RMM - Calculus Marathon 301 - 400

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ROMANIAN MATHEMATICAL MAGAZINE

Founding Editor
DANIEL SITARU

Available online
www.ssmrmh.ro

ISSN-L 2501-0099

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301. **For $z_1, z_2 \in \mathbb{C}$, satisfy: $|z_1 + z_2| = |z_1| + |z_2|$. Prove:**

$$|z_1 - z_2| = max\{|z_1|; |z_2|\} - min\{|z_1|; |z_2|\}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution by Ravi Prakash-New Delhi-India

$$|z_1 + z_2| = |z_1| + |z_2| \Rightarrow z_1 = kz_2 \text{ for some } k \geq 0. \text{ Now,}$$

$$|z_1 - z_2| = |(k-1)z_2| = |(k-1)z_2|$$

$$\text{If } k \geq 1, \text{ then } |z_1| = k|z_2| \geq |z_2|,$$

$$\text{and } |z_1 - z_2| = (k-1)|z_2| = k|z_2| - |z_2| = |z_1| - |z_2|$$

$$= \max\{|z_1|, |z_2|\} - \min\{|z_1|, |z_2|\}$$

$$\text{If } 0 \leq k < 1, |z_1| = k|z_2| < |z_2| \text{ and}$$

$$|z_1 - z_2| = |k-1||z_2| = (1-k)|z_2| = |z_2| - k|z_2|$$

$$= |z_2| - |z_1| = \max\{|z_1|, |z_2|\} - \min\{|z_1|, |z_2|\}$$

302. **If $z \in \mathbb{C}$, $\alpha \geq 2$ then:**

$$|Rez|^\alpha + |Imz|^\alpha \geq 2^{1-\frac{\alpha}{2}} \cdot |z|^\alpha$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution by Daniel Sitaru-Romania

$$\sqrt[\alpha]{\frac{1}{2}|Rez|^\alpha + \frac{1}{2}|Imz|^\alpha} \overset{POWER\ MEANS}{\geq} \sqrt{\frac{1}{2}|Rez|^2 + \frac{1}{2}|Imz|^2}, (\alpha \geq 2)$$

$$\frac{1}{2}|Rez|^\alpha + \frac{1}{2}|Imz|^\alpha \geq \frac{1}{2^{\frac{\alpha}{2}}}(|Rez|^2 + |Imz|^2)^\alpha = \frac{1}{2^{\frac{\alpha}{2}}}|z|^\alpha$$

$$\frac{1}{2}|Rez|^\alpha + \frac{1}{2}|Imz|^\alpha \geq \frac{1}{2^{\frac{\alpha}{2}}}|z|^\alpha \to |Rez|^\alpha + |Imz|^\alpha \geq 2^{1-\frac{\alpha}{2}}|z|^\alpha$$

303. **If $A \in M_2(\mathbb{R})$ then:**

$$det(A^2 + 2A + 2I_2) \geq (2 + Tr\ A)^2$$

Proposed by Marian Ursarescu-Romania

Solution 1 by Serban George Florin-Romania

$$x^2+2x+2=0, \Delta=4-8=-4<0, x_1=\frac{-2+2i}{2}=-1+i, x_2=-1-i$$

$$\det(A^2+2A+2I_2)=\det(A-x_1I_1)\cdot\det(A-x_2I_2)$$

$$P(x)=\det(A-xI_2)=x^2-(Tr\,A)x+\det A$$

$$\det(A^2+2A+2I_2)=P(x_1)\cdot P(x_2)=[(-1+i)^2-(Tr\,A)(-1+i)+\det A]\cdot$$

$$\cdot[(-1-i)^2-(Tr\,A)(-1-i)+\det A]=(-2i+Tr\,A-i\,TrA+\det A)\cdot$$

$$\cdot(2i+Tr\,A+i\,Tr\,A+\det A)=(Tr\,A+\det A)^2-(2i+i\,Tr\,A)^2=$$

$$=Tr^2\,A+2\,Tr\,A\det A+\det^2 A+4+4\,Tr\,A+Tr^2A=$$

$$=2Tr^2A+2Tr\,A\det A+\det^2 A+4\,Tr\,A+4=$$

$$=(Tr\,A+2)^2+Tr^2A+2\,Tr\,A\det A+\det^2 A$$

$$=(Tr\,A+2)^2+(Tr\,A+\det A)^2\ge(Tr\,A+2)^2, \textit{true.}$$

Solution 2 by Ravi Prakash-New Delhi-India

$$Let\ A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}, a,b,c,d\in\mathbb{R}$$

$$A^2+2A+2I_2=(A+I_2)^2+I_2=(A+I_2+iI_2)(A+I_2-iI_2)$$

$$=(A+I_2+iI_2)(\overline{A+I_2+\imath I_2})$$

$$\det(A^2+2A+2I_2)=\det(A+(1+i)I_2)\left(\overline{A+(1+\imath)I_2}\right)=$$

$$=\det(A+(1+i)I_2)\,\overline{\det(A+(1+\imath)I_2)}=|\det(A+(1+i)I_2)|^2=$$

$$=\left|\begin{vmatrix}a+(1+i) & b\\ c & d+(1+i)\end{vmatrix}\right|^2=\left|(1+i)^2+(a+d)(1+i)+ad-bc\right|^2$$

$$=|(a+d+ad-bc)+(2+a+d)i|^2\ge\left(2+(a+d)\right)^2=(2+tr\,A)^2$$

304. $A,B\in M_2(\mathbb{R}), det\,A\ne 0, det\,B\ne 0, Tr(AB^{-1})=det(AB^{-1})=1$

Find: $\Omega=det(I_2+A^{-1}B)$

Proposed by Marian Ursarescu-Romania

Solution by Ravi Prakash-New Delhi-India

$$Let\ X = AB^{-1}.\ As\ tr(X) = 1,\ we\ take\ X = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}$$

$$1 = \det(X) = a(1-a) - bc$$

$$\det(I + AB^{-1}) = \det(I + X) = \begin{vmatrix} a+1 & b \\ c & 2-a \end{vmatrix} = (a+1)(2-a) - bc$$

$$= 2 + a - a^2 - bc = 3.\ Now\ \det(I + A^{-1}B) = \det\{A^{-1}(AB^{-1} + I)B\} =$$

$$= \det(A^{-1}\det(AB^{-1} + I))\det(B) = (\det(A))^{-1}(\det B)\det(X)$$

$$= [\det(A)\,(\det(B))^{-1}]^{-1}(3) = (\det(AB^{-1}))^{-1}(3) = (1)(3) = 3$$

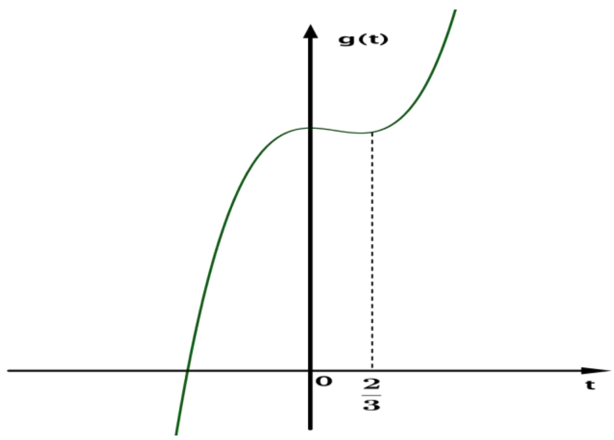
305. If $A, B \in M_5(\mathbb{R}), A^3 + 7I_5 = A^2, B^3 + 9I_5 = B^2$ then:

$$det(AB) > 0$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

A polynomial satisfied by A is $g(t) = t^3 - t^2 + 7$. *Graph of* $g(t)$ *is as follows:*



This shows $g(t)$ *has a negative root and two imaginary roots*

Let α *be negative root and* $\beta + i\gamma, \beta - i\gamma, \beta, \gamma \in \mathbb{R}$ *be imaginary roots.*

Note tat $\alpha, \beta + i\gamma, \beta - i\gamma$ *are distinct eigen values of A.*

Also, $g(t) = (t - \alpha)(\alpha - \beta - i\gamma)(t - \beta + i\gamma)$ *is minimal polynomial of A.*

As minimal polynomial and characteristic polynomials have same zeros, and k is real or $k = \alpha(\beta + i\gamma)^2(\beta - i\gamma)^2 < 0$. *Also,* $\det(A) = k$

$\therefore \det(A) < 0$ ***Similarly*** $\det(B) < 0 \Rightarrow \det(AB) = \det(A)\det(B) > 0.$

Solution 2 by Marian Ursarescu-Romania

$$A^2 - A^3 = 7I_5 \Rightarrow \det[A^2(I_5 - A)] = \det(7I_5) \Rightarrow (\det A)^2 \cdot \det(I_5 - A) = 7^5 \neq 0 \Rightarrow \det A \neq 0 \quad (1)$$

$$B^2 - B^3 = 9I_5 \Rightarrow \det[B^2(I_5 - B)] = \det(9I_5) \Rightarrow (\det B)^2 \cdot \det(I_5 - B) = 9^5 \neq 0 \Rightarrow \det B \neq 0 \quad (2)$$

Now, $A^3 = A^2 - 7I_5 \Rightarrow A^4 = A^3 - 7A \Rightarrow A^4 = A^2 - 7I_5 - 7A \Rightarrow A^4 = A^2 - 7A - 7I_5 \Rightarrow$

$$\Rightarrow A^5 = A^3 - 7A^2 - 7A = A^2 - 7I_5 - 7A^2 - 7A \Rightarrow$$

$$\Rightarrow A^5 = -6A^2 - 7A - 7I_5 \Rightarrow A^5 = -6\left(A^2 + \frac{7}{6}A + \frac{7}{6}I_5\right) \Rightarrow$$

$$\det A^5 = \det\left[-6\left(A^2 + \frac{7}{6}A + \frac{7}{6}I_5\right)\right] \Rightarrow \left.\begin{array}{c}(\det A)^5 = (-6)^5 \cdot \det\left(A^2 + \frac{7}{6}A + \frac{7}{6}I_5\right) \\ \textit{But } \det\left(A^2 + \frac{7}{6}A + \frac{7}{6}I_5\right) \geq 0\end{array}\right\} \Rightarrow$$

$\Rightarrow (\det A)^5 \leq 0 \Rightarrow \det A \leq 0$ ***(3). Now,*** $B^3 = B^2 - 9I_5 \Rightarrow B^4 = B^3 - 9B \Rightarrow$

$$\left(B^5 = B^4 - 9B^2\right) \quad B^4 = B^2 - 9B - 9I_5 \Rightarrow B^5 = B^3 - 9B^2 - 9B = B^2 - 9I_5 - 9B^2 - 9B$$

$$\Rightarrow B^5 = -8B^2 - 9B - 9I_5 \Rightarrow B^5 = -8\left(B^2 + \frac{9}{8}B + \frac{9}{8}I_5\right) \Rightarrow$$

$$\Rightarrow \det\left(B^5\right) = \det\left[-8\left(B^2 + \frac{9}{8}B + \frac{9}{8}I_5\right)\right] \Rightarrow$$

$$\left.\begin{array}{c}(\det B)^5 = (-8)^5 \det\left(B^2 + \frac{9}{8}B + \frac{9}{8}I_5\right) \\ \det\left(B^2 + \frac{9}{8}B + \frac{9}{8}I_5\right) \geq 0\end{array}\right\} \Rightarrow (\det B)^5 \leq 0 \Rightarrow (\det B) \leq 0 \quad (4)$$

From (1)+(2)+(3)+(4) $\Rightarrow \det A < 0$ ***and*** $\det B < 0 \Rightarrow \det(AB) > 0.$

Observation: $A \in M_n(\mathbb{R}), p \in (0,4) \Rightarrow \det(A^2 + pA + pI_n) \geq 0$

because $\Leftrightarrow \det\left(A^2 + pA + \frac{p^2}{4}I_n - \frac{p^2}{4}I_n + pI_n\right) =$

$$= \det\left[\left(A + \frac{p}{2}I_n\right)^2 + \frac{-p^2 + 4p}{4}I_n\right] = \det\left[\left(A + \frac{p}{2}I_n\right)^2 + \left(\frac{\sqrt{4p - p^2}}{2}\right)^2 I_n^2\right] =$$

$$= \det\left[\left(A + \frac{p}{2}I_n\right)^2 - i^2\left(\frac{\sqrt{4p - p^2}}{2}\right)^2 I_n^2\right] =$$

$$= \det\left[\left(A + \frac{p}{2}I_n + i\frac{\sqrt{4p - p^2}}{2}I_n\right)\overline{\left(A + \frac{p}{2} + i\frac{\sqrt{4p - p^2}}{2}I_n\right)}\right] \geq 0$$

306. If $A, B \in M_5(\mathbb{R}), A^3 - 2I_5 = A^2, B^3 - 3I_5 = B^2$ then:

$$det(AB) > 0$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursarescu-Romania

$$A^3 \cdot A^2 = 2I_5 \Rightarrow A^2(A - I_5) = 2I_5 \Rightarrow$$

$$\Rightarrow (\det A)^2 \cdot \det(A - I_5) = 2^5 \Rightarrow \det A \neq 0 \quad (1)$$

$$A^3 = A^2 + 2I_5 \Rightarrow \det A^3 = \det(A^2 + 2I_5)\,(\det A)^3 = \det\left(A + \sqrt{2}iI_5\right)\left(A - \sqrt{2}iI_5\right) \Rightarrow$$

$$(\det A)^3 = \det\left(A + \sqrt{2}iI_5\right) \cdot \overline{\det\left(A + \sqrt{2}\imath I_5\right)} \geq 0 \quad (2)$$

From (1)+(2)$\Rightarrow \det A > 0$ *(3)*

$$B^3 - B^2 = 3I_5 \Rightarrow B^2(B - I_5) = 3I_5 \Rightarrow$$

$$\Rightarrow (\det B)^2 \cdot \det(B - I_5) = 3^5 \Rightarrow \det B \neq 0 \quad (4)$$

$$B^3 = B^2 + 3I_5 \Rightarrow \det B^3 = \det(B^2 + 3I_5) \Rightarrow$$

$$\Rightarrow (\det B)^3 = \det\left(B + \sqrt{3}iI_5\right)\left(B - \sqrt{3}iI_5\right) \Rightarrow$$

$$\Rightarrow (\det B)^3 = \det\left(B + \sqrt{3}iI_5\right) \cdot \overline{\det\left(B + \sqrt{3}\imath I_5\right)} \geq 0 \quad (5)$$

From (4)+(5)$\Rightarrow \det(B) > 0$ *(6)*

From (3)+(6) $\Rightarrow \det(AB) > 0$

Solution 2 by Ravi Prakash-New Delhi-India

$$A^3 = A^2 + 2I = \left(A - \sqrt{2}iI\right)\left(A + \sqrt{2}iI\right)$$

$$\Rightarrow \det(A^3) = \det\left(A - \sqrt{2}iI\right)\det\left(A + \sqrt{2}iI\right) = \det\left(\overline{A + \sqrt{2}\imath I}\right)\det\left(A + \sqrt{2}iI\right)$$

$$= \overline{\det\left(A + \sqrt{2}\imath I\right)}\det\left(A + \sqrt{2}iI\right)$$

$$(\det(A))^3 = \left|\det\left(A + \sqrt{2}iI\right)\right|^2 \geq 0$$

$$\Rightarrow \det(A) \geq 0 \quad (1)$$

Also $A^3 - A^2 = 2I \Rightarrow A^2(A - I) = 2I \Rightarrow \det(A^2)\det(A - I) = \det(2I) = 2^5 > 0$

$$\Rightarrow (\det(A))^2 \det(A - I) > 0 \Rightarrow \det(A) \neq 0 \quad (2)$$

In view of (1), (2) $\det(A) > 0$

Similarly, $\det(B) > 0$*. Thus,* $\det(AB) = \det(A)\det(B) > 0$*.*

307. Find $A, B \in M_2(\mathbb{R})$ such that:

$$det\,A < 0, det(A-B) > 0, det(A+B) < 0, det(2A+B) > 0$$

Proposed by Marian Ursarescu-Romania

Solution by Omran Kouba-Damascus-Syria

Suppose that A and B satisfy the proposed conditions. Let $C = A^{-1}B$ and let

$$\chi(\lambda) = \det(\lambda I_2 - C) = \lambda^2 - tr\,(A)\lambda + \det(C)$$

be the characteristic polynomial of C. The proposed inequalities yields

$$\chi(1) = \frac{\det(A-B)}{\det A} < 0$$

$$\chi(-1) = \frac{\det(-A-B)}{\det A} = \frac{\det(A+B)}{\det A} > 0$$

$$\chi(-2) = \frac{\det(-2A-B)}{\det A} = \frac{\det(2A+B)}{\det A} < 0$$

But $\chi(\lambda)$ is positive for large $|\lambda|$, so the above conditions imply the second degree polynomial χ has at least 4 zeros and this is absurd. Thus, no such matrices exist.

