m \geq 2$ *the next inequality holds:*

$$\prod_{k=2}^{m}(kn-1) > \frac{n^m m!}{n \cdot m^{\frac{1}{m}}} \quad (1)$$

Applying (1) with $n = m = a$ *and* $n = m = b$ *and using the AM-GM inequality we get*

$$\prod_{k=2}^{a}(ka-1) + \prod_{k=2}^{b}(kb-1) \geq 2\sqrt{\prod_{k=2}^{a}(ka-1) \cdot \prod_{k=2}^{b}(kb-1)} > 2\sqrt{\frac{a^a a!}{a \cdot a^{\frac{1}{a}}} \cdot \frac{b^b b!}{b \cdot b^{\frac{1}{b}}}}$$

Which is equivalent to the proposed inequality.

283. If $m, n \in \mathbb{N}, a, b, c > 0, u \geq 0$ – fixed then:

$$\sum(m + a^{m+1})\left(n + \frac{1}{(b + c + u)^{m+1}}\right) \geq \frac{3(m + 1)(n + 1)(a + b + c)}{2(a + b + c) + 3u}$$

Proposed by D.M.Batinetu-Giurgiu, Neculai Stanciu-Romania

Solution by Marian Ursarescu-Romania

$$m + a^{m+1} = 1 + 1 + \dots + 1 + a^{m+1} \geq (m + 1)\sqrt[m+1]{1 \cdot 1 \cdot \ldots \cdot 1 \cdot a^{m+1}} \Rightarrow$$

$$m + a^{m+1} \geq (m + 1)\, a \quad \textit{(1)}$$

$$n + \frac{1}{(b + c + u)^{n+1}} = 1 + 1 + \dots + 1 + \frac{1}{(b + c + n)^{n+1}} \geq (n + 1)\sqrt[n+1]{\frac{1 \cdot 1 \cdot \ldots \cdot 1}{(b + c + u)^{n+1}}}$$

$$\Rightarrow n + \frac{1}{(b+c+u)^{n+1}} \geq \frac{n+1}{(b+c+u)} \quad \textit{(2)}$$

From (1) and (2) inequlaity becomes: $\sum(m + a^{m+1})\left(n + \frac{1}{(b+c+u)^{n+1}}\right) \geq$

$\geq (m + 1)(n + 1)\sum\frac{a}{b+c+u}$. ***We must show this:*** $\sum\frac{a}{b+c+u} \geq \frac{3(a+b+c)}{2(a+b+c)+3u}$ ***(3). From***

Cauchy's inequality $\Rightarrow \sum\frac{a}{b+c+u} = \sum\frac{a^2}{a(b+c+u)} \cdot \sum(ab + ac + au) \geq (a + b + c)^2 \Rightarrow$

$$\Rightarrow \sum\frac{a}{b+c+u} \geq \frac{(a+b+c)^2}{2(ab+bc+ac)+(a+b+c)^u} \quad \textit{(4)}$$

From (3)+(4) we must show: $\frac{(a+b+c)^2}{2(ab+ac+bc)+(a+b+c)u} \geq \frac{3(a+b+c)}{2(a+b+c)+3u} \Leftrightarrow$

$$\Leftrightarrow \frac{(a + b + c)}{2(ab + ac + bc) + (a + b + c)u} \geq \frac{3}{2(a + b + c) + 3u} \Leftrightarrow$$

$$\Leftrightarrow 2(a + b + c)^2 + 3u(a + b + c) \geq 6(ab + ac + bc) + 3u(a + b + c) \Leftrightarrow$$

$$\Leftrightarrow (a + b + c)^2 \geq 3(ab + ac + bc) \Leftrightarrow a^2 + b^2 + c^2 \geq ab + ac + bc \quad \textit{(true)}$$

284. If $a, b, c > 1$ then:

$$\frac{1}{\log_a c + 2\log_a b} + \frac{1}{\log_b a + 2\log_b c} + \frac{1}{\log_c b + 2\log_c a} \geq 1$$

Proposed by Marian Ursărescu – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc}\frac{1}{\log_a c+2\log_a b}=\sum_{cyc}\frac{\log a}{\log c+2\log b}=\sum_{cyc}\frac{(\log a)^2}{\log a\log c+2\log a\log b}\geq$$

$$\geq\frac{\left(\sum_{cyc}\log a\right)^2}{3\sum_{cyc}\log a\log b}\geq 1 \textit{ (proved)}$$

285. If $x,y,z\in\mathbb{R}, x+y+z=0$ then:

$$\frac{|2x+3|+|2y+3|+|2z+3|+9}{2}\geq|x-3|+|y-3|+|z-3|$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution by Daniel Sitaru-Romania

$$|2x+3+2y+3+2z+3|+\sum_{cyc(x,y,z)}|2x+3|\overset{HLAWKA}{\geq}\sum_{cyc(x,y,z)}|2x+3+2y+3|$$

$$|2(x+y+z)+9|+\sum_{cyc(x,y,z)}|2x+3|\geq 2\sum_{cyc(x,y,z)}|x+y+3|$$

$$\frac{1}{2}\left(\sum_{cyc(x,y,z)}|2x+3|+9\right)\geq\sum_{cyc(x,y,z)}|-z+3|=\sum_{cyc(x,y,z)}|x-3|$$