308. If $A \in M_4(\mathbb{C}), det\,A \neq 0, Tr\,A = 0$ then:

$$Tr\,(A^3) = 3(det\,A)(Tr\,A^{-1})$$

Proposed by Marian Ursarescu-Romania

Solution 1 by Andrew Okukura-Romania

If $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues of A we have:

$$Tr\,A = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0 \Leftrightarrow x_1 + x_2 = -(\lambda_3 + \lambda_4) \Leftrightarrow$$

$$\Leftrightarrow \lambda_1^3 + 3\lambda_1^2\lambda_2 + 3\lambda_1\lambda_2^2 + \lambda_2^3 = -(\lambda_3^3 + 3\lambda_3^2\lambda_4 + 3\lambda_3\lambda_4^2 + \lambda_4^3) \Leftrightarrow$$

$$\Leftrightarrow \lambda_1^3 + \lambda_2^3 + \lambda_3^3 + \lambda_4^3 = -3[\lambda_1\lambda_2(\lambda_1 + \lambda_2) + \lambda_3\lambda_4(\lambda_3 + \lambda_4)] =$$

$$= 3[\lambda_1\lambda_2(\lambda_3 + \lambda_4) + \lambda_3\lambda_4(\lambda_1 + \lambda_2)]. \textit{ But } Tr(A^3) = \lambda_1^3 + \lambda_2^3 + \lambda_3^3 + \lambda_4^3 \Rightarrow$$

$$\Rightarrow Tr(A^3) = 3[\lambda_1\lambda_2(\lambda_3 + \lambda_4) + \lambda_3\lambda_4(\lambda_1 + \lambda_2)] \textit{ (1)}$$

$$\det A = \lambda_1\lambda_2\lambda_3\lambda_4 \textit{ (2)}$$

$$Tr(A^{-1})=\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}+\frac{1}{\lambda_4}\ (3)$$

$$(2)\wedge(3)\Rightarrow 3(\det A)(Tr\,A^{-1})=3\lambda_1\lambda_2\lambda_3\lambda_4\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}+\frac{1}{\lambda_4}\right)=$$

$$=3\lambda_1\lambda_2\lambda_3+3\lambda_1\lambda_2\lambda_4+3\lambda_1\lambda_3\lambda_4+3\lambda_2\lambda_3\lambda_4=3[\lambda_1\lambda_2(\lambda_3+\lambda_4)+\lambda_3\lambda_4(\lambda_1+\lambda_2)]\ (4)$$

$$(1)+(4)\Rightarrow \text{conclusion}$$

Solution 2 by Ravi Prakash-New Delhi-India

Let $A=(a_{ij})_{4\times 4}\in M_4(\mathbb{C})$ and $Tr\,(A)=0,\det(A)\neq 0$.

Let $f(t)=t^4-\alpha t^3+\beta t^2-\gamma t+\delta$ be the characteristic polynomial of A.

Then $\alpha=Tr\,(A)=0$ and $\delta=\det(A)\neq 0$. $\therefore f(t)=t^4+\beta t^2-\gamma t+\delta$. We have

$$A^4=-\beta A^2+\gamma A-\delta I_4\ (1)$$

$$\Rightarrow A^3=-\beta A-\gamma I-\delta A^{-1}$$

$$Tr\,(A^3)=-\beta Tr(A)+4\gamma-\delta Tr\,(A^{-1})=4\gamma-\delta Tr(A^{-1})\ (1)$$

Let $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ be eigen values of A, then $\sum\lambda_i=0,\sum\lambda_i\lambda_j=B$

Let λ be an eigenvalue of $A\Rightarrow\exists a\ x\neq 0$ such that $Ax=\lambda x\Rightarrow$

$$\Rightarrow A^2(x)=A(Ax)=A(\lambda x)=\lambda Ax=\lambda(\lambda x)=\lambda^2 x$$

Similarly, $A^3=\lambda^3 x\Rightarrow\lambda^3$ is an eigenvalue of A^3. If A^{-1} exists, then

$A^{-1}(Ax)=A^{-1}(\lambda x)\Rightarrow\lambda^{-1}x=A^{-1}x\ \therefore\ \lambda^{-1}$ is an eigenvalue of A^{-1}.

If $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ eigenvalues of A, then $\lambda_1+\lambda_2+\lambda_3+\lambda_4=Tr\,(A)=0$.

Now, $Tr(A^3)=\lambda_1^3+\lambda_2^3+\lambda_3^3+\lambda_4^3=(\lambda_1+\lambda_2)^3-3\lambda_1\lambda_2(\lambda_1+\lambda_2)+(\lambda_3+\lambda_4)^3-$

$-3\lambda_3\lambda_4(\lambda_3+\lambda_4)=(-\lambda_3-\lambda_4)^3+3\lambda_1\lambda_2(\lambda_3+\lambda_4)+(\lambda_3+\lambda_4)^3+3\lambda_3\lambda_4(\lambda_1+\lambda_2)$

$[\because\lambda_1+\lambda_2+\lambda_3+\lambda_4=0]$

$$=3\lambda_1\lambda_2\lambda_3\lambda_4\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}+\frac{1}{\lambda_4}\right)=3\det(A)\,Tr\,(A^{-1})$$

$$\left[\because\lambda_1\lambda_2\lambda_3\lambda_4=\det(A)\ \text{and}\ Tr(A^{-1})=\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}+\frac{1}{\lambda_4}\right]$$

$\sum\lambda_i\lambda_j\lambda_k=\gamma;\ \lambda_1\lambda_2\lambda_3\lambda_4=\delta$. Note

$$\gamma=\delta\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}+\frac{1}{\lambda_4}\right)=\det(A)\,Tr\,(A^{-1})\ (2)$$

From (1), (2): $Tr(A^3)=3\det(A)\,Tr\,(A^{-1})$

309. If $A, B \in M_5(\mathbb{R}), A^3 - 2I_5 = A^2, B^3 - 3I_5 = B^2$ then:

$$det(AB) > 0$$

Proposed by Daniel Sitaru – Romania

Solution by Marian Ursarescu-Romania

$$A^3 \cdot A^2 = 2I_5 \Rightarrow A^2(A - I_5) = 2I_5 \Rightarrow$$

$$(\det A)^2 \cdot \det(A - I_5) = 2^5 \Rightarrow \det A \neq 0 \quad (1)$$

$$A^3 = A^2 + 2I_5 \Rightarrow \det A^3 = \det(A^2 + 2I_5)\,(\det A)^3 = \det\left(A + \sqrt{2}iI_5\right)\left(A - \sqrt{2}iI_5\right) \Rightarrow$$

$$(\det A)^3 = \det\left(A + \sqrt{2}iI_5\right) \cdot \overline{\det\left(A + \sqrt{2}\imath I_5\right)} \geq 0 \quad (2)$$

From (1)+(2)$\Rightarrow \det A > 0$ *(3)*

$$B^3 - B^2 = 3I_5 \Rightarrow B^2(B - I_5) = 3I_5 \Rightarrow$$

$$(\det B)^2 \cdot \det(B - I_5) = 3^5 \Rightarrow \det B \neq 0 \quad (4)$$

$$B^3 = B^2 + 3I_5 \Rightarrow \det B^3 = \det(B^2 + 3I_5) \Rightarrow$$

$$(\det B)^3 = \det\left(B + \sqrt{3}iI_5\right)\left(B - \sqrt{3}iI_5\right) \Rightarrow$$

$$(\det B)^3 = \det\left(B + \sqrt{3}iI_5\right) \cdot \overline{\det\left(B + \sqrt{3}\imath\sqrt{5}\right)} \geq 0 \quad (5)$$

From (4)+(5)$\Rightarrow \det(B) > 0$ *(6)*

From (3)+(6)$\Rightarrow \det(AB) > 0$

310. If $A, B \in M_2(\mathbb{C}), det(A + B) = 1$ then

$$det(A \cdot det\, B + B \cdot det\, A) = det(AB)$$

Proposed by Marian Ursarescu-Romania

Solution 1 by Omran Kouba-Damascus-Syria

The polynomial $P(X, Y) = \det(XA + YB)$ *is homogenous of degree* 2, *so it has the form* $P(X, Y) = aX^2 + bXY + cY^2$. *Testing* $(X, Y) \in \{(1, 0), (0, 1), (1, 1)\}$ *and using the hypothesis* $\det(A + B) = 1$ *we see taht* $a = \det(A) \triangleq \beta, b = 1 - \alpha - \beta$. *It follows that*

$$\det(\beta A + \alpha B) = P(\beta, \alpha) = \alpha\beta^2 + (1 - \alpha - \beta)\alpha\beta + \beta\alpha^2 = \alpha\beta = \det(ABC)$$

Solution 2 by Serban George Florin-Romania

If $\det B = 0 \Rightarrow \det A = 1 \Rightarrow \det(A \cdot 0 + B) = \det A \cdot \det B \Rightarrow \det B = 1 \cdot 0, \det B = 0,$ *true. Because* $\det B \neq 0, \det A \neq 0, \det(A + \det B + B \cdot \det A) = \det A \cdot \det B$

$$\Rightarrow \det\left(\det B \cdot \left(A + \frac{\det A}{\det B} \cdot B\right)\right) = \det A \cdot \det B$$

$$\det^2 B \cdot \det\left(A + \frac{\det A}{\det B} \cdot B\right) = \det A \cdot \det B \,|: \det B, \frac{\det A}{\det B} = k$$

$$\det(A + k \cdot B) = k = k^2 \det B + \alpha \cdot k + \det A$$

$$\det(A + B) = 1 \Rightarrow \frac{\det A}{\det B} = \frac{\det^2 A}{\det^2 B} \cdot \det B + \alpha \cdot \frac{\det A}{\det B} + \det A$$

$$\det A = \det^2 A + \alpha \det A + \det A \cdot \det B \,|: \det A$$

$$1 = \det A + \alpha + \det B \Rightarrow \alpha = 1 - \det A - \det B$$

From $k = 1 \Rightarrow \det(A + B) = 1 = \det B + \alpha + \det A$ *(A)*

Solution 3 by Ravi Prakash-New Delhi-India

If $\det(A) = 0$ *or* $\det(B) = 0$, *then* $\det(\det(A)B + \det(B)A) = 0 = \det(A)\det(B)$.

Suppose $\det(A) \neq 0, \det(B) \neq 0$. *Let* $A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, B = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$

Let $\alpha = \det A, \beta = \det(B)$. *Now,* $I = \det(A + B) = \det[A(B^{-1} + A^{-1})B] =$

$$= \det(A)\det(B)\det(B^{-1} + A^{-1}) \quad (1)$$

But $A^{-1} = \frac{1}{\alpha}\begin{pmatrix} d_1 & -b_1 \\ -c_1 & a_1 \end{pmatrix}, B^{-1} = \frac{1}{\beta}\begin{pmatrix} d_2 & -b_2 \\ -c_2 & a_2 \end{pmatrix}$

$$\therefore B^{-1} + A^{-1} = \begin{bmatrix} \frac{d_1}{\alpha} + \frac{d_2}{\beta} & -\left(\frac{b_1}{\alpha} + \frac{b_2}{\beta}\right) \\ -\left(\frac{c_1}{\alpha} + \frac{c_2}{\beta}\right) & \frac{a_1}{\alpha} + \frac{a_2}{\beta} \end{bmatrix}$$

Now, note $\det(B^{-1} + A^{-1}) = \det\begin{bmatrix} \frac{a_1}{\alpha} + \frac{a_2}{\beta} & \frac{b_1}{\alpha} + \frac{b_2}{\beta} \\ \frac{c_1}{\alpha} + \frac{c_2}{\beta} & \frac{d_1}{\alpha} + \frac{d_2}{\beta} \end{bmatrix}$

$$\therefore \det(B^{-1} + A^{-1}) = \det\left(\frac{1}{\alpha}A + \frac{1}{\beta}B\right) \quad (2)$$

Thus, from (1), (2): $1 = \alpha\beta \det\left(\frac{1}{\alpha}A + \frac{1}{\beta}B\right) = \frac{1}{\alpha\beta}\det\left[\frac{\alpha\beta}{\alpha}A + \frac{\alpha\beta}{\beta}B\right]$

$$[\because A, B\ are\ 2\times\ 2\ matrices] \Rightarrow \det(\beta A+\alpha\beta)=\alpha\beta$$

$$or\ \det[(\det B)A+(\det A)B]=\det A\det B=\det(AB)$$

311. If $A\in M_2(\mathbb{R}), B\in M_3(\mathbb{R}), C\in M_4(\mathbb{R})$,

$A^2-A=I_2, B^2-B=I_3, C^2-C=I_4$ then: $|det\ A+det\ B+det\ C|<28$

Proposed by Daniel Sitaru – Romania

Solution by Marian Ursărescu – Romania

$$Let\ f(x)=x^2-x-1, f(x)=0\Rightarrow x_{1,2}=\tfrac{1\pm\sqrt{5}}{2}. Now\ the\ own\ values\ for\ A\ is\ \lambda_1,\lambda_2\Rightarrow$$

$$from\ McCoy\ theorem\Rightarrow \lambda_1,\lambda_2\in\left\{\tfrac{1-\sqrt{5}}{2},\tfrac{1+\sqrt{5}}{2}\right\}\Rightarrow|\lambda_i|\le\tfrac{1+\sqrt{5}}{2}, i=1,2\Rightarrow$$

$$|\det A|=|\lambda_1\lambda_2|=|\lambda_1|\cdot|\lambda_2|\le\left(\tfrac{1+\sqrt{5}}{2}\right)^2\quad (1)$$

$$Let\ \lambda_1,\lambda_2,\lambda_3\ the\ own\ values\ for\ B\Rightarrow from\ McCoy\ theorem\Rightarrow\{\lambda_1,\lambda_2,\lambda_3\}\in\left\{\tfrac{1-\sqrt{5}}{2},\tfrac{1+\sqrt{5}}{2}\right\}$$

$$\Rightarrow|\lambda_i|\le\frac{1+\sqrt{5}}{2}, i=1,2,3\Rightarrow$$

$$|\det B|=|\lambda_1||\lambda_2||\lambda_3|\le\left(\tfrac{1+\sqrt{5}}{2}\right)^3\quad (2)$$

$$Let\ \lambda_1,\lambda_2,\lambda_3,\lambda_4\ the\ own\ values\ for\ C\Rightarrow\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}\in\left\{\tfrac{1-\sqrt{5}}{2},\tfrac{1+\sqrt{5}}{2}\right\}\Rightarrow|\lambda_i|\le\tfrac{1+\sqrt{5}}{2},$$

$$i=1,2,3,4\Rightarrow|\det C|=|\lambda_1|\cdot|\lambda_2|\cdot|\lambda_3|\cdot|\lambda_4|\le\left(\tfrac{1+\sqrt{5}}{2}\right)^4\quad (3)$$

$$From\ (1)+(2)+(3)\Rightarrow|\det A+\det B+\det C|\le|\det A|+|\det B|+|\det C|\le$$

$$\le\left(\frac{1+\sqrt{5}}{2}\right)^2+\left(\frac{1+\sqrt{5}}{2}\right)^3+\left(\frac{1+\sqrt{5}}{2}\right)^4=7+3\sqrt{5}<28$$

312. If $A,B,C,D\in M_n(\mathbb{C}), n\in\mathbb{N}, n\ge 2, det(ABCD)\neq 0$ then:

$$rank(AB\cdot det(CD)+CD\cdot det(AB))=rank\left(\frac{1}{det\ C\cdot det\ D}B^{-1}A^{-1}+\frac{1}{det\ A\cdot det\ B}D^{-1}C^{-1}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Marian Ursărescu – Romania

We use two properties: (1) $rank\ (\alpha\cdot A)=rank\ A,\forall\alpha\neq 0$ *(obvious)*

(2) $rank\ (A)=rank\ (A\cdot B^{-1}),\forall B=$ *invertible (from Sylvester)*

$$rank\ (AB\cdot\det(CD)+CD\cdot\det(AB))=rank(B\det(CD)+A^{-1}C\cdot D\det(AB))=$$

$$=rank\left(\det(CD)\,I_n+B^{-1}A^{-1}C\cdot D\det(AB)\right)=rank\left(\det(CD)\,D^{-1}+B^{-1}A^{-1}C\det(AB)\right)=$$

$$=rank(\det(CD)\,D^{-1}\cdot C^{-1}+B^{-1}A^{-1}\cdot\det(AB))=$$

$$=rank(\det D\cdot D^{-1}\det C\cdot C^{-1}+\det B^{-1}\cdot\det A\cdot A^{-1})$$

$=rank\ (D^*C^*+B^*A^*)$ *(3). Now,* $rank\ \left(\frac{1}{\det C\cdot\det D}B^{-1}A^{-1}+\frac{1}{\det A\cdot\det B}D^{-1}C^{-1}\right)$

$$=rank\left(\frac{1}{\det A\det B\det C\det D}B^*A^*+\frac{1}{\det A\det B\det C\det D}D^*C^*\right)=$$

$=rank(B^*A^*+D^*C^*)$ *(4). From (3) + (4)* $\Rightarrow$ *relation from hypothesis.*

313. If $A,B\in M_2(\mathbb{C}),det(A+B)=1$ **then:**

$$det(A\cdot det\ B+B\cdot det\ A)=det(AB)$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Omran Kouba-Damascus-Syria

The polynomial $P(X,Y)=\det(XA+YB)$ *is homogenous of degree* 2*,*

so it has the form $P(X,Y)=aX^2+bXY+cY^2$*. Testing* $(X,Y)\in\{(1,0),(0,1),(1,1)\}$

and using the hypothesis

$\det(A+B)=1$ *we see that* $a=\det(A)\triangleq\alpha,c=\det(B)\triangleq\beta,b=1-\alpha-\beta$*. It follows*

that $\det(\beta A+\alpha B)=P(\beta,\alpha)=\alpha\beta^2+(1-\alpha-\beta)\alpha\beta+\beta\alpha^2=\alpha\beta=\det(AB)$

Solution 2 by Ravi Prakash-New Delhi-India

If $\det(A)=0$ *or* $\det(B)=0$*, then* $\det(\det(A)\,B+\det(B)A)=0=\det(A)\det(B)$

Suppose $\det(A)\neq 0,\det(B)\neq 0$*. Let* $A=\begin{pmatrix}a_1 & b_1\\ c_1 & d_1\end{pmatrix},B=\begin{pmatrix}a_2 & b_2\\ c_2 & d_2\end{pmatrix}$

Let $\alpha=\det(A)\,,\beta=\det(B)$*. Now,* $I=\det(A+B)=\det[A(B^{-1}+A^{-1})B]$

$$=\det(A)\det(B)\det(B^{-1}+A^{-1})\quad (1)$$

$$But\ A^{-1}=\frac{1}{\alpha}\begin{pmatrix}d_1 & -b_1\\ -c_1 & a_1\end{pmatrix}, B^{-1}=\frac{1}{\beta}\begin{pmatrix}d_2 & -b_2\\ -c_2 & a_2\end{pmatrix}$$

$$\therefore B^{-1}+A^{-1}=\begin{bmatrix}\frac{d_1}{\alpha}+\frac{d_2}{\beta} & -\left(\frac{b_1}{\alpha}+\frac{b_2}{\beta}\right)\\ -\left(\frac{c_1}{\alpha}+\frac{c_2}{\beta}\right) & \frac{a_1}{\alpha}+\frac{a_2}{\beta}\end{bmatrix}$$

$$Now,\ note\ \det(B^{-1}+A^{-1})=\det\begin{bmatrix}\frac{a_1}{\alpha}+\frac{a_2}{\beta} & \frac{b_1}{\alpha}+\frac{b_2}{\beta}\\ \frac{c_1}{\alpha}+\frac{c_2}{\beta} & \frac{d_1}{\alpha}+\frac{d_2}{\beta}\end{bmatrix}$$

$$\therefore \det(B^{-1}+A^{-1})=\det\left(\frac{1}{\alpha}A+\frac{1}{\beta}B\right)\ (2)$$

$$Thus,\ from\ (1), (2): I=\alpha\beta\det\left(\frac{1}{\alpha}A+\frac{1}{\beta}B\right)=\frac{1}{\alpha\beta}\det\left[\frac{\alpha\beta}{\alpha}A+\frac{\alpha\beta}{\beta}B\right]$$

$$[\because A, B\ are\ 2\times 2\ matrices]\Rightarrow \det(\beta A+\alpha B)=\alpha\beta$$

$$or\ \det[(\det B)A+(\det A)B]=\det A\det B=\det(AB)$$

314. $A\in M_n(\mathbb{R}), det\ A\neq 0, \alpha\in(-1,1), A^2+A^{-2}=\alpha(A+A^{-1})$

Find: $|det\ A|$

Proposed by Marian Ursărescu – Romania

Solution by Omran Kouba-Damascus-Syria

Let $A\in M_n(\mathbb{R})$ *be an invertible matrix with*

$$A^2+A^{-2}=\alpha(A+A^{-1}), for\ some\ \alpha\in(-1,1)\qquad (H)$$

Find $|\det(A)|$

Step 1. If $\alpha\in(-1,1)$ *then all the complex roots of the polynomial* $P(x)=X^4-\alpha X^3-\alpha X+1$ *belong to the unit circle. Indeed,* $P(z)=0$ *is equivalent to*

$$z^3=\frac{\alpha z-1}{z-\alpha}\ thus\ |z|^6-1=\left|\frac{\alpha z-1}{z-\alpha}\right|^2-1=\frac{(1-\alpha^2)(1-|z|^2)}{|z-\alpha|^2}\ and\ consequently$$

$$(|z|^2-1)\underbrace{\left[1-|z|^2+|z|^4+\frac{1-\alpha^2}{|z-\alpha|^2}\right]}_{positive}=0. Thus, |z|=1$$

Step 2 $|\det A|=1$*. Consider* A *as a complex matrix. If* $\lambda\in\mathbb{C}$ *is an eigenvalue of* A *then according to* (H)*,* λ *satisfies*

$\lambda^2+\frac{1}{\lambda^2}=\alpha\left(\lambda+\frac{1}{\lambda}\right)$. *Equivalently* $P(\lambda)=0$, *hence* $|\lambda|=1$ *according to Step 1. But* $\det A$ *is the product of all the eigenvalues of A, (each one is repeated according to its multiplicity), so* $|\det A|=1$.

315. Solve for real numbers:

$$\begin{vmatrix} 1 & 3+\sin x & 2+3\sin x & 2\sin x \\ 1 & 2+\sin x+\cos x & 2\sin x+\sin x\cos x & \sin 2x \\ 1 & 1+\sin x+\cos x & \sin x+\cos x+\sin x\cos x & \sin x\cos x \\ 1 & 3+\cos x & 2+3\cos x & 2\cos x \end{vmatrix}=0$$

Proposed by Daniel Sitaru – Romania

Solution by Srinivasa Raghava-AIRMC-India

Solve

$$\det\left[\begin{pmatrix} 1 & 3+\sin x & 2+3\sin x & 2\sin x \\ 1 & 2+\sin x+\cos x & 2\sin x+\sin x\cos x+2\cos x & \sin 2x \\ 1 & 1+\sin x+\cos x & \sin x+\cos x+\sin x\cos x & \sin x\cos x \\ 1 & 3+\cos x & 2+3\cos x & 2\cos x \end{pmatrix}\right]=0$$

After simplification we have:

$$\left|\begin{pmatrix} 1 & \sin x+3 & 3\sin x+2 & 2\sin x \\ 1 & \cos x+\sin x+2 & \sin x\cos x+2\cos x+2\sin x & \sin 2x \\ 1 & \cos x+\sin x+1 & \sin x\cos x+\cos x+\sin x & \sin x\cos x \\ 1 & \cos x+3 & 3\cos x+2 & 2\cos x \end{pmatrix}\right|=$$

$-\frac{1}{4}(\sin x-2)(\sin x+\cos x-1)^2(4\sin x+\cos 2x-2(\sin x+2)\cos x+1)$. *Solve for x:*

$$-\frac{1}{4}(\sin x-2)(-1+\cos x+\sin x)^2(1+\cos 2x+4\sin x-2\cos x)(\sin x+2)=0$$

Multiply both sides by a constant to simplify the equation. Multiply both sides by -4:

$$(\sin x-2)(-1+\cos x+\sin x)^2(1+\cos 2x+4\sin x-2\cos x)(\sin x+2)=0$$

Find the roots of each term in the product separately. Split into three equations:

$$\sin x-2=0 \text{ or } (-1+\cos x+\sin x)^2=0 \text{ or}$$

$$1+\cos 2x+4\sin x-2\cos x\,(\sin x+2)=0$$

Isolate terms with x to the left hand side. Add 2 to both sides: $\sin x=2$ *or*

$(-1+\cos x+\sin x)^2=0$ *or* $1+\cos 2x+4\sin x-2\cos x(\sin x+2)=0$. *After solving each equation separately and some calculations we have the following solutions* $x=\pi\left(\frac{n-7}{4}\right)$; $x=2\pi n$; $x=2\pi n+\frac{\pi}{2}$; $x=2\pi n+\frac{\pi}{4}$; $x=2\pi n-\frac{3\pi}{4}$

$$x=2\pi n-2i\tanh^{-1}\frac{1}{\sqrt{3}};\ x=2\pi n+2i\tanh^{-1}\frac{1}{\sqrt{3}};\ x=2\pi n+\pi-\sin^{-1}2$$

316. If $A,B,C\in M_n(\mathbb{Z}), n\geq 3, (A^*B^*)^*=BA, (B^*C^*)^*=CB$ then:

$$\det A+\det B+\det C<\sqrt{10}$$

Proposed by Daniel Sitaru – Romania

Solution by Marian Ursărescu – Romania

If $\det A=0$ *or* $\det B=0$ *or* $\det C=0$ *obvious. Let* $\det A\neq 0, \det B\neq 0, \det C\neq 0$.

Lemma 1: $(AB)^*=B^*A^*$ *(1)*

Lemma 2: $(A^*)^*=(\det A)^{n-2}A$ *(2)*

From $(A^*B^*)^*=BA^{(1)}\Rightarrow((BA)^*)^*=BA\overset{(2)}{\Rightarrow}$

$$\left.\begin{array}{c}(\det BA)^{n-2}BA=BA\\ BA\ \textit{invertible}\end{array}\right\}\Rightarrow(\det BA)^{n-2}=1\Rightarrow$$

$$\Rightarrow\left.\begin{array}{c}\det BA=\pm 1\Rightarrow\det A\cdot\det B=\pm 1\\ \textit{but}\ \det A\ \textit{and}\ \det B\in\mathbb{Z}\end{array}\right\}\Rightarrow\det A,\det B\in\{-1,1\}\quad (3)$$

Similarly: $\det B,\det C\in\{-1,1\}$ *(4)*

From (3)+(4)$\Rightarrow\det A+\det B+\det C\leq 3<\sqrt{10}$

317. If $X,Y,Z\in M_n(\mathbb{R}), n\geq 2, n\in\mathbb{N}, XY=YX, YZ=ZY, ZX=XZ$ then:

$$\det(9X^2+5Y^2+5Z^2+12XY+6YZ+12ZX)\geq 0$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

We have: $9X^2+5Y^2+5Z^2+12XY+6YZ+12ZX=$

$$=[3X+(2+i)Y+(2-i)Z][3X+(2-i)Y+(2+i)Z]\Rightarrow$$

$$\Rightarrow \det(9X^2+5Y^2+5Z^2+12XY+6YZ+12ZX)=$$

$$=\det[(3X+(2+i)Y+(2-i)Z)(3X+(2-i)Y+(2+i)Z)]$$

$$=\det\left[(3X+(2+i)Y+(2-i)Z)\overline{(3X+(2+\imath)Y+(2-\imath)Z)}\right]=$$

$$=(\det[3X+(2+i)Y+(2-i)Z])\left(\det\left(\overline{3X+(2+i)Y}+\overline{(2-i)Z}\right)\right)$$

$$=(\det(3X+(2+i)Y+(2-i)Z))\,\overline{\det(3X+(2+\imath)Y+(2-\imath)Z)}$$

$$=|\det(3x+(2+i)Y+(2-i)Z|^2\geq 0$$

Solution 2 by Marian Ursărescu – Romania

We use: $\det(A\cdot\overline{A})\geq 0,\forall A\in M_n(R)$ *(1)*

Because $XY=YX, YZ=ZY$ *and* $ZX=XZ$ *we can make algebraic calculus:*

$$\det\left[(3A+(2+i)B+(2-i)C)\overline{(3A+(2+\imath B+(2-\imath)C))}\right]\geq 0\quad (2)$$

(From (1))