286. If $x\in\left(0,\frac{\pi}{2}\right)$ then:

$$\left|\frac{2}{\sin x+\cos x}+\frac{(\sin x-\cos x)(1-\tan x)}{1+\tan x}\right|\leq\sqrt{2}$$

Proposed by Daniel Sitaru – Romania

Solution by Amit Dutta-Jamshedpur-India

Let $P=\left|\frac{2}{\sin x+\cos x}+\frac{(\sin x-\cos x)(1-\tan x)}{1+\tan x}\right|$

$$P=\left|\frac{2}{\sin x+\cos x}+\frac{(\sin x-\cos x)(\cos x-\sin x)}{\cos x\frac{(\sin x+\cos x)}{\cos x}}\right|$$

$$P=\left|\frac{2-(\cos x-\sin x)^2}{(\sin x+\cos x)}\right|;\ P=\left|\frac{2-1+\sin 2x}{\sin x+\cos x}\right|;\ P=\left|\frac{1+\sin 2x}{\sin x+\cos x}\right|$$

$$P=\left|\frac{(\sin x+\cos x)^2}{\sin x+\cos x}\right|;\ P=|\sin x+\cos x|;\ P=\sqrt{2}\left|\sin\left(x+\frac{\pi}{4}\right)\right|\le\sqrt{2}$$

287. If $x,y,z>0$ then:

$$\frac{1}{\sqrt{x+y}}+\frac{1}{\sqrt{y+z}}+\frac{1}{\sqrt{z+x}}+\frac{3\sqrt{3}}{\sqrt{2(x+y+z)}}\ge 2\sqrt{2}\left(\frac{1}{\sqrt{x+2y+z}}+\frac{1}{\sqrt{y+2x+z}}+\frac{1}{\sqrt{x+2z+y}}\right)$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution by Daniel Sitaru-Romania

$$f:(0,\infty)\to(0,\infty), f(a)=a^{-\frac{1}{2}}, f'(a)=-\frac{1}{2}a^{-\frac{3}{2}}, f''(a)=\frac{3}{4}a^{-\frac{5}{2}}>0, f-convexe$$

$$\frac{1}{3}\sum f(a)+f\left(\frac{a+b+c}{3}\right)\overset{POPOVICIU}{\geq}\frac{2}{3}\sum f\left(\frac{a+b}{2}\right)$$

$$a=x+y, b=y+z, c=z+x$$

$$\frac{1}{3}\sum f(x+y)+f\left(\frac{2x+2y+2z}{3}\right)\overset{POPOVICIU}{\geq}\frac{2}{3}\sum f\left(\frac{x+2y+z}{2}\right)$$

$$\frac{1}{3}\sum\frac{1}{\sqrt{x+y}}+\frac{1}{\sqrt{\frac{2(x+y+z)}{3}}}\ge\frac{2}{3}\sum\frac{1}{\sqrt{\frac{x+2y+z}{2}}}$$

$$\sum\frac{1}{\sqrt{x+y}}+\frac{3\sqrt{3}}{\sqrt{2(x+y+z)}}\ge 2\sqrt{2}\sum\frac{1}{\sqrt{x+2y+z}}$$

288. If $a<b<c<d<e<f<g<h, a,b,c,d,e,f,g,h\in\mathbb{R}$ then:

$$(a+b+c+d+e+f+g+h)^2\ge 16(ah+bg+cf+de)$$

Proposed by Marian Ursărescu – Romania

Solution by Soumava Chakraborty-Kolkata-India

$\because a<b<c<d<e<f<g<h$, **we can consider** $b=a+x, c=a+x+y$,

$d=a+x+y+z, e=a+x+y+z+u, f=a+x+y+z+u+v$,

$g=a+x+y+z+u+v+w, h=a+x+y+z+u+v+w+t$, **where**

$x,y,z,u,v,w,t>0$ $\therefore$ ***by these substitutions, given inequality transforms into:***

$$(8a+7x+6y+5z+4u+3v+2w+t)^2-16a(a+x+y+z+u+v+w+t)-$$

$$-16(a+x)(a+x+y+z+u+v+w)-16(a+x+y)(a+x+y+z+u+v)-$$

$$-16(a+x+y+z)(a+x+y+z+u)\ge 0 \Leftrightarrow t^2+8tu+6tv+4tw+14tx+12ty+$$

$$+10tz+16u^2+24uv+16uw+8ux+16uy+24uz+9v^2+12vw+10vx+20vy+$$

$$+30vz+4w^2+12wx+24wy+20wz+x^2+4xy+6xz+4y^2+12yz+9z^2>0\rightarrow$$

$\rightarrow$ ***true*** $\because x,y,z,u,v,w,t>0$ ***(proved)***

289. If $b>a\ge e$ then: $\frac{\pi^b-\pi^a}{e\cdot \log\frac{b}{a}}>\pi^e$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan

Solution by Daniel Sitaru-Romania

$$f:[a,b]\to\mathbb{R}, f(x)=\pi^x, f'(x)=\pi^x\cdot\log\pi, g:[a,b]\to\mathbb{R}, g(x)=\log x, g'(x)=\frac{1}{x}$$

$$\frac{\pi^b-\pi^a}{\log\frac{b}{a}}=\frac{f(b)-f(a)}{g(b)-g(a)}\overset{CAUCHY}{=}\frac{f'(c)}{g'(c)}=\frac{\pi^c\log\pi}{\frac{1}{c}}>c\cdot\pi^c>e\cdot\pi^e, b>c>a\ge e$$

290. If $x,y,z\ge 0, x+y+z=\frac{\pi}{4}$ then:

$$\sum\tan x(1+\tan y)\ge 2\sqrt{\tan x\cdot\tan y\cdot\tan z}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Amit Dutta-Jamshedpur-India

$$\rightarrow x+y+z=\frac{\pi}{4}; x+y=\frac{\pi}{4}-2; \tan(x+y)=\tan\left(\frac{\pi}{4}-z\right); \frac{\tan x+\tan y}{1-\tan x\tan y}=\frac{1-\tan z}{1+\tan z}$$

$$(\tan x+\tan y)(1+\tan z)=(1-\tan z)(1-\tan x\tan y)\Rightarrow \tan x+\tan y+\tan z+$$

$$+\tan x\tan y+\tan y\tan z+\tan x\tan z=1+\tan x\tan y\tan z\Rightarrow$$

$$\Rightarrow\sum\tan x\,(1+\tan y)=1+\tan x\tan y\tan z.$$

Using AM-GM: $\frac{1+\tan x\tan y\tan z}{2}\geq\sqrt{\tan x\cdot\tan y\cdot\tan z}\Rightarrow 1+\tan x+\tan y\tan z\geq$

$$\geq 2\sqrt{\tan x\tan y\tan z}\Rightarrow\sum\tan x\,(1+\tan y)\geq 2\sqrt{\tan x\tan y\tan z}$$

Solution 2 by Serban George Florin-Romania

$$\tan x=a,\tan y=b,\tan z=c$$

$$\tan(x+b+z)=\tan\frac{\pi}{4}=1=\frac{\sum\tan x-\prod\tan x}{1-\sum\tan x\tan y}\Rightarrow a+b+c-abc=1-ab-bc-ac$$

$$\Rightarrow a+b+c+ba+bc+ac=1+abc$$

$$\sum\tan x\,(1+\tan y)=\sum a\,(1+b)=a+b+c+ab+bc+ac=1+abc\overset{(M_a\geq M_g)}{\geq}$$

$$\geq 2\sqrt{1\cdot abc}=2\sqrt{abc}$$

291. Let $x,y,z\in\mathbb{R}$ and $\sum x=0$. Prove:

$$4^x+4^y+4^z\geq 2(2^{x+y}+2^{y+z}+2^{z+x})-3$$

Propsed by Nho Nguyen Van-Nghe An-Vietnam

Solution by Daniel Sitaru – Romania

$$f:\mathbb{R}\to\mathbb{R},f(x)=4^x,f''(x)=4^x\log^2 4>0,f\text{ – convexe}$$

By Popoviciu's inequality: $\frac{1}{3}\sum f(x)+f\left(\frac{x+y+z}{3}\right)\geq\frac{2}{3}\sum f\left(\frac{x+y}{2}\right)\leftrightarrow$

$$\leftrightarrow\frac{1}{3}\sum 4^x+4^0\geq\frac{2}{3}\sum 4^{\frac{x+y}{2}}\leftrightarrow\sum 4^x\geq 2\sum 2^{x+y}-3$$

292. Let $x,y,z\in\mathbb{R}\wedge x+y+z=0$. Prove:

$$2\sqrt{2(1+e^x)(1+e^y)(1+e^z)}\geq\left(1+\frac{1}{\sqrt{e^x}}\right)\left(1+\frac{1}{\sqrt{e^y}}\right)\left(1+\frac{1}{\sqrt{e^z}}\right)$$