But $\det\left[(3A+(2+i)B+(2-i)C)\overline{(3A+(2+\imath)B+(2-\imath)C)}\right]=$

$$=\det[(3A+(2+i)B+(2-i)C)(3A+(2-i)B+(2+i)C)]=$$

$$=\det(9X^2+5Y^2+5Z^2+12XY+6YZ+12ZX)\quad (3)$$

From (2)+(3)$\Rightarrow \det(9X^2+5Y^2+5Z^2+12XY+6YZ+12XZ)\geq 0$

318. $A,B\in M_2(\mathbb{R}), Tr((AB)^2)=Tr(A^2B^2), n\in\mathbb{N}, n\geq 2$. **Find:**

$$\Omega=Tr[(AB-BA)^n]$$

Proposed by Marian Ursărescu – Romania

Solution by Ravi Prakash-New Delhi-India

If X *and* Y *are two* $n\times n$ *matrices, then:* $Tr(XY)=Tr(YX)$

$Tr(X\pm Y)=Tr(X)\pm Tr(Y)$. *We are given:* $Tr((AB)^2)=Tr(A^2B^2)\Rightarrow$

$$\Rightarrow Tr\{ABAB-AABB\}=0\Rightarrow Tr\{A(BA-AB)B\}=0\Rightarrow$$

$$\Rightarrow Tr\{BA(BA-AB)\}=0\quad (1)$$

$$\Rightarrow Tr((BA)^2)=Tr(BA^2B)=Tr(BBA^2)=Tr(B^2A^2)\Rightarrow$$

$$\Rightarrow Tr\{BABA-BBAA\}=0\Rightarrow Tr\{B(AB-BA)A\}=0\Rightarrow$$

$$\Rightarrow Tr\{AB(AB-BA)\}=0 \quad (2)$$

$$\text{Now, } Tr\{(AB-BA)^2\}=Tr\{AB(AB-BA)+BA(BA-AB)\}=$$

$$=Tr\big(AB(AB-BA)\big)+Tr\big(BA(BA-AB)\big)=0+0=0 \text{ [from (1), (2)]}$$

$$\text{Let } x=AB-BA, \text{ then } Tr(x)=Tr(AB)-Tr(BA)=0.$$

$$\text{Also, } Tr(X^2)=0 \text{ [Prove above]. Let } X=\begin{pmatrix} a & b\\ c & -a\end{pmatrix} \ [\because Tr(X)=0]$$

$$X^2=\begin{pmatrix} a^2+bc & 0\\ 0 & a^2+bc\end{pmatrix};\ Tr(X^2)=0\Rightarrow 2(a^2+bc)=0\Rightarrow a^2+bc=0$$

$$\therefore X^2=\begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}\Rightarrow Tr(X^n)=0\ \ \forall n\ge 2$$

319. Find $(a_n)\subset\mathbb{N}$ such that:

$$\sum_{k=0}^{n} a_k\binom{n}{k}\binom{n+1}{k+1}=(n+1)\binom{2n}{n}, n\in\mathbb{N}$$

Proposed by Marian Ursărescu – Romania

Solution by Ravi Prakash-New Delhi-India

$$\binom{n+1}{k+1}=\frac{(n+1)!}{(k+1)!\,(n-k)!}=\frac{n+1}{k+1}\cdot\frac{n!}{k!\,(n-k)!}=\frac{n+1}{k+1}\binom{n}{k}\Rightarrow\binom{n}{k}\binom{n+1}{k+1}=$$

$$=(n+1)\cdot\frac{1}{k+1}\binom{n}{k}\binom{n}{k}\Rightarrow\sum_{k=0}^{n}a_k\binom{n}{k}\binom{n+1}{k+1}=(n+1)\sum_{k=0}^{n}\frac{a_k}{k+1}\binom{n}{k}^2$$

$$\text{We know } \binom{2n}{n}=\sum_{k=0}^{n}\binom{n}{k}^2$$

$$\therefore\sum_{k=0}^{n}a_k\binom{n}{k}\binom{n+1}{k+1}=(n+1)\binom{2n}{n}\Rightarrow(n+1)\sum_{k=0}^{n}\frac{a_k}{k+1}\binom{n}{k}^2=(n+1)\sum_{k=0}^{n}\binom{n}{k}^2$$

Thus a possible sequence is $a_k=k+1, \forall k\in\mathbb{N}$.

320. Solve for real numbers:

$$(x+\sin x+\cos x)^3=(x+\sin x-\cos x)^3+(x+\cos x-\sin x)^3+(\sin x+\cos x-3)^3$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Amit Dutta-Jamshedpur-India

$$Let\ (x+\sin x-\cos x)=a;(x+\cos x-\sin x)=b;(\sin x+\cos x-x)=c$$

$$Hence,\ given\ equation\ reduces\ to\ (a+b+c)^3=a^3+b^3+c^3\Rightarrow$$

$$\Rightarrow (a+b+c)^3-a^3-b^3-c^3=0\Rightarrow 3(a+b)(b+c)(c+a)=0$$

$$Putting\ the\ values\ of\ a,b,c: 3(2x)(2\cos x)(2\sin x)=0$$

$$\begin{array}{c|c|c} x=0 & \cos x=0 & \sin x=0 \\ cosx=0 & x=(2n+1)\frac{\pi}{2} & x=m\pi \\ & n\in I & m\in I \end{array}$$

$$Real\ solutions\ are\ x=0,x=(2n+1)\frac{\pi}{2};x=m\pi$$

Solution 2 by Sagar Kumar-Kolkata-India

$$(x+\sin x+\cos x)^3=(x+\sin x-\cos x)^3+(x+\cos x-\sin x)^3+(\sin x+\cos x-x)^3$$

$$(x+y+z)^3-x^3-y^3-z^3=3(x+y)(y+z)(z+x)\Rightarrow 3(x)(\cos x)(\sin x)=0$$

$$x=0,x=n\pi,\frac{(2n+1)\pi}{2}.\ Combining\ these\ values:\ x=\frac{m\pi}{2},m\in I$$

321. **Solve for real numbers:**

$$\frac{1}{1+8^x}+\frac{1}{1+27^x}+\frac{1}{1+64^x}=\frac{3}{1+24^x}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution by Daniel Sitaru-Romania(using a result of Omran Kouba-Damascus-Syria)

$$Let\ be\ f:[0,\infty)\to\mathbb{R}, f(x)=\frac{1}{1+e^x}, f''(x)=\frac{e^x(e^x-1)}{(1+e^x)^3}\ge 0, f-convexe$$

$$If\ u,v,w\ge 0\ then\ by\ Jensen's\ inequality:\ f\left(\frac{u+v+w}{3}\right)\le\frac{1}{3}(f(u)+f(v)+f(w))$$

$$\frac{1}{1+e^{\frac{u+v+w}{3}}}\le\frac{1}{3}\left(\frac{1}{1+e^u}+\frac{1}{1+e^v}+\frac{1}{1+e^w}\right).\ Denote\ a=e^u,b=e^v,c=e^w$$

$$\frac{1}{1+\sqrt[3]{abc}}\le\frac{1}{3}\left(\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}\right)$$

$$\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}\le\frac{3}{1+\sqrt[3]{abc}}.\ Equality\ holds\ if\ a=b=c.\ Denote\ a=8^x,b=27^x,c=64^x$$

$$\frac{1}{1+8^x}+\frac{1}{1+27^x}+\frac{1}{1+64^x}\le\frac{3}{1+\sqrt[3]{8^x\cdot 27^x\cdot 64^x}}=\frac{3}{1+24^x}$$

Equality holds for $8^x=27^x=64^x\to x=0$

322. Find all continuous functions:

$$f:\mathbb{R}\to\mathbb{R},\ f(x^3)-f(y^3)=(x^2+xy+y^2)f(x-y),\forall x,y\in\mathbb{R}$$

Proposed by Marian Ursărescu – Romania

Solution by Omran Kouba-Damascus-Syria

Consider a continuous function f satisfying the proposed property. Let $P(x,y)$ be the property $f(x^3)-f(y^3)=(x^2+xy+y^2)f(x-y)$. From $P(1,1)$ we conclude that $f(0)=0$. From $P(x,0)$ we conclude that $f(x^3)=x^2f(x)$ for every x

From $P(tx,x)$ for $x\neq 0$ we get

$$t^2f(tx)-f(x)=(t^2+t+1)f\big((t-1)x\big)\qquad (1)$$

Which is also true when $x=0$ according to the first point. Setting $t=0$ in (1) we conclude that f is odd. Setting $t=2$ in (1) we conclude that $f(2x)=2f(x)$ for all x. Now suppose that $f(nx)=nf(x)$ for some positive integer n and for all x. Applying (1) with $t=n+1$ we get

$$(n+1)^2f\big((n+1)x\big)=f(x)+(n^2+3n+3)nf(x)=(n+1)^3f(x)$$

that is $f\big((n+1)x\big)=(n+1)f(x)$ for all x. Thus, since f is odd, we have proved that

$$\forall x\in\mathbb{R},\forall n\in\mathbb{Z}, f(nx)=nf(x)\qquad (2)$$

Applying (2) with positive n and $\frac{x}{n}$ instead of x we get also

$$\forall x\in\mathbb{R},\forall n\in\mathbb{N}^*, f\left(\frac{x}{n}\right)=\frac{1}{n}f(x)\qquad (3)$$

Combining (2) and (3) we get for $n\in\mathbb{N}^*, m\in\mathbb{Z}$ and $x\in\mathbb{R}$ the following

$$f\left(\frac{m}{n}x\right)=\frac{1}{n}f(mx)=\frac{m}{n}f(x)\qquad (4)$$

Thus $f(r)=f(1)r$ for all $r\in\mathbb{Q}$. Now, the continuity of f shows that $f(x)=f(1)x$ for all real x. Conversely, any function of the form $x\to ax$ satisfies the proposed functional equation.

323. **Find all continuous functions $f:\mathbb{R}\to(0,\infty)$ such that:**

$$f(x)f(2x)f(4x)=2^x,\forall\, x\in\mathbb{R}$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Abdallah El Farissi-Bechar-Algerie

We put $x=0$ *in* $f(x)f(2x)f(4x)=2^x$, *we get* $f^3(0)=1$ *then* $f(0)=1$.

We have $\begin{cases} f\left(\frac{x}{2}\right)f(x)f(2x)=2^{\frac{x}{2}} \\ f(x)f(2x)f(4x)=2^x \end{cases}$ *it follows that* $f(4x)=2^{\frac{x}{2}}f\left(\frac{x}{2}\right)$ *then* $f(x)=2^{\frac{x}{8}}f\left(\frac{x}{8}\right)$ *by induction we get* $f(x)=2^{\frac{x}{8}}2^{\frac{x}{8^2}}\cdots2^{\frac{x}{8^n}}f\left(\frac{x}{8^n}\right)=2^{\frac{x}{8}\left(\frac{1-\left(\frac{1}{8}\right)^n}{1-\frac{1}{8}}\right)}f\left(\frac{x}{8^n}\right)$ *for all* $n\in\mathbb{N}$ *then*

$$f(x)=\lim_{n\to+\infty}f(x)=\lim_{n\to+\infty}2^{\frac{x}{8}\left(\frac{1-\left(\frac{1}{8}\right)^n}{1-\frac{1}{8}}\right)}f\left(\frac{x}{8^n}\right)=2^{\frac{x}{7}}f(0)=2^{\frac{x}{7}}$$

Solution 2 by Ravi Prakash-New Delhi-India

$$f(x)f(2x)f(4x)=2^x,\forall x\in\mathbb{R}$$

$$f(2x)f(4x)f(8x)=2^{2x}\Rightarrow\frac{f(2x)f(4x)f(8x)}{f(x)f(2x)f(4x)}=\frac{2^{2x}}{2^x}=2^x\Rightarrow f(8x)=2^xf(x)\Rightarrow$$

$$\Rightarrow f(x)=2^{\frac{x}{8}}f\left(\frac{x}{8}\right)=2^{\frac{x}{8}}\cdot2^{\frac{x}{8^2}}f\left(\frac{x}{8^2}\right)=2^{\frac{x}{8}+\frac{x}{8^2}+\frac{x}{8^3}}f\left(\frac{x}{8^3}\right)$$

$f(x)=2^{\frac{x}{8}+\frac{x}{8^2}+\frac{x}{8^3}+\cdots+\frac{x}{8^n}}f\left(\frac{x}{8^n}\right)=2^{\frac{x}{7}\left(1-\left(\frac{1}{8}\right)^n\right)}f\left(\frac{x}{8^n}\right)$. *Taking limit as* $n\to\infty$ *we get*

$$f(x)=2^{\frac{x}{7}}f(0)\ [\because f\text{ is continuous}].$$

Also, $f(x)f(2x)f(4x)=2^x\Rightarrow f(0)f(0)f(0)=1\Rightarrow f(0)=1$. *Thus,* $f(x)=2^{\frac{x}{7}}$

324.

$$\int_0^1\int_0^1 \ln\Gamma\,(x+y+1)\,dx\,dy$$

Proposed by Shafiqur Rahman-Bangladesh

Solution by Togrul Ehmedov-Baku-Azerbaidian

$$\int_0^1\int_0^1 \ln\Gamma(x+y+1)\,dx\,dy = \int_0^1\int_{x+1}^{x+2} \ln\Gamma(u)\,du\,dx$$

$$= \left[x\int_{x+1}^{x+2} \ln\Gamma(u)\,du\right]_0^1 - \int_0^1 x\ln\frac{\Gamma(x+2)}{\Gamma(x+1)}dx = \int_0^2 \ln\Gamma(u)\,du - \int_0^1 x\ln(x+1)\,dx$$

$$I_1 = \int_1^2 \ln\Gamma(u)\,du;\ I_1(a) = \int_a^{a+1} \ln\Gamma(\mathrm{u})\,du$$

$$I_1'(a) = \ln\Gamma(a+1) - \ln\Gamma(a) = \ln a\,, \qquad I_1(a) = a\ln a - a + C$$

$$I_1(0) = \int_0^1 \ln\Gamma(u)\,du = \ln\left(\sqrt{2\pi}\right) = C\,, \qquad I_1(a) = a\ln a - a + \ln\left(\sqrt{2\pi}\right)$$

$$I_1 = -1 + \ln\left(\sqrt{2\pi}\right)\,, \qquad I_2 = \int_0^1 x\ln(x+1)\,dx = \frac{1}{4}$$

$$I = \frac{3}{4} + \ln\left(\sqrt{2\pi}\right)$$

325. Find:

$$\Omega = \int_a^{\frac{\pi}{2}} \left(\frac{\sin 7x}{\sin x}\right)^2 dx, 0 < a \le \frac{\pi}{2}$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

$$For\ 0 < \alpha < x \le \frac{\pi}{2}$$

$$\frac{\sin 7x}{\sin x} = \frac{\sin 7x - \sin 5x + \sin 5x - \sin 3x + \sin 3x - \sin x + \sin x}{\sin x}$$

$$= \frac{2\cos 6x\sin x + 2\cos 4x\sin x + 2\cos 2x\sin x + \sin x}{\sin x}$$

$$= 2\cos 6x + 2\cos 4x + 2\cos 2x + 1$$

$$\Rightarrow \left(\frac{\sin 7x}{\sin x}\right)^2 = 4\cos^2 6x + 4\cos^2 4x + 4\cos^2 2x + 1 + 8\cos 6x\cos 4x +$$

$$+8\cos 6x\cos 2x + 4\cos 6x + 8\cos 4x\cos 2x + 4\cos 4x + 4\cos 2x$$

$$= 2(1-\cos 12x) + 2(1+\cos 8x) + 2(1+\cos 4x) + 1 + 4(\cos 10x + \cos 2x) +$$

$$+4(\cos 8x + \cos 4x) + 4\,(\cos 6x + \cos 2x) + 4\cos 6x + 4\cos 4x + 4\cos 2x$$

$$= 7 + 12\cos 2x + 10\cos 4x + 8\cos 6x + 6\cos 8x + 4\cos 10x + 2\cos 12x$$

$$\int_{\alpha}^{\frac{\pi}{2}} \left(\frac{\sin 7x}{\sin x}\right)^2 dx = \int_{\alpha}^{\frac{\pi}{2}} \begin{pmatrix} 7 + 12\cos 2x + 10\cos 4x + 8\cos 6x + \\ +6\cos 8x + 4\cos 10x + 2\cos 12x \end{pmatrix} dx$$

$$= \left[7x - 6\sin 2x - \frac{5}{2}\sin 4x - \frac{4}{3}\sin 6x - \frac{3}{4}\sin 8x - \frac{2}{5}\sin 10x - \frac{1}{6}\sin 12x\right]_{\alpha}^{\frac{\pi}{2}}$$

$$= 7\left(\frac{\pi}{2}\right) + 6\sin 2\alpha + \frac{5}{2}\sin 4\alpha + \frac{4}{3}\sin 6\alpha + \frac{3}{4}\sin 8\alpha + \frac{2}{5}\sin 10\,\alpha + \frac{1}{6}\sin 12\alpha - 7\alpha$$

326. Evaluate:

$$\int_0^{\infty} \frac{1}{(1+y^{(20n)!!})(1+y^2)}dy, n \in \mathbb{N}$$

where *"!!"* means double factorial

Proposed by Ibrahim Abdulazeez-Zaria-Nigeria

Solution by Yen Tung Chung-Taichung-Taiwan

$$\int_0^{\infty} \frac{1}{(1+y^k)(1+y^2)}dy = \int_0^{1} \frac{1}{(1+y^k)(1+y^2)}dy + \underbrace{\int_1^{\infty} \frac{1}{(1+y^k)(1+y^2)}dy}_{let\ y=\frac{1}{x}\Rightarrow dy=-\frac{1}{x^2}dx}$$

$$= \int_0^{1} \frac{1}{(1+y^k)(1+y^2)}dy + \int_1^{0} \frac{1}{\left(1+\frac{1}{x^k}\right)\left(1+\frac{1}{x^2}\right)} \cdot \left(-\frac{1}{x^2}dx\right)$$

$$=\int_0^1 \frac{1}{(1+y^k)(1+y^2)}dy+\int_0^1 \frac{x^k}{(x^k+1)(x^2+1)}dx$$

$$=\int_0^1 \frac{1}{(1+y^k)(1+y^2)}dy+\int_0^1 \frac{y^k}{(y^k+1)(y^2+1)}dx=\int_0^1 \frac{1+y^k}{(1+y^k)(1+y^2)}dy$$

$$=\int_0^1 \frac{1}{1+y^2}dy=(\tan^1 y)|_0^1=\frac{\pi}{4}.\ So,\ \int_0^\infty \frac{1}{(1+y^{(20n)!!})(1+y^2)}dy=\frac{\pi}{4}$$

327. **Prove that:**

$$\int_0^1 \frac{x^{p-1}(1-x)^{q-1}}{(a+bx)^{p+q}}dx=\frac{\beta(p,q)}{a^q(a+b)^{p+q}}$$

Proposed by Amit Dutta-Jamshedpur-India

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$\Omega=\int_0^1 \frac{x^{p-1}(1-x)^{q-1}}{(a+bx)^{p+q}}dx,\ let\ t=\frac{bx}{a(1-x)}\Rightarrow x=\frac{at}{at+b}$$

$$dx=\frac{ab}{(at+b)^2}dt;\ when\ x=0,t=0;\ when\ x=1,t\to\infty$$

$$\Omega=\int_0^\infty \frac{\left(\frac{at}{at+b}\right)^{p-1}\left(1-\frac{at}{at+b}\right)^{q-1}}{\left(a+\frac{abt}{at+b}\right)^{p+q}}\cdot\frac{ab}{(at+b)^2}dt=$$

$$=\frac{ab}{a^{p+q}}\int_0^\infty \frac{\left(\frac{at}{at+b}\right)^{p-1}\left(\frac{b}{at+b}\right)^{q-1}}{\left(1+\frac{bt}{at+b}\right)^{p+q}}\cdot\frac{dt}{(at+b)^2}$$

$$=\frac{b^q}{a^q}\int_0^\infty \frac{t^{p-1}}{\left(b+t(a+b)\right)^{p+q}}dt=\frac{1}{a^q b^p}\int_0^\infty \frac{t^{p-1}}{\left(1+\frac{a+b}{b}\cdot t\right)^{p+q}}dt$$

$$=\frac{1}{a^q b^p}\cdot\frac{b}{a+b}\cdot\left(\frac{b}{a+b}\right)^{p-1}\int_0^\infty \frac{z^{p-1}}{(1+z)^{p+q}}dz\ \left[we\ put\ z=\frac{a+b}{b}t\right]=\frac{\beta(p,q)}{a^q(a+b)^{p+q}}\quad (proved)$$

Solution 2 by Tobi Joshua-Lagos-Nigeria

$$\text{We know that } B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$$

$$\text{Then Let } I = \int_0^1 \frac{x^{p-1}(1-x)^{q-1}}{(a+bx)^{p+q}}dx. \text{ Putting } x = \frac{1}{1+y}; dx = -\frac{dy}{(1+y)^2}$$

$$I = \int_\infty^0 \frac{\left(1-\frac{1}{1+y}\right)^{q-1}\left(\frac{1}{1+y}\right)^{p-1}}{\left(a+b\left(\frac{1}{1+y}\right)\right)^{p+q}} \cdot \frac{-dy}{(1+y)^2}$$

$$I = \int_0^\infty \frac{y^{q-1}dy}{(a+ay+b)^{p+q}\frac{(1+y)^{p+q-2}}{(1+y)^{p+q}}(1+y)^2} \Rightarrow \int_0^\infty \frac{y^{q-1}}{0\left((a+b)+ay\right)^{p+q}}dy$$

$$I = \int_0^\infty \frac{y^{q-1}dy}{(a+b)^{p+q}\left(1+\frac{ay}{(a+b)}\right)^{p+q}}. \text{ Put } u = \frac{ay}{a+b}, du = \frac{ady}{a+b}$$

$$I = \frac{1}{(a+b)^{p+q}}\int_0^\infty \frac{(a+bc)^{q-1}y^{q-1}}{a^{q-1}(1+u)^{p+q}} \cdot du\frac{(a+b)}{a}$$

$$I = \frac{(a+b)^q}{(a+b)^{p+q}(a^q)}\int_0^\infty \frac{u^{q-1}}{(1+u)^{p+q}}du \Rightarrow \frac{(q+bc)^q}{a^q(a+b)^{p+q}}\int_0^\infty \frac{u^{q-1}}{(1+u_i)^{p+q}}du$$

$$I = \frac{1}{a^7(a+b)^p}\int_0^\infty \frac{u^{q-1}}{(1+u)^{p+q}}du \Rightarrow \frac{B(p,q)}{a^q(a+b)^p}$$

Q.E.D.

328. For any complex number m & $Re(m) > -\frac{1}{2}$

$$H_m = H_{m-\frac{1}{2}} + \frac{d}{dm}\left(\int_0^1 \frac{x^{2m}-x}{ln(x)} - \frac{dx}{1+x}\right)$$

Where H_m – Harmonic Number.

Proposed by K.Srinivasa Raghava-AIRMC-India

Solution by Kartick Chandra Betal-India

$$H_{m-\frac{1}{2}}+\frac{d}{dm}\int_0^1\left(\frac{x^{2m}-x}{\ln x}\cdot\frac{dx}{1+x}\right)=H_{m-\frac{1}{2}}+\frac{d}{dm}\int_0^1\frac{1}{1+x}\left(\int_1^{2m}x^y\,dy\right)dx$$

$$=H_{m-\frac{1}{2}}+\frac{d}{dm}\int_1^{2m}\left(\frac{x^y}{1+x}dx\right)dy=H_{m-\frac{1}{2}}+2\left(\int_0^1\frac{x^{2m}}{1+x}dx\right)$$

$$=\int_0^1\frac{1-x^{m-\frac{1}{2}}}{1-x}dx+2\int_0^1\frac{x^{2m}}{1+x}dn=\int_0^1\frac{1-x^{2m-1}}{1-x^2}\cdot 2xdx+2\int_0^1\frac{x^{2m}}{1+x}dn$$

$$=2\int_0^1\frac{x-x^{2m}}{(1+x)(1-x)}dn+2\int_0^1\frac{x^{2m}}{1+x}=2\int_0^1\frac{x-x^{2m}+x^{2m}(1-x)}{1-x^2}dx$$

$$=2\int_0^1\frac{x-x^{2m-1}}{1-x^2}dn=\int_0^1\frac{1-x^{2m}}{1-x^2}d(x^2)=\int_0^1\frac{1-x^m}{1-x}dx=H_m$$

(proved)

329. Find:

$$\Omega=\int\frac{x(\tan x+2\tan 2x+4\tan 4x)}{\cot x-8\cot 8x}dx$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursarescu-Romania

$$\textit{We know } \tan x=\cot x-2\cot 2x\ \textit{(easy to prove)}\Rightarrow$$

$$\left.\begin{array}{c}\tan x=\cot 2x-2\cot 2x\\ 2\tan 2x=2\cot 2x-4\cot 4x\\ 4\tan 4x=4\cot 4x-8\cot 8x\end{array}\right\}\Rightarrow$$

$$\tan x+2\tan 2x+4\tan 4x=\cot 2x-8\cot 8x\Rightarrow\Omega=\int x\,dx=\frac{x^2}{2}+C$$

Solution 2 by Ravi Prakash-New Delhi-India

$$\cot x-(\tan x+2\tan 2x+4\tan 4x)=(\cot x-\tan x)-2\tan 2x-4\tan 4x$$

$$=2\left(\frac{1-\tan^2 x}{2\tan x}\right)-2\tan 2x-4\tan 4x=2\cot 2x-2\tan 2x-4\tan 4x$$

$$=2(\cot 2x-\tan 2x)-4\tan 4x=(2)(2\cot 4x)-4\tan 4x=$$

$$=4(\cot 4x-\tan 4x)=(4)(2\cot 8x)=8\cot 8x\Rightarrow \cot x-8\cot 8x=$$

$$=\tan x+2\tan 2x+4\tan 4x \therefore \Omega=\int\frac{x(\cot x-8\cot 8x)}{\cot x-8\cot 8x}dx=\int x\,dx=\frac{1}{2}x^2+c$$

Solution 3 by Hasan Bostanlik-Sarkisla-Turkey

$$\sin 8x=2\sin 4x-\cos 4x=4\cdot\sin 2x\cos 2x\cos 4=8\cdot\sin x\cos x\cdot\cos 2x-\cos 4x$$

$$8\cot 8x=\cot x-\tan x-2\tan 2x-4\tan 4x$$

$$\cot x-8\cot 8x=\tan x+2\tan 2x+4\tan 4x;\ \Omega=\int x\,dx\Rightarrow\frac{x^2}{2}+c$$

Solution 4 by Soumitra Mandal-Chandar Nagore-India

$$8\cot 8x+4\tan 4x+2\tan 2x+\tan x=\frac{8}{\tan 8x}+4\tan 4x+2\tan 2x+\tan x$$

$$=\frac{4(1-\tan^2 4x)}{\tan 4x}+4\tan 4x+2\tan 2x+\tan x=\frac{4}{\tan 4x}+2\tan 2x+\tan x$$

$$=\frac{2(1-\tan^2 2x)}{\tan 2x}+2\tan 2x+\tan x=\frac{2}{\tan 2x}+\tan x=\frac{1-\tan^2 x}{\tan x}+\tan x$$

$$=\cot x\ \textit{then}\ \int\frac{x(\tan x+2\tan 2x+4\tan 4x)}{\cot x-8\cot 8x}dx=\int x\,dx=\frac{x^2}{2}+c$$

330. Evaluate:

$$\int\frac{\tan x+\ln(1-x)^{\ln x}}{x}dx$$

Proposed by Nawar Alasadi-Babylon-Iraq

Solution by Shivam Sharma-New Delhi-India

As we know, the series representation of $\tan(x)$.

$$\tan(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}x^{2n-1}\quad (1)$$

$$\Rightarrow \int \frac{\tan(x)}{x} dx + \int \frac{\ln(x)\ln(2-x)}{x} dx. \textit{ Using (1), we get,}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}}{(2n)!} \int x^{2n-2}\,dx + \int \sum_{n=1}^{\infty} \frac{-x^{n-1}}{n} \ln(x)\,dx \Rightarrow$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}}{(2n)!} \left[\frac{x^{2n-1}}{2n-1}\right] - \sum_{n=1}^{\infty} \frac{1}{n} \int x^{n-1}\ln(x)\,dx \Rightarrow$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}x^{2n-1}}{(2n)!\,(2n-1)} - \sum_{n=1}^{\infty} \frac{1}{n}\left[\frac{\partial}{\partial n}\left(\int x^{n-1}\partial x\right)\right]$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}x^{2n-1}}{(2n)!\,(2n-1)} - \sum_{n=1}^{\infty} \frac{1}{n}\left[\frac{x^n\ln(x)}{n} - \frac{x^n}{n^2}\right] + c \Rightarrow$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}x^{2n}-1}{(2n)!\,(2n-1)} + \sum_{n=1}^{\infty} \left(\frac{x^n}{n^3}\right) + c$$

(OR)

$$I = \sum_{n=1}^{\infty} \left(\frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}x^{2n-1}}{(2n)!\,(2n-1)}\right) + Li_3(x) + c$$

(Answer)

331. Find:

$$\Omega = \int \tan^2 x\,(\tan x + 2\tan 2x + 4\tan 4x + 8\cot 8x)dx,\ x \in \left(0, \frac{\pi}{2}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursarescu-Romania

Because for any $x \in \mathbb{R}$ *we have:* $\tan x = \cot x - 2\cot 2x \Rightarrow$

$$\left.\begin{aligned} \tan x &= \cot x - 2\cot 2x \\ 2\tan 2x &= 2\cot 2x - 4\cot 4x \\ 4\tan 4x &= 4\cot 4x - 8\cot 8x \end{aligned}\right\} \Rightarrow \tan x + 2\tan 2x + 4\tan 4x + 8\cot 8x = \cot x \Rightarrow$$

$$\Rightarrow \Omega = \int \tan^2 x \cdot \cot x\,dx = \int \tan x\,dx = -\ln|\cos x| + C$$