Propsed by Nho Nguyen Van-Nghe An-Vietnam

Solution by Daniel Sitaru – Romania

$$\sqrt{1+e^x} \overset{QM-AM}{\gtreqless} \frac{1}{\sqrt{2}}\left(1+\sqrt{e^x}\right) \to \prod \sqrt{1+e^x} \geq \frac{1}{2\sqrt{2}}\prod\left(1+\sqrt{e^x}\right) \leftrightarrow$$

$$2\sqrt{2}\prod\sqrt{1+e^x} \geq \frac{1}{2\sqrt{2}} \cdot \frac{1}{\sqrt{e^{x+y+z}}} \cdot \prod\left(1+\sqrt{e^x}\right) \leftrightarrow$$

$$2\sqrt{2(1+e^x)(1+e^y)(1+e^z)} \geq \prod\left(1+\frac{1}{\sqrt{e^x}}\right)$$

293. Let $x, y, z \in (0, +\infty)$. Prove:

$$\frac{(x+y+z)\sqrt{xyz(x+y+z)}}{(x+y)(y+z)(z+x)} \leq \frac{3\sqrt{3}}{8}$$

Propsed by Nho Nguyen Van-Nghe An-Vietnam

Solution by Daniel Sitaru – Romania

$$a = y+z, b = z+x, c = x+y, s = x+y+z, S = \sqrt{xyx(x+y+z)}$$

$$s \overset{MITRINOVIC}{\leq} \frac{3\sqrt{3}R}{2} \leftrightarrow \frac{sS}{4RS} \leq \frac{3\sqrt{3}}{8} \leftrightarrow \frac{sS}{abc} \leq \frac{3\sqrt{3}}{8} \leftrightarrow$$

$$\leftrightarrow \frac{(x+y+z)\sqrt{xyz(x+y+z)}}{(x+y)(y+z)(z+x)} \leq \frac{3\sqrt{3}}{8}$$

294.

$$-1 < a, b, c < 1, \Omega(a) = \int_0^{\pi} \frac{\log(1+a\cos x)}{\cos x} dx$$

Prove that:

$$\frac{1}{\pi^2}\left(\Omega^2(a)+\Omega^2(b)+\Omega^2(c)\right) \geq \sum\left(\sin^{-1} a \cdot \sin^{-1} b\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursarescu-Romania

Let $f(a) = \frac{\ln(1+a\cos x)}{\cos x}$ *is a continuous function in* $a \Rightarrow \Omega'(a) = \int_0^{\pi} \frac{1}{1+a\cos x} dx$

$$\left.\begin{matrix} Let\ \tan\frac{x}{2} = t \Rightarrow x = 2\arctan t \Rightarrow dx = \frac{2}{1+t^2}dt \\ x = 0 \Rightarrow t = 0; x = \pi \Rightarrow t = \infty \end{matrix}\right\} \Rightarrow \Omega'(a) = \int_0^\infty \frac{1}{1+a\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}dt$$

$$= 2\int_0^\infty \frac{1}{1+t^2+a-at^2}dt = 2\int_0^\infty \frac{1}{(1-a)t^2+1+a}dt = \frac{2}{1-a}\int_0^\infty \frac{1}{t^2+\left(\sqrt{\frac{1+a}{1-a}}\right)^2}dt =$$

$$= \frac{2}{1-a} \cdot \frac{1}{\sqrt{\frac{1+a}{1-a}}}\arctan\frac{t}{\sqrt{\frac{1+a}{1-a}}}\Bigg|_0^\infty = \frac{\pi}{\sqrt{1-a^2}} \Rightarrow$$

$$\left.\begin{matrix} \Omega(a) = \pi\int\frac{1}{\sqrt{1-a^2}}da = \pi\arcsin a + c \\ But\ \Omega(a) = 0 \Rightarrow c = 0 \end{matrix}\right\} \Rightarrow \Omega(a) = \pi\arcsin a \Rightarrow \textit{we must show:}$$

$\sum(\arcsin a)^2 \ge \sum\arcsin a \cdot \arcsin b$, ***which its true because*** $\sum x^2 \ge \sum xy$

Solution 2 by Sagar Kumar-Kolkata-India

$$I(a) = \int_0^\pi\int_0^{a\cos x} \frac{dy}{1+y} \cdot \frac{dx}{(\cos x)} = \int_0^\pi \frac{dy}{1+y}\log\left(\frac{\sin(a\cos x)}{\sin(a)}\right)$$

$$I(a) = \int_0^\pi \frac{\log(1+a\cos x)}{\cos x}dx;\ I'(a) = \int_0^\pi \frac{dx}{1+a\cos x}$$

$$\textit{Put}\ \tan\frac{x}{2} = t; \sin^2\frac{x}{2}dx = 2t$$

$$I'(a) = 2\int_0^\infty \frac{dt}{(1+a)+(1-a)t^2};\ I'(a) = \frac{\pi}{\sqrt{1-a^2}}; I(a) = \pi\sin^{-1}(a) + C$$

$$I(0) = 0 \Rightarrow c = 0;\ I(a) = \pi\sin^{-1}(a)$$

$$\frac{\sum I^2(a)}{\pi^2} = (\sin^{-1}(a))^2 + (\sin^{-1}(b))^2 + \sin^{-1}(c)^2$$

$$\textit{as}\ x^2 + y^2 + z^2 \ge xy + yz + zx \Rightarrow \frac{\sum I^2(a)}{\pi^2} \ge \sum\sin^{-1}(a)\sin^{-1}(b)$$

(proved)