Solution 2 by Ravi Prakash-New Delhi-India

$$\cot x - \{\tan x + 2\tan 2x + 4\tan 4x\} = (\cot x - \tan x) - 2\tan 2x - 4\tan 4x$$

$$= 2\left(\frac{1-\tan^2 x}{2\tan x}\right) - 2\tan 2x - 4\tan 4x = 2\cot 2x - 2\tan 2x - 4\tan 4x =$$

$$= 2(\cot 2x - \tan 2x) - 4\tan 4x = (2)(2\cot 4x) - 4\tan 4x =$$

$$= 4(\cot 4x - \tan 4x) = (4)(2\cot 8x) = 8\cot 8x$$

$$\therefore \tan x + 2\tan 2x + 4\tan 4x + 8\cot 8x = \cot x$$

$$\textit{Thus, } \Omega = \int \tan^2 x \cot x\, dx = \log|\sec x| + C$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$8\cot 8x + 4\tan 4x + 2\tan 2x + \tan x = 4\frac{1-\tan^2 4x}{\tan 4x} + 4\tan 4x + 2\tan 2x + \tan x$$

$$= \frac{4}{\tan 4x} + 2\tan 2x + \tan x = 2\frac{1-\tan^2 2x}{\tan 2x} + 2\tan 2x + \tan x = \frac{2}{\tan 2x} + \tan x$$

$$= \frac{1-\tan^2 x}{\tan x} + \tan x = \cot x$$

$$\int \tan^2 x\,(8\cot 8x + 4\tan 4x + 2\tan 2x + \tan x)\,dx = \int \tan x\,dx = \log|\sec x| + C$$

Solution 4 by Amit Dutta-Jamshedpur-India

$$\tan x + 2\tan 2x + 4\tan 4x + 8\cot 8x \Rightarrow \tan x + 2\tan 2x + 4\tan 4x + \frac{8}{\tan 8x}$$

$$\textit{Using } \tan 2\theta = \left(\frac{2\tan\theta}{1-\tan^2\theta}\right) \Rightarrow \tan x + 2\tan 2x + 4\tan 4x + \frac{8(1-\tan^2 4x)}{2\tan 4x} \Rightarrow$$

$$\Rightarrow \tan x + 2\tan 2x + 4\tan 4x + \frac{4(1-\tan^2 4x)}{\tan 4x} \Rightarrow \tan x + 2\tan 2x + 4\tan 4x +$$

$$+ \frac{4}{\tan 4x} - 4\tan 4x \Rightarrow \tan x + 2\tan 2x + \frac{4(1-\tan^2 2x)}{2\tan 2x} \Rightarrow$$

$$\Rightarrow \tan x + 2\tan 2x + \frac{2}{\tan 2x} - 2\tan 2x \Rightarrow \tan x + \frac{2(1-\tan^2 x)}{2\tan x} \Rightarrow$$

$$\Rightarrow \tan x + \frac{1}{\tan x} - \tan x = \frac{1}{\tan x} = \cot x$$

$$\Omega = \int \tan^2 x\,(\cot x)dx = \int \tan x\,dx \Rightarrow -\ln|\cos x| + C$$

332. Find:

$$\Omega = \int \frac{tanh^2 x + tanh^2 x\,(1 + tanh^2 x)^2}{(1 + tanh^2 x)^2} dx, x \in \mathbb{R}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\Omega = \int \frac{\tanh^2 x + \tanh^2 x\,(1 + \tanh^2 x)^2}{(1 + \tanh^2 x)^2} dx$$

$\Omega = \int \frac{t^2+t^2(1+t^2)^2}{(1+t^2)^2} \cdot \frac{dt}{1-t^2}$ *where* $t = \tanh x$. *To split into partial fractions, put* $t^2 = y$

$$\frac{y + y(1+y)^2}{(1+y)^2(1-y)} = -1 + \frac{A}{1+y} + \frac{B}{(1+y)^2} + \frac{C}{1-y} \Rightarrow y + y(1+y)^2 =$$

$$= -(1-y)(1+y)^2 + A(1+y)(1-y) + B(1-y) + C(1+y)^2$$

Put $y = 1; 1 + 2^2 = C(1+1)^2 \Rightarrow C = \frac{5}{2}$. *Put* $y = -1; -1 = -2B \Rightarrow B = \frac{1}{2}$. *Put* $y = 0$

$0 = -1 + A + B + C \Rightarrow A = 1 - B - C = 1 - 3 = -2$. *Thus*

$, \Omega = \int \left[-1 - \frac{2}{1+t^2} + \frac{1}{2} \cdot \frac{1}{(1+t^2)^2} + \frac{5}{2} \cdot \frac{1}{1-t^2}\right] dt = -t - 2\tan^{-1} t + \frac{1}{2} I_1 + \frac{5}{2} \cdot \frac{1}{2} \ln\left|\frac{1+t}{1-t}\right| + C$

where $I_1 = \int \frac{1}{(1+t^2)^2} dt$. *Let* $I_2 = \int \frac{dt}{1+t^2} = \frac{t}{1+t^2} + \int \frac{t \cdot 2t}{(1+t^2)^2} = \frac{t}{1+t^2} + 2\int \frac{t^2+1-1}{(1+t^2)^2} dt =$

$$= \frac{t}{1+t^2} + 2I_2 - 2I_1 \Rightarrow -2I_1 = -I_2 - \frac{t}{1+t^2} = -\tan^{-1} t - \frac{t}{1+t^2}$$

$$\therefore \Omega = -t - 2\tan^{-1} t - \frac{1}{4}\tan^{-1} t - \frac{1}{4} \cdot \frac{t}{1+t^2} + \frac{5}{4}\ln\left|\frac{1+t}{1-t}\right| + C = -t - \frac{9}{4}\tan^{-1} t -$$

$-\frac{1}{4} \cdot \frac{t}{1+t^2} + \frac{5}{4}\ln\left|\frac{1+t}{1-t}\right| + C$ *where* $t = \tanh x$.

Solution 2 by Ibrahim Abdulazeez-Zaria-Nigeria

$$\Omega = \int \left[\frac{\tanh^2 x}{(1 + \tanh^2 x)^2} + \frac{\tanh^2 x}{x}\right] dt;\ \Omega = \int \frac{\tanh^2 x}{(1 + \tanh^2 x)^2} dx +$$

$$+ \int \tanh^2 x\, dx = \frac{1}{4}\int \tanh^2 2x\, dx + (x - \tanh x) = \frac{1}{8}(2x - \tanh 2x) + x - \tanh x + C$$

$$\Omega = \frac{x}{4} + x - \tanh 2x - \tanh x + C;\ \Omega = \frac{5x}{4} - \tanh x - \tanh 2x + C$$

333. Find:

$$\Omega = \int_0^{\frac{\pi}{6}} \frac{\tan^4 x\,(\tan^2 x - 2)}{(1 - \tan^2 x)^2}\,dx$$

Proposed by Daniel Sitaru – Romania

Solution by Omran Kouba-Damascus-Syria

Note that $\frac{\tan^4 x(\tan^2 x-2)}{(1-\tan^2 x)^2} = \frac{\tan^2 x\left((\tan^2 x-1)^2-1\right)}{(1-\tan^2 x)^2} = \tan^2 x - \frac{\tan^2 x}{(1-\tan^2 x)^2} = \tan^2 x - \frac{1}{4}\tan^2 2x$

$$= (1+\tan^2 x) - \frac{1}{4}(\tan^2 2x + 1) - \frac{3}{4} = \left(\tan x - \frac{1}{8}\tan 2x - \frac{3x}{4}\right)'$$

$$So,\ \Omega = \left[\tan x - \frac{1}{8}\tan 2x - \frac{3x}{4}\right]_0^{\frac{\pi}{6}} = \frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{8} - \frac{\pi}{8} = \frac{5\sqrt{3}}{24} - \frac{\pi}{8}$$

334. Find:

$$\Omega = \int \frac{\tan^2 x \cdot \tan^4 x \cdot \tan^6 x \cdot \tan^8 x \cdot \ldots \cdot \tan^{2n} x}{\sin^2 x\sqrt{1-\tan^{n^2+n+1}x} - \sqrt{1-\tan^{n^2+n+1}x} - \cos^2 x\sqrt{1-\tan^{n^2+n+1}x}}\,dx$$

Proposed by Vural Ozap-Turkey

Solution by Daniel Sitaru-Romania

$$\Omega = \int \frac{\tan^{2\cdot\frac{n(n+1)}{2}}}{-2\cos^2 x\sqrt{1-\tan^{n^2+n+1}x}}\,dx = \int \frac{(\tan x)'\tan^{n^2+n}x}{-2\sqrt{1-\tan^{n^2+n+1}x}}\,dx =$$

$$= \frac{1}{n^2+n+1}\int \frac{\left(1-\tan^{n^2+n+1}x\right)'}{2\sqrt{1-\tan^{n^2+n+1}x}}\,dx = \frac{1}{n^2+n+1}\sqrt{1-\tan^{n^2+n+1}x} + C$$

335. If $0 < a < b < 1$ then:

$$\frac{1}{2}+\frac{2}{a+b}<\frac{2(\ln b-\ln a)}{b-a}+\frac{1}{b-a}\int_a^b \frac{dx}{\ln(1-x)}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$\text{Let } f(x)=x-\ln(1-x) \text{ for all } x\in(0,1) \text{ then } f'(x)=1+\frac{1}{1-x}>0$$

$$\text{hence } f \text{ is increasing on } (0,1) \text{ then } f(x)>f(0)=0 \Rightarrow x>\ln(1-x)$$

$$\frac{1}{b-a}\int_a^b \frac{dx}{\ln(1-x)}+\frac{2(\ln b-\ln a)}{b-a}>\frac{1}{b-a}\int_a^b \frac{dx}{x}+\frac{2}{b-a}\int_a^b \frac{dx}{x}=\frac{3}{b-a}\int_a^b \frac{dx}{x}$$

$$>\frac{6}{a+b}\left[\begin{array}{c}\because \frac{1}{x} \text{ is convex then applying}\\ \text{Hermite Hadamard Inequality}\end{array}\right]$$

$$\text{We need to prove, } \frac{6}{a+b}>\frac{2}{a+b}+\frac{1}{2}\Leftrightarrow \frac{4}{a+b}>\frac{1}{2}\Leftrightarrow 2>a+b$$

$$\text{Which is true since } 1>a>b. \text{ Hence true.}$$

336. Prove that:

$$\int_2^3\int_2^3\left(\frac{e^x+e^y-4}{\sqrt{xy}-1}\right)^5 dxdy \geq 32$$

Proposed by Sameer Shihab-Saudi Arabia

Solution by Daniel Sitaru-Romania

$$e^x+e^y-4\geq x+1+y+1-4 \overset{AM-GM}{\geq} 2\sqrt{xy}-2$$

$$\left(\frac{e^x+e^y-4}{\sqrt{xy}-1}\right)^5\geq 2^5\to \int_2^3\int_2^3\left(\frac{e^x+e^y-4}{\sqrt{xy}-1}\right)^5 dxdy\geq \int_2^3\int_2^3 32dxdy=32$$

337. If $0 < a < b < 1$ then:

$$\frac{2(\ln b - \ln a)}{b-a} + \frac{1}{b-a}\int_a^b \frac{dx}{\ln(1-x)} < 1 + \frac{1}{\sqrt{ab}}$$

Proposed by Daniel Sitaru – Romania

Solution by Chris Kyriazis-Athens-Greece

We know that: $\ln x \le x - 1, \forall x > 0$

Setting $x \to \frac{1}{x} > 0$, *we have that,* $\ln x \ge \frac{x-1}{x}, \forall x > 0$ *(1)*

So, setting $x \to 1 - x > 0$, *we have that*

$$0 > \ln(1-x) \ge -\frac{x}{1-x} \Rightarrow \frac{1}{\ln(1-x)} \le \frac{x-1}{x} \Rightarrow \frac{1}{\ln(1-x)} \le 1 - \frac{1}{x} \Rightarrow$$

$$\Rightarrow \int_a^b \frac{1}{\ln(1-x)}dx \le b - a - (\ln b - \ln a)$$

$$\Rightarrow 2\frac{\ln b - \ln a}{b-a} + \frac{1}{b-a}\int_a^b \frac{1}{\ln(1-x)}dx \le \frac{\ln b - \ln a}{b-a} + 1$$

So, it suffices to prove that $\frac{\ln b - \ln a}{b-a} + 1 < 1 + \frac{1}{\sqrt{ab}}$ *or* $\sqrt{ab} < \frac{b-a}{\ln b - \ln a}$ *which holds as a fundamental property of the Lograrithmic Mean!*

338. If $0 < a \le b$ then:

$$2\int_a^b \frac{1}{1+e^{2x^2}}dx \ge \frac{\tan^{-1}(e^{b^2}) - \tan^{-1}(e^{a^2})}{be^{b^2}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

For $0 < a \le x \le b \Rightarrow xe^{x^2} \le be^{b^2}$. *Now,*

$$2\int_a^b \frac{be^{b^2}}{1+e^{2x^2}}dx \ge 2\int_a^b \frac{xe^{x^2}}{1+e^{2x^2}}dx = \tan^{-1}(e^{x^2})]_a^b = \tan^{-1}(e^{b^2}) - \tan^{-1}(e^{a^2})$$

$$\Rightarrow 2\int_a^b \frac{dx}{1+e^{2x^2}} \geq \frac{\tan^{-1}(e^{b^2}) - \tan^{-1}(e^{a^2})}{be^{b^2}}$$

Solution 2 by Kartick Chandra Betal-India

$$2\int_a^b \frac{1}{1+e^{2x^2}}dx = \int_a^b \frac{2e^{x^2}x}{1+e^{2x^2}} \cdot \left(\frac{e^{-x^2}}{x}\right)dx =$$

$$= \int_a^b \frac{e^{x^2}2x}{1+e^{2x^2}} \cdot \frac{1}{xe^{x^2}}dx \geq \int_a^b \frac{e^{x^2}\cdot 2x}{1+e^{2x^2}} \cdot \frac{1}{be^{b^2}} \text{ as } b > a$$

$$\geq \frac{1}{be^{b^2}}\left[\tan^{-1}(e^{x^2})\right]_a^b = \frac{1}{be^{b^2}}\left[\tan^{-1}(eb^2) - \tan^{-1}(e^{a^2})\right]$$

$$\therefore \frac{1}{be^{b^2}} \leq \frac{1}{xe^{x^2}} \leq \frac{I}{ae^{x^2}} \text{ for } a < x \,\&\, b \geq a$$

$$\therefore 2\int_a^b \frac{1}{1+e^{2x^2}}dx \geq \frac{1}{be^{b^2}}\left[\tan(e^{b^2}) - \tan^{-1}(e^{a^2})\right]$$