295. If $F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n, m, n, p, q \in \mathbb{N}$ then:

$$\frac{F_m^2}{(F_qF_n+F_{q+1}F_p)^2}+\frac{F_n^2}{(F_qF_p+F_{q+1}F_m)^2}+\frac{F_p^2}{(F_qF_m+F_{q+1}F_n)^2}\ge\frac{3}{F_{q+2}^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by Marian Ursarescu-Romania

From Cauchy's inequality $\Rightarrow\left(\frac{F_m}{F_2F_n+F_{q+1}F_p}\right)^2+\left(\frac{F_n}{F_qF_p+F_{q+1}F_m}\right)^2+\left(\frac{F_p^2}{F_qF_m+F_{q+1}F_n}\right)^2\ge$

$$\ge\frac{1}{3}\left(\frac{F_m}{F_qF_n+F_{q+1}F_p}+\frac{F_n}{F_qF_p+F_{q+1}F_m}+\frac{F_p}{F_qF_m+F_{q+1}F_n}\right)^2$$

Then we must show this: $\left(\frac{F_m}{F_qF_n+F_{q+1}F_p}+\frac{F_n}{F_qF_p+F_{q+1}F_m}+\frac{F_p}{F_qF_m+F_{q+1}F_n}\right)^2\ge\frac{9}{F_{q+2}^2}\Leftrightarrow$

$$\Leftrightarrow\frac{F_m}{F_qF_n+F_{q+1}F_p}+\frac{F_n}{F_qF_p+F_{q+1}F_m}+\frac{F_p}{F_qF_m+F_{q+1}F_n}\ge\frac{3}{F_{q+2}}\quad(1)$$

But from Cauchy's inequality we have $\frac{F_m}{F_qF_n+F_{q+1}F_p}+\frac{F_n}{F_qF_p+F_{q+1}F_m}+\frac{F_p}{F_qF_m+F_{q+1}F_m}=$

$$=\frac{F_m^2}{F_mF_qF_n+F_mF_{q+1}F_p}+\frac{F_n^2}{F_nF_qF_p+F_nF_{q+1}F_m}+\frac{F_p^2}{F_pF_qF_m+F_pF_{q+1}F_m}\ge$$

$$\ge\frac{(F_m+F_n+F_p)^2}{F_q(F_mF_n+F_nF_p+F_pF_m)+F_{q+1}(F_mF_p+F_nF_m+F_pF_m)}=$$

$$=\frac{(F_m+F_n+F_p)^2}{(F_mF_n+F_nF_p+F_pF_m)(F_q+F_{q+1})}=\frac{(F_m+F_n+F_p)^2}{(F_mF_n\cdot F_nF_p+F_pF_m)\cdot F_{q+2}}\quad(2)$$

From (1)+(2) we must show: $\frac{(F_m+F_n+F_p)^2}{(F_mF_n+F_nF_p+F_pF_m)F_{q+2}}\ge\frac{3}{F_{q+2}}\Leftrightarrow$

$$\Leftrightarrow(F_m+F_n+F_p)^2\ge3(F_mF_n+F_nF_p+F_pF_m)\Leftrightarrow(F_m^2+F_n^2+F_p^2)$$
$$\ge F_mF_n+F_nF_p+F_pF_m$$

(true)

Solution 2 by Soumava Chakraborty-Kolkata-India

$$LHS\overset{Chebyshev}{\ge}\frac{1}{3}\left(\frac{F_m}{F_qF_n+F_{q+1}F_p}+\frac{F_n}{F_qF_p+F_{q+1}F_m}+\frac{F_p}{F_qF_m+F_{q+1}F_n}\right)^2\overset{?}{\ge}\frac{3}{F_{q+2}^2}$$

$$\Leftrightarrow \frac{F_m}{F_qF_n + F_{q+1}F_p} + \frac{F_n}{F_qF_p + F_{q+1}F_m} + \frac{F_p}{F_qF_m + F_{q+1}F_n} \underset{(1)}{\overset{?}{\geq}} \frac{3}{F_{q+2}}$$

$$\textit{LHS of (1)} = \frac{F_m^2}{F_qF_mF_n+F_{q+1}F_pF_m} + \frac{F_n^2}{F_qF_pF_n+F_{q+1}F_mF_n} + \frac{F_p^2}{F_qF_mF_p+F_{q+1}F_nF_p} \overset{Bergstrom}{\geq}$$

$$\geq \frac{(F_m + F_n + F_p)^2}{F_q(F_mF_n + F_nF_p + F_pF_m) + (F_{q+1})(F_mF_n + F_nF_p + F_pF_m)} =$$

$$= \frac{(F_m+F_n+F_p)^2}{\{F_q+(F_{q+2}-F_q)\}(F_mF_n+F_nF_p+F_pF_m)} \geq \frac{3(F_mF_n+F_nF_p+F_pF_m)}{F_{q+2}(F_mF_n+F_nF_p+F_pF_m)} = \frac{3}{F_{q+2}} = \textit{RHS of (1)}$$

(proved)