339. Prove that:

$$\int_2^3 \ln x \ln(x^2-1)\, dx < \frac{35}{8} + \ln\frac{3}{2}$$

Proposed by Mihály Bencze – Romania

Solution by Tobi Joshua-Lagos-Nigeria

$$I = \int_2^3 \log x \log(x^2-1)\, dx$$

$$I = \left[\left(\log(x^2-1)\,(x\log x - x)\right)\right]_2^3 - \int_2^3 \frac{(2x)(x\log x - x)}{x^2-1}dx$$

$$I=\left[\left(\log(8)\,(3\log 3-3)\right)-\left(\log(3)\,(2\log 2-2)\right)\right]-2\int_2^3\frac{(x^2)(\log x-1)}{x^2-1}dx$$

$$I=[2\log(3)+\log(2)\,(7\log(3)-9)]-2\int_2^3\frac{(x^2\log x)-x^2}{x^2-1}dx$$

$$I=[2\log(3)+\log(2)\,(7\log(3)-9)]-2\int_2^3\frac{(x^2\log x)}{x^2-1}dx+2\int_2^3\frac{x^2}{x^2-1}dx$$

$$I=[2\log(3)+\log(2)\,(7\log(3)-9)]-2\int_2^3\log x\,dx-2\int_2^3\frac{\log x}{x^2-1}dx+2\int_2^3 dx+2\int_2^3\frac{dx}{x^2-1}$$

$$I=[2\log(3)+\log(2)\,(7\log(3)-9)]-2[x\log x-x]_2^3-\int_2^3\frac{\log x}{x-1}dx+\int_2^3\frac{\log x}{x+1}dx+\left[\log\left|\frac{x-1}{x+1}\right|\right]_2^3$$

$$I=[2\log(3)+\log(2)\,(7\log(3)-9)]-2[x\log x-2]_2^3-\int_2^3\frac{\log x}{x-1}dx+\int_2^3\frac{\log x}{x+1}+\left[\log\left|\frac{1}{2}\right|-\log\left|\frac{1}{3}\right|\right]$$

$$\textit{Let } A=\int_2^3\frac{\log x}{x-1}dx;\ A=[-Li_2(1-x)+c]_2^3;\ A=-Li_2(-2)+Li_2(-1)$$

$$A=-Li_2(-2)-\frac{\pi^2}{12}\cong 0.614279\quad (1)$$

$$B=\int_2^3\frac{\log x}{x+1}dx=\sum_{k=0}^{\infty}(-1)^k\int_2^3 x^k\log x\,dx;\ B=\sum_{k=0}^{\infty}(-1)^k\left[\frac{x^{k+1}\log x}{(x+1)}-\frac{x^{k+1}}{(k+1)^2}\right]_2^3$$

$$B=\sum_{k=0}^{\infty}(-1)^k\left[\frac{3(3^k\log 3)}{(k+1)}-\frac{3(3^k)}{(k+1)^2}-\frac{2(2^k\log 2)}{(k+1)}+\frac{2(2^k)}{(k+1)^2}\right]$$

$$B=-\sum_{k=1}^{\infty}(-1)^k\left[\frac{(3^k\log 3)}{(k)}-\frac{(3^k)}{(k)^2}-\frac{(2^k\log 2)}{(k)}+\frac{(2^k)}{(k)^2}\right]$$

$$B=-[\log 3\,Li_1(-3)-Li_2(-3)-\log 2\,Li_1(-2)+Li_2(-2)]$$

$$B=[Li_2(-3)-Li_2(-2)+\log(2)\log(3)]\cong 0.258871\quad (2)$$

$$\Rightarrow I=[2\log(3)+\log(2)\,(7\log(3)-9)]-2[x\log x-x]_2^3-\left(-Li_2(-2)-\frac{\pi^2}{12}\right)+$$

$$+([Li_2(-3)-Li_2(-2)+\log(2)\log(3)])+\left[\log\left|\frac{1}{2}\right|-\log\left|\frac{1}{3}\right|\right]$$

$$\Rightarrow I = [2\log(3) + \log(2)\,(7\log(3) - 9)] - 2[3\log 3 - 3 - 2\log 2 + 2] -$$

$$-\left(-Li_2(-2) - \frac{\pi^2}{12}\right) + ([Li_2(-3) - Li_2(-2) + \log(2) + \log(2)\log(3)]) +$$

$$+\left[\log\left|\frac{1}{2}\right| - \log\left|\frac{1}{3}\right|\right]$$

$$\Rightarrow I = [2\log(3) + \log(2)\,(7\log(3) - 9)] - 2[3\log 3 - 3 - 2\log 2 + 2] -$$

$$-\left(-Li_2(-2) - \frac{\pi^2}{12}\right) + ([Li_2(-3) - Li_2(-2) + \log(2)\log(3)]) + \left[\log\left|\frac{3}{2}\right|\right]$$

$$\Rightarrow I = [2\log(3) + \log(2)\,(7\log(3) - 9)] - 2\left[\log\left(\frac{27}{4}\right) - 1\right] - (0.614279) + (0.258871) + \left[\log\left|\frac{3}{2}\right|\right]$$

$$\Rightarrow I = \log\left(\frac{9}{512}\right) + 7(\log 2\log 3) - 2\log\left(\frac{27}{4}\right) + \log\left(\frac{3}{2}\right) + 1.644592 < \frac{35}{8} + \log\left(\frac{3}{2}\right)$$

340. If $a, b, c \in \left(0, \frac{1}{2}\right)$ then:

$$b\int_a^{2a} \frac{15x+2}{36x^3+1}dx + c\int_a^{2b} \frac{15x+2}{36x^3+1}dx + a\int_a^{2c} \frac{15x+2}{36x^3+1}dx \le (a+b+c)\ln 2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Rovsen Pirguliyev-Sumgait-Azerbaidian

$$b\int_a^{2a} \frac{15x+2}{36x^3+1}dx + c\int_a^{2b} \frac{15x+2}{36x^3+1}dx + a\int_a^{2c} \frac{15x+2}{36x^3+1}dx \le (a+b+c)\ln 2$$

Since $36x^3 + 1 \ge 15x^2 + 2x \Leftrightarrow (3x-1)^2(4x+1) \ge 0$. *We have:* $\frac{15x+2}{36x^3+1} \le \frac{1}{x}$.

$$LHS \le b\int_a^{2a} \frac{1}{x}dx + c\int_a^{2b} \frac{1}{x}dx + a\int_a^{2c} \frac{1}{x}dx = b \cdot \ln|_a^{2a} + c\ln x|_a^{2b} + a\ln x|_a^{2c} =$$

$$= (a+b+c)\ln 2 \quad (q.e.d.)$$

Solution 2 by Lazaros Zachariadis-Thessaloniki-Greece

$$f(x) = \frac{15x^2+2x}{36x^3+1}, x \in \left(0, \frac{1}{2}\right)$$

$$f'(x)=\frac{(30x+2)(36x^3+1)-(15x^2+2x)\cdot 108x^2}{(36x^3+1)^2}$$

$$=\frac{1080x^4+30x+72x^3+2-1620x^4-216x^3}{(36x^3+1)^2}=\frac{-540x^4-144x^3+30x+2}{(36x^3+1)^2}$$

$$\Rightarrow f'(x)=\frac{-2(3x-1)(90x^3+54x^2+18x+1)}{(36x^3+1)^2}$$

x	0 $\qquad\qquad$ $\frac{1}{3}$ $\qquad\qquad$ $\frac{1}{2}$
$-(3x-1)$	$+++++++++0----------$
$1+90x^3+54x^2+18x$	$+++++++++++++++++++$
$f'(x)$	$+++++++++0----------$
$f(x)$	$\nearrow \qquad \searrow$

$$thus\ f(x)\geq f\left(\frac{1}{3}\right)=1;\ f\left(\frac{1}{3}\right)=\frac{\frac{15}{9}+\frac{2}{3}}{\frac{36}{27}+1}=\frac{\frac{30+12}{18}}{\frac{63}{27}}=\frac{42\cdot 27}{18\cdot 63}=1$$

$$f(x)\leq 1\Leftrightarrow \frac{x(15x+2)}{36x^3+1}\leq 1\overset{x>0}{\Leftrightarrow}\frac{15x+2}{36x^3+1}\leq\frac{1}{x}$$

$$\sum_{cyc}b\int_a^{2a}\frac{15x+2}{36x^3+1}dx\leq\sum_{cyc}b\int_a^{2a}\frac{1}{x}dx=\sum_{cyc}b\,(\ln x)_a^{2a}=\sum_{cyc}b\cdot\ln\frac{2x}{a}=(b+c+a)\ln 2$$

Solution 3 by Ravi Prakash-New Delhi-India

$$Let\ f(x)=(15x+2)x-(36x^3+1)=-(4x+1)(9x^2-6x+1)$$

$$=-(4x+1)(3x-1)^2\leq 0\ \ for\ all\ x>0\ \therefore\ \frac{15x+2}{36x^3+1}\leq\frac{1}{x}\ \ for\ x>0$$

$$\Rightarrow\int_\alpha^{2\alpha}\frac{15x+2}{36x^3+1}dx\leq\ln x]_\alpha^{2\alpha}=\ln 2\,,\forall\alpha>0$$

$$Thus\ b\int_a^{2a}\frac{15x+2}{36x^3+1}dx+c\int_a^{2b}\frac{15x+2}{36x^3+1}dx+a\int_a^{2c}\frac{15x+2}{36x^3+1}dx$$

$$\leq(b+c+a)\ln 2=(a+b+c)\ln 2$$

Solution 4 by Soumitra Mandal-Chandar Nagore-India

$$Let\ f(x)=36x^3-15x^2-2x+1\ for\ all\ x\in\left(0,\frac{1}{2}\right)$$

$$f'(x)=108x^2-30x-2=2(6x+1)(3x-1). \text{ Let } f'(\alpha)=0 \text{ then } \alpha=\frac{1}{3} \text{ or } \alpha=-\frac{1}{6}$$

but we will rejective value hence $\alpha=\frac{1}{3}$

$f''(\alpha)=216x-30=42>0$ *hence the function f attains minimum value at*

$$x=\alpha=\frac{1}{3}\in\left(0,\frac{1}{2}\right) \text{ so, } f(x)\geq f\left(\frac{1}{3}\right)>0$$

$$\therefore\sum_{cyc}b\int_a^{2a}\frac{15x+2}{36x^3+1}dx\leq\sum_{cyc}b\int_a^{2a}\frac{dx}{x}=\sum_{cyc}b(\ln 2a-\ln a)=\ln 2\sum_{cyc}a$$

proved

341. If $0\leq a<b$ then:

$$(1+ab-a^2)e^{a^2}<\frac{1}{b-a}\int_a^b e^{x^2}dx<(1-ab+b^2)\,e^{b^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Chris Kyriazis-Athens-Greece

$$f(x)=e^{x^2}, x\in[a,b], a\geq 0;\; f'(x)=2xe^{x^2}\geq 0, \forall x\in[a,b]$$

$$f''(x)=2e^{x^2}(1+2x^2)>0, \forall x\in[a,b]$$

So, f' is strictly increasing in $[a,b]$. Using the mean value theorem, we have that:

$$f'(f)=\frac{f(x)-f(a)}{x-a}, 0<x\leq b, f\in(a,x)$$

$$\text{So, } f>a\Rightarrow f'(f)>f'(a)\Rightarrow\frac{f(x)-f(a)}{x-a}>2ae^{a^2}\Rightarrow f(x)>f(a)+2ae^{a^2}(x-a)\Rightarrow$$

$$\int_a^b f(x)\,dx>f(a)\int_a^b dx+2ae^{a^2}\int_a^b(x-a)dx\Rightarrow$$

$$\int_a^b f(x)>f(a)(b-a)+2ae^{a^2}\left[\frac{(x-a)^2}{2}\right]_a^b\Rightarrow$$

$$\int_a^b f(x)\,dx > e^{a^2}(b-a) + ae^{a^2}(b-a)^2 \Rightarrow \int_a^b f(x)\,dx > e^{a^2}(b-a)(1+ab-a^2)$$

and we are done!. We can work exactly the same way for the R.H.S.

$$a \le x < b \Rightarrow f'(x) < f'(b) \Rightarrow \cdots$$

Solution 2 by Ravi Prakash-New Delhi-India

$$\text{Let } f(x) = e^{x^2} - [1 + 2a(x-a)]e^{a^2}, a \le x \le b, a > 0$$

$$f'(x) = 2xe^{x^2} - 2ae^{a^2}$$

$$\text{If } x > 0, e^{x^2} > e^{a^2} \Rightarrow 2xe^{x^2} > 2ae^{a^2} \therefore f'(x) > 0 \text{ for } 0 < x < b \Rightarrow$$

$$\Rightarrow f(x) \text{ increases on } [a,b] \Rightarrow f(x) > f(a) = 0 \text{ for } a < x \le b \Rightarrow$$

$$\Rightarrow \int_a^b \left[e^{x^a} - [1 + 2a(x-a)]e^{a^2}\right] dx > 0 \Rightarrow$$

$$\Rightarrow \int_a^b e^{x^2} dx > e^{a^2}(b-a) + 2ae^{a^2}(b-a)^2$$

$$\Rightarrow \frac{1}{b-a}\int_a^b e^{x^2} dx > [1 + a(b-a)]e^{a^2} \quad (1)$$

$$\text{Let } g(x) = [1 + 2b(b-x)]e^{b^2} - e^{x^2}, a \le x \le b$$

$$g'(x) = -2be^{b^2} - 2xe^{x^2} < 0 \ [\because 0 < a \le x < b]$$

$$\Rightarrow g(x) \text{ is strictly decreasing on } [a,b]. \text{ As } g(b) = e^{b^2} - e^{b^2} = 0, \text{ we get}$$

$$g(x) > g(b) = 0 \text{ for } a \le x < b \Rightarrow [1 + 2b(b-x)]eb^2 > e^{x^2}$$

$$\Rightarrow \int_a^b [1 + 2b(b-x)]\, e^{b^2} dx > \int_a^b e^{x^2}\, dx$$

$$\Rightarrow \int_a^b e^{x^2} dx < [x - b(b-x)^2]e^{b^2}]_a^b = [(b-a) + b(a-b)^2]e^{b^2} =$$

$$= (b-a)[1 + ab - b^2]e^{b^2}$$

$$\Rightarrow \frac{1}{b-a}\int_a^b e^{x^2} dx < (1 + ab - b^2)e^{b^2} \quad (2)$$

From (1), (2) we get the desired inequality.