296. $F_0 = 0, F_1 = 1, F_n + F_{n+1} = F_{n+2}, n \in \mathbb{N}$. ***Prove that:***

$$\frac{\sin^3 t}{\sin t \cdot F_n + \cos t \cdot F_{n+1}} + \frac{\cos^3 t}{\cos t \cdot F_n + \sin t \cdot F_{n+1}} \geq \frac{1}{F_{n+2}}, n \in \mathbb{N}^*, t \in \left(0, \frac{\pi}{2}\right)$$

Proposed by D.M Bătinețu Giurgiu, Neculai Stanciu – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc} \frac{\sin^3 t}{\sin t \cdot F_n + \cos t \cdot F_{n+1}} \overset{Bergstrom}{\geq}$$

$$\geq \frac{(\sin^2 t + \cos^2 t)^2}{\sin t\,(\sin t \cdot F_n + \cos t \cdot F_{n+1}) + \cos t\,(\cos t \cdot F_n + \sin t \cdot F_{n+1})} =$$

$$= \frac{1}{F_n + 2\sin t \cos t \cdot F_{n+1}} \geq \frac{1}{F_n + (\sin^2 t + \cos^2 t)F_{n+1}} = \frac{1}{F_{n+2}}$$

(proved)

297. $A(a,b,c), B(d,e,f), C(g,h,i)$ **belongs to** $S: x^2 + y^2 + z^2 = R^2$. **Prove that:**

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}^2 \leq R^6$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

Let $\hat{a}$ be unit vector along $\overrightarrow{OA}$, $\hat{b}$ along $\overrightarrow{OB}$ and $\hat{c}$ along $\overrightarrow{OC}$, then

$$\overrightarrow{OA} = a\hat{i} + b\hat{j} + c\hat{k} = R\hat{a};\ \overrightarrow{OB} = d\hat{i} + e\hat{j} + f\hat{k} = R\hat{b};\ \overrightarrow{OC} = g\hat{i} + h\hat{j} + i\hat{k} = R\hat{c}$$

Now, $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}^2 = \left[\overrightarrow{OA}\,\overrightarrow{OB}\,\overrightarrow{OC}\right]^2 = \left[R\hat{a}\,R\hat{b}\,R\hat{c}\right]^2 = R^6\left[\hat{a}\hat{b}\hat{c}\right]^2$*. But* $\left[\hat{a}\hat{b}\hat{c}\right] = \pm$ *volume*

of parallelipiped with sides $\hat{a}, \hat{b}, \hat{c} \Rightarrow \left[\hat{a}\hat{b}\hat{c}\right]^2 \le 1 \therefore \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}^2 \le R^6$

298. In $\Delta\, ABC$ the following relationship holds:

$$\begin{vmatrix} 1 & 0 & a^2 & b^2 \\ 0 & 1 & 1 & 1 \\ 1 & a^2 & 0 & c^2 \\ 1 & b^2 & c^2 & 0 \end{vmatrix} \le 4abcR\sqrt{\left(\sum \sin^2 A\right)\left(\sum \cos^2 A\right)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu – Romania

$$\begin{vmatrix} 1 & 0 & a^2 & b^2 \\ 0 & 1 & 1 & 1 \\ 1 & a^2 & 0 & c^2 \\ 1 & b^2 & c^2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & a^2 & b^2 \\ 0 & 1 & 1 & 1 \\ 0 & a^2 & -a^2 & c^2 - b^2 \\ 0 & b^2 & c^2 - a^2 & -b^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & -a^2 & c^2 - b^2 \\ b^2 & c^2 - a^2 & -b^2 \end{vmatrix} =$$

$$a^2b^2 + a^2c^2 - a^4 + b^2c^2 - b^4 + a^2b^2 + a^2b^2 - (c^2 - a^2)(c^2 - b^2) =$$

$$= 2(a^2b^2 + b^2c^2 + a^2c^2) - (a^4 + b^4 + c^4) \quad \textbf{(1)}$$

From (1) we must show this:

$$2(a^2b^2 + b^2c^2 + a^2c^2) - (a^4 + b^4 + c^4) \le 4abcR\sqrt{(\sum \sin 2A)\sum \cos^2 A} \quad \textbf{(2)}$$

From Cauchy inequality $\Rightarrow$

$$\sqrt{\sum \sin^2 A} \ge \frac{1}{\sqrt{3}}(\sum \sin A) \text{ and } \sqrt{\sum \cos^2 A} \ge \frac{1}{\sqrt{3}}(\sum \cos A) \quad \textbf{(3)}$$

From (2)+(3) we must show this:

$$2(a^2b^2 + a^2c^2 + b^2c^2) - (a^4 + b^4 + c^4) \ge \frac{4}{3}abcR(\sum \sin A)(\sum \cos A) \quad \textbf{(4)}$$

But $\sum \sin A = \frac{a+b+C}{2R}$ *(5)*

$$\sum \cos A = \sum \frac{b^2+c^2-a^2}{2bc} = \frac{\sum a(b^2+c^2-a^2)}{2abc} \quad (6)$$

From (4)+(5)+(6) we must show this: $2(a^2b^2+a^2c^2+b^2c^2)-(a^4+b^4+c^4) \geq$

$$\geq \frac{1}{3}(a+b+c)(ab^2+ac^2-a^3+ba^2+bc^2-b^3+ca^2+cb^2-c^3) \Leftrightarrow$$

$$6(a^2b^2+a^2c^2+b^2c^2)-3(a^4+b^4+c^4) \geq -a^4-b^4-c^4+a^3(b+c)+$$

$$+b^3(a+c)+c^3(a+b)-a(b^3+c^3)-b(a^3+c^3)-c(a^3+b^3)+$$

$$+a^2(b^2+c^2)+b^2(a^2+c^2)+c^2(a^2+b^2)+abc(b+c)+abc(a+c)+$$

$$+abc(a+b) \Leftrightarrow 2(a^4+b^4+c^4)-4(a^2b^2+b^2c^2+a^2c^2)+2abc(a+b+c) \geq 0 \Leftrightarrow$$

$$\Leftrightarrow a^4+b^4+c^4-2(a^2b^2+b^2c^2+a^2c^2)+abc(a+b+c) \geq 0 \quad (7)$$

$$\Leftrightarrow a^4+b^4+c^4+abc(a+b+c) \geq 2(a^2b^2+b^2c^2+a^2c^2) \quad (8)$$

By Schur's inequality we have:

$$a^4+b^4+c^4+abc(a+b+c) \geq ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2) \quad (9)$$

From (8)+(9) we must show:

$$ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2) \geq 2(a^2b^2+b^2c^2+a^2c^2) \quad (10)$$

But $ab(a^2+b^2) \geq 2a^2b^2 \Leftrightarrow a^2+b^2 \geq 2ab$ *which its true. Similarly:*

$bc(b^2+c^2) \geq 2bc$ *and* $ac(a^2+c^2) \geq 2a^2c^2 \Rightarrow$ *(10) its true.*

Solution 2 by Soumava Chakraborty-Kolkata-India

$$LHS = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & 0 & c^2 \\ b^2 & c^2 & 0 \end{vmatrix} + a^2 \begin{vmatrix} 0 & 1 & 1 \\ 1 & a^2 & c^2 \\ 1 & b^2 & 0 \end{vmatrix} - b^2 \begin{vmatrix} 0 & 1 & 1 \\ 1 & a^2 & 0 \\ 1 & b^2 & c^2 \end{vmatrix} = 1(-c^4)-1(-b^2c^2)+$$

$$+1(a^2c^2)+a^2\{(-1)(-c^2)+b^2-a^2\}-b^2\{(-1)c^2+b^2-a^2\} =$$

$$= -c^4+b^2c^2+a^2c^2+a^2c^2+a^2b^2-a^4+b^2c^2-b^4+a^2b^2 = 2\sum a^2b^2-\sum a^4 =$$

$$= 4\sum a^2b^2-\left(\sum a^2\right)^2 = 4\left(\sum ab\right)^2-4(s^2-4Rr-r^2)^2-64Rrs^2 =$$

$$= 4(s^2+4Rr+r^2+s^2-4Rr-r^2)(s^2+4Rr+r^2-s^2+4Rr+r^2)-64Rrs^2$$

$$= 4(2s^2)(8Rr+2r^2)-64Rrs^2 \overset{(1)}{=} 16s^2r^2$$

$$RHS = 16R^2rs\sqrt{\frac{1}{2}\left(\sum 2\sin^2 A\right)\cdot\frac{1}{2}\left(\sum 2\cos^2 A\right)} =$$

$$= 8R^2rs\sqrt{\left\{\sum(1-\cos 2A)\right\}\left\{\sum(1+\cos 2A)\right\}} =$$

$$= 8R^2rs\sqrt{\left(3-\sum\cos 2A\right)\left(3+\sum\cos 2A\right)} = 8R^2rs\sqrt{9-\left(\sum\cos 2A\right)^2} =$$

$$= 8R^2rs\sqrt{9-(1+4\pi\cos A)^2} = 8R^2rs\sqrt{9-\left(1+\frac{s^2-(2R+r)^2}{R^2}\right)^2} =$$

$$= 8R^2rs\sqrt{9-\frac{(s^2+r^2-4R^2-4Rr-r^2)^2}{R^4}} = 8rs\sqrt{9R^4-(3R^2+4Rr+r^2-S^2)^2} =$$

$$= 8rs\sqrt{9R^4-(3R^2+4Rr+r^2)^2-s^4+2s^2(3R^2+4Rr+r^2)}$$

$$\overset{(2)}{=} 8rs\sqrt{(6R^2+4Rr+r^2)(-4Rr-r^2)-s^4+2s^2(3R^2+4Rr+r^2)}$$

(1), (2) ⇒ given inequality is:

$$4r^2s^2 \le -(4Rr+r^2)(6R^2+4Rr+r^2)-s^4+2s^2(3R^2+4Rr+r^2)$$

$$\Leftrightarrow s^4 \overset{(3)}{\le} -(4Rr+r^2)(6R^2+4Rr+r^2)+2s^2(3R^2+4Rr-r^2)$$

Now, LHS of (3) $\overset{Gerretsen}{\le} s^2(4R^2+4Rr+3r^2) \overset{?}{\le} 2s^2(3R^2+4Rr-r^2)-$

$$-(4Rr+r^2)(6R^2+4Rr+r^2) \Leftrightarrow s^2(2R^2+4Rr-5r^2) \underset{(4)}{\overset{?}{\ge}} (4Rr+r^2)(6R^2+4Rr+r^2)$$

Now, LHS of (4) $\overset{Gerretsen}{\ge} (16Rr-5r^2)(2R^2+4Rr-5r^2) \overset{?}{\ge}$

$$\ge (4Rr+r^2)(6R^2+4Rr+r^2) \Leftrightarrow 2t^3+8t^2-27t+6 \overset{?}{\ge} 0 \quad \left(t=\frac{R}{r}\right) \Leftrightarrow$$

$\Leftrightarrow (t-2)(2t^2+12(t-2)+21) \overset{?}{\ge} 0 \rightarrow$ ***true*** $\because t \ge 2$ ***(Euler) (Proved)***

299. In $\Delta\, ABC$ the following relationship holds:

$$\begin{vmatrix} a & 0 & c & b \\ 0 & a & b & c \\ b & c & 0 & a \\ c & b & a & 0 \end{vmatrix} \geq 432r^4$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$LHS = a \times \begin{vmatrix} a & b & c \\ c & 0 & a \\ b & a & 0 \end{vmatrix} + c \times \begin{vmatrix} 0 & a & c \\ b & c & a \\ c & b & 0 \end{vmatrix} - b \begin{vmatrix} 0 & a & b \\ b & c & 0 \\ c & b & a \end{vmatrix} =$$

$$= a\{a(-a^2) - b(-ab) + c \cdot ca\} + c\{-a(-ac) + c(b^2 - c^2)\} -$$

$$-b\{-a(ab) + b(b^2 - c^2)\} = a(-a^3 + ab^2 + ac^2) + c(a^2c + b^2c - c^3) +$$

$$+b(-a^2b + b^3 - bc^2) = a^2(b^2 + c^2 - a^2) + c^2(a^2 + b^2 - c^2) + b^2(c^2 + a^2 - b^2)$$

$$= 2a^2bc\cos A + 2c^2abc\cos C + 2b^2ca\cos B = 2abc\left(\sum a\cos A\right) =$$

$$= 2Rabc(\sin 2A + \sin 2B + \sin 2C) = 2Rabc \cdot 4\sin A\sin B\sin C = 2R \cdot 4Rrs\left(4\frac{abc}{8R^3}\right)$$

$$= 16\frac{R^2rs \cdot Rrs}{R^3} = 16r^2s^2 \overset{s\geq 3\sqrt{3}r}{\geq} 16 \cdot 27r^4 = 432r^4$$

(Proved)

300. If $a, b, c, d, e, f > 0$ then:

$$64\begin{vmatrix} 1 & b & b & b & b \\ a & c & 0 & 0 & 0 \\ a & 0 & d & 0 & 0 \\ a & 0 & 0 & e & 0 \\ a & 0 & 0 & 0 & f \end{vmatrix} \leq (a+f)^2(b+e)^2(c+d)^2\left(\frac{1}{ab} - \frac{1}{c} - \frac{1}{d} - \frac{1}{e} - \frac{1}{f}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Amit Dutta-Jamshedpur-India

Let $P = 64\begin{vmatrix} 1 & b & b & b & b \\ a & c & 0 & 0 & 0 \\ a & 0 & d & 0 & 0 \\ a & 0 & 0 & e & 0 \\ a & 0 & 0 & 0 & f \end{vmatrix}$***. Expanding this determinant, we get***

$$P = 64(cdef - abdef - abcef - abcdf - abcde).$$

$$P = 64abcdef\left(\frac{1}{ab} - \frac{1}{c} - \frac{1}{d} - \frac{1}{e} - \frac{1}{f}\right)$$

$P = (4af)(4be)(4cd)\left(\frac{1}{ab} - \frac{1}{c} - \frac{1}{d} - \frac{1}{e} - \frac{1}{b}\right)$***. By AM-GM:*** $\sqrt{af} \le \frac{a+f}{2} \Rightarrow 4af \le (a+f)^2$

$$\sqrt{bc} \le \frac{b+e}{2} \Rightarrow 4be \le (b+e)^2; \sqrt{cd} \le \frac{c+d}{2} \Rightarrow 4cd \le (c+d)^2 \Rightarrow$$

$$\Rightarrow P \le (a+f)^2(b+e)^2(c+d)^2\left(\frac{1}{ab} - \frac{1}{c} - \frac{1}{d} - \frac{1}{e} - \frac{1}{f}\right)$$

(Proved)

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To be continued!

Daniel Sitaru