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$\text{Let } f(x) = e^{x^2} \text{ for all } x \geq 0, f'(x) = 2xe^{x^2}$$

$$f''(x) = 2e^{x^2} + 4x^2e^{x^2} \geq 0 \text{ for all } x \geq 0, \text{ hence } f \text{ is convex}$$

$$\text{Applying Hermite – Hadamard Inequality } e^{\left(\frac{a+b}{2}\right)^2} \leq \frac{1}{b-a}\int_a^b e^{x^2}\,dx \leq \frac{e^{b^2}+e^{a^2}}{2}, \text{ we will}$$

$$\text{prove, } e^{\left(\frac{a+b}{2}\right)^2 - a^2} \geq 1 + a(a-b) \Leftrightarrow e^{\left(\frac{a+b}{2}\right)^2 - a^2} \geq e^{a(a-b)} [e^m \geq 1 + m]$$

$$\Leftrightarrow \left(\frac{a+b}{2}\right)^2 - a^2 \geq a^2 - ab \Leftrightarrow b^2 + 6ab - 7a^2 \geq 0 \Leftrightarrow (b-a)(b+7a) \geq 0$$

$$\text{Which is true. Again, we will prove, } \frac{e^{a^2}+e^{b^2}}{2} \leq e^{b^2}\big(1 + b(b-a)\big) \Leftrightarrow$$

$$\Leftrightarrow e^{a^2-b^2} + 1 \leq 2 + 2b(b-a) \Leftrightarrow 1 \leq e^{b^2-a^2}\big(1 + 2b(b-a)\big), \text{ which is true.}$$

$$\text{Since, } b > a \geq 0 \text{ hence } (1 + a^2 - ab)e^{a^2} \leq \frac{1}{b-a}\int_a^b e^{x^2}\,dx \leq e^{b^2}(1 + b^2 - ab)$$

342. If $0 < a < \frac{\pi}{2}$ then:

$$a\pi + \pi\int_a^{\frac{\pi}{2}} \frac{\sin x}{x}\,dx > (\pi - 2)(1 + a - \sin a)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraboty-Kolkata-India

$$\because a \leq x \leq \frac{\pi}{2}, \therefore 0 < x \leq \frac{\pi}{2} \ (\because a > 0). \text{ By Jordan's inequality, } \forall x \in \left(0, \frac{\pi}{2}\right],$$

$$\frac{\sin x}{x} \geq \frac{2}{\pi} \Rightarrow \pi\int_a^{\frac{\pi}{2}} \frac{\sin x}{x}\,dx \geq \pi\int_a^{\frac{\pi}{2}} \frac{2}{\pi}\,dx = 2\left(\frac{\pi}{2} - a\right) = \pi - 2a \Rightarrow LHS \geq a\pi + \pi - 2a$$

$$\overset{?}{>} \pi + a\pi - \pi\sin a - 2 - 2a + 2\sin a \Leftrightarrow \pi\sin a + 2 \underset{(1)}{\overset{?}{\geq}} 2\sin a$$

$$\because 2 > 2\sin a, \therefore \pi\sin a + 2 > 2\sin a \ (\because \sin a > 0) \Rightarrow \text{(1) is true (proved)}$$

Solution 2 by Srinivasa Raghava-AIRMC-India

$$\textit{Let } f(a)=\frac{\left(a\pi+\pi\int_a^{\frac{\pi}{2}}\frac{\sin(x)}{x}dx\right)}{\pi-2}+\sin(a)\ \textit{ provided } 0<a<\frac{\pi}{2} \textit{ then we have}$$

$$f(a)=\sin(a)+\frac{\pi\left(a-Si(a)+Si\left(\frac{\pi}{2}\right)\right)}{\pi-2}\ \textit{if we see the series expansion at } a=0,$$

$$f(a)=\frac{\pi Si\left(\frac{\pi}{2}\right)}{\pi-2}+a+\frac{(\pi-3)}{9\cdot(\pi-2)}a^3+O\left(a^5\right) \textit{ and numerically we have}$$

$$\frac{\pi Si\left(\frac{\pi}{2}\right)}{\pi-2}=3.77225\ldots\ \&\ \frac{(\pi-3)}{9(\pi-2)}=0.0137812\ \ldots \textit{ approximately.}$$

Clearly the higher power coeffs tends to zero, so we can justify that

$$f(a)\approx a+3.7725\Rightarrow f(a)<a+4, \textit{ hence, } f(a)>a+k \textit{ for } k=0,1,2,3.$$

343.

$$a,b,c\in\mathbb{N}^*, \Omega(a)=\lim_{n\to\infty} n\int_0^1\frac{x^n}{(x+\pi)^a}dx$$

Prove that:

$$(1+\pi)^b\Omega(a)+(1+\pi)^c\Omega(b)+(1+\pi)^a\Omega(c)\ge 3$$

Proposed by Daniel Sitaru – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$1\ge x\ge 0\Rightarrow\frac{1}{\pi^a}\ge\frac{1}{(x+\pi)^a}\ge\frac{1}{(1+\pi)^a},\frac{1}{\pi^b}\ge\frac{1}{(x+\pi)^b}\ge\frac{1}{(1+\pi)^b}$$

$$\frac{1}{\pi^c}\ge\frac{1}{(x+\pi)^c}\ge\frac{1}{(1+\pi)^c},\Omega(a)=\lim_{n\to\infty} n\int_0^1\frac{x^n}{(x+\pi)^a}dx\ge\frac{1}{(1+\pi)^a}\lim_{n\to\infty} n\int_0^1 x^n\,dx$$

$$=\frac{1}{(1+\pi)^a}\lim_{n\to\infty}\frac{n}{n+1}\left[x^{n+1}\right]_{x=0}^{x=1}=\frac{1}{(1+\pi)^a}$$

$$\sum_{cyc}(1+\pi)^b\Omega(a)\ge\sum_{cyc}\frac{(1+\pi)^b}{(1+\pi)^a}\overset{AM\ge GM}{\ge}3$$

(proved)

344. If $a,b,c>0$

$$\Omega(a)=\int_a^{2a}\int_a^{2a}\frac{(x+y)^2+1}{xy+(x+y)\sqrt{3}}dx\,dy\Rightarrow\Omega(a)+\Omega(b)+\Omega(c)\ge ab+bc+ca$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\because xy\le\frac{(x+y)^2}{4}\ \&\ x,y\ge a,b,c>0;\therefore\frac{(x+y)^2+1}{xy+(x+y)\sqrt{3}}\ge\frac{t^2+1}{\frac{t^2}{4}+\sqrt{3}t}\quad(t=x+y)$$

$$\ge 1\Leftrightarrow 4t^2+4\ge t^2+4\sqrt{3}t\Leftrightarrow 3t^2+4-4\sqrt{3}t\ge 0\Leftrightarrow\left(\sqrt{3}t-2\right)^2\ge 0$$

$$\to \textit{true}\ \therefore\frac{(x+y)^2+1}{xy+(x+y)\sqrt{3}}\ge 1\quad (1)$$

$$\therefore\Omega(a)\overset{(i)}{\ge}\int_a^{2a}\int_a^{2a}dx\,dy=\int_a^{2a}a\left(\int_a^{2a}dx\right)dy=\int_a^{2a}a\,dy=a\int_a^{2a}dy=a^2$$

$$\textit{Similarly, }\Omega(b)\overset{(ii)}{\ge}b^2\ \&\ \Omega(c^2)\overset{(iii)}{\ge}c^2$$

$$(i)+(ii)+(iii)\Rightarrow\Omega(a)+\Omega(b)+\Omega(c)\ge\sum a^2\ge\sum ab\ \ \textit{(Proved)}$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\textit{Because for all } x,y,z>0$$

$$\frac{3}{4}(x+y)+\frac{1}{(x+y)}\ge\sqrt{3}\Rightarrow\frac{3}{4}(x+y)^2+1\ge\sqrt{3}(x+y)\Rightarrow$$

$$\Rightarrow x^2+xy+y^2+1\ge\sqrt{3}(x+y)\Rightarrow x^2+2xy+y^2+1\ge xy+\sqrt{3}(x+y)\Rightarrow$$

$$\Rightarrow\frac{(x+y)^2+1}{xy+\sqrt{3}(x+y)}\ge 1$$

$$\textit{Hence }\Omega(a)=\int_a^{2a}\int_a^{2a}\frac{(x+y)^2+1}{xy+(x+y)\sqrt{3}}dx\,dy\ge\int_a^{2a}\int_a^{2a}1\,dx\,dy=xy|_a^{2a}|_a^{2a}=$$

$$=(2a-a)(2a-a)=a^2.\ \textit{Similarly }\Omega(b)\ge b^2\ \textit{ and }\Omega(c)\ge c^2$$

$$\textit{Hence }\Omega(a)+\Omega(b)+\Omega(c)\ge a^2+b^2+c^2\ge ab+bc+ca$$

Therefore it is to be true

345. If $f:[0,a]\to[0,\infty), a\geq 0, f$ – continuous then:

$$\int_0^a\int_0^a \sqrt{f^2(x)+f^2(y)}\,dx\,dy+\int_0^a\int_0^a \sqrt{2f(x)f(y)}\,dx\,dy\leq 2a\sqrt{2}\int_0^a f(x)\,dx$$

Proposed by Daniel Sitaru – Romania

Solution by Chris Kyriazis-Athens-Greece

Using Cauchy – Schwarz inequality, we have that:

$$\sqrt{f^2(x)+f^2(y)}+\sqrt{2f(x)f(y)}\leq\sqrt{2}\cdot\sqrt{\left(\sqrt{f^2(x)+f^2(y)}\right)^2+\left(\sqrt{2f(x)+2f(y)}\right)^2}$$

$$=\sqrt{2}\cdot\sqrt{f^2(x)+f^2(y)+2f(x)f(y)}=\sqrt{2}\cdot\sqrt{\left(f(x)+f(y)\right)^2}=$$

$$=\sqrt{2}\left(f(x)+f(y)\right)\underset{in\ [0,a]\times[0,a)}{\overset{integrate}{\Rightarrow}}$$

$$\int_0^a\int_0^a \sqrt{f^2(x)+f^2(y)}\,dx\,dy+\int_0^a\int_0^a \sqrt{2f(x)f(y)}\leq\sqrt{2}\int_0^a\int_0^a\left(f(x)+f(y)\right)$$

$$=\sqrt{2}\cdot 2a\int_0^a f(x)\,dx$$

346. If $0<a\leq b<\frac{\pi}{2}$ then:

$$\int_a^b\frac{\cos x\cdot\sin^2(\sin x)}{\sin^2 x}dx\geq\tan^{-1}\left(\frac{\sin b-\sin a}{1+\sin a\cdot\sin b}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$0<a\leq b<\frac{\pi}{2}\ \text{now}\ \int_a^b\frac{\cos x\cdot\sin^2(\sin x)}{\sin^2 x}dx=\int_{\sin a}^{\sin b}\frac{\sin^2 y}{y^2}dy$$

Where $1\geq y\geq 0$, *we need to prove,* $\sin^2 y\geq\frac{y^2}{1+y^2}$ *or* $\sin^2 y+\frac{1}{1+y^2}-1\geq 0$

$$Let\ f(y)=\sin^2 y+\frac{1}{1+y^2}-1\ for\ all\ 1\geq y\geq 0, f'(y)=\sin 2y-\frac{2y}{(1+y^2)^2}$$

$$Now\ we\ know, \sin 2y\geq 2y-\frac{4y^3}{3}. So, we\ will\ prove, 2y-\frac{4y^2}{3}\geq\frac{2y}{(1+y^2)^2}$$

$$\Leftrightarrow (1+y^2)^2(3-2y^2)\geq 0\Leftrightarrow y^2(4-y^2-2y^4)\geq 0, which\ is\ true$$

$$[since, 1\geq y\geq 0\Rightarrow 4>3\geq y^2+2y^4]$$

$$\therefore \sin 2y-\frac{2y}{(1+y^2)^2}\geq 0\Rightarrow f'(y)\geq 0\Rightarrow f(y)\geq f(0)=0\Rightarrow\frac{\sin^2 y}{y^2}\geq\frac{1}{1+y^2}$$

$$\Rightarrow\int_{\sin a}^{\sin b}\frac{\sin^2 y}{y^2}dy\geq\int_{\sin a}^{\sin b}\frac{dy}{1+y^2}=\tan^{-1}(\sin b)-\tan^{-1}(\sin a)$$

$$=tan^{-1}\left(\frac{\sin b-\sin a}{1+\sin a\cdot\sin b}\right)\ (proved)$$

347. For $0<a<b$. Prove:

$$\int_a^b sin(\sqrt[3]{x})\,dx\leq(b-a)\sqrt[3]{b}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Amit Dutta-Jamshedpur-India

$$\because \sin\left((x)^{\frac{1}{3}}\right)<(x)^{\frac{1}{3}}$$

$$\int_a^b\sin\left((x)^{\frac{1}{3}}\right)dx<\int_a^b(x)^{\frac{1}{3}}dx<\left(\frac{3}{4}\right)\left(b^{\frac{4}{3}}-a^{\frac{4}{3}}\right)\ \ (1)$$

$$Let\ b(x)=x^{\frac{4}{3}}. By\ Lagrange's\ mean\ value\ theorem, \frac{b(b)-b(a)}{b-a}=b'(c)$$

$$\frac{(b)^{\frac{4}{3}}-(a)^{\frac{4}{3}}}{(b-a)}=\frac{4}{3}(c)^{\frac{1}{3}}; (b)^{\frac{4}{3}}-(a)^{\frac{4}{3}}=\frac{4}{3}(b-a)(c)^{\frac{1}{3}}. But\ a<c<b\Rightarrow a^{\frac{1}{3}}<c^{\frac{1}{3}}<b^{\frac{1}{3}}\Rightarrow$$

$$\Rightarrow(b)^{\frac{4}{3}}-(a)^{\frac{4}{3}}<\frac{4}{3}(b-a)b^{\frac{1}{3}}\ \ (2). From\ (1)\ \&\ (2):$$

$$\int_a^b\sin\left((x)^{\frac{1}{3}}\right)dx<(b-a)(b)^{\frac{1}{3}}$$

Solution 2 by Chris Kyriazis-Athens-Greece

The function $g(x) = \sqrt[3]{x}$ is strictly increasing in $[a,b], a > 0$. (easy to check). So, it's well known that $\sin x < x$, for $x > 0$, so getting $x \to \sqrt[3]{x}$ we have that:

$\sin\sqrt[3]{x} < \sqrt[3]{x} \le \sqrt[3]{b}$ for every $x \in [a,b], a > 0$. Integrate this inequality we have

$$\int_0^1 \sin\left(\sqrt[3]{x}\right) dx < \sqrt[3]{b}\int_0^1 dx = \sqrt[3]{b}(b-a)$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

Let $f(x) = \sqrt[3]{x}$ for all $x \ge 0, f'(x) = \frac{x^{-\frac{2}{3}}}{3}, f''(x) = -\frac{2x^{-\frac{5}{3}}}{9} \le 0$, f is a concave function,

Applying Hermite – Hadamard,

$$\frac{1}{b-a}\int_a^b \sqrt[3]{x}dx \le \sqrt[3]{\left(\frac{a+b}{2}\right)} \le \sqrt[3]{b}, \text{ since } 0 \le a < b, \int_a^b \sqrt[3]{x}\,dx \le (b-a)\sqrt[3]{b}$$

$$\int_a^b \sin\left(\sqrt[3]{x}\right) dx \le \int_a^b \sqrt[3]{x}\,dx \le (b-a)\sqrt[3]{b}$$

348.

$$F(a) = \int_0^a \frac{\cos^7 x}{(\cos 6x + 6\cos 4x + 15\cos 2x + 10)}dx, a \in \left[0, \frac{\pi}{2}\right]$$

Prove that:

$$F(a)F(b)F(c)[F(a)+F(b)+F(c)] \le 2^{-20}(a^4+b^4+c^4)$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Amit Dutta-Jamshedpur-India

$$\cos 6x + 6\cos 4x + 15\cos 2x + 10 = (1+\cos 6x) + 9(1+\cos 2x) + 6(\cos 2x + \cos 4x)$$

$$= 2\cos^2 3x + 18\cos^2 x + 12\cos x\cos 3x = 2\{\cos^2 3x + 9\cos^2 x + 6\cos x\cos 3x\}$$

$$= 2(\cos 3x + 3\cos x)^2 = 2(4\cos^3 x)^2 = 32\cos^6 x \quad \{\because \cos 3x = 4\cos^3 x - 3\cos x\}$$

$$F(a)=\int_0^a \frac{\cos^7 x}{(32\cos^6 x)}dx$$

$$F(a)=\frac{1}{32}\int_0^a (\cos x)\,dx=\frac{1}{32}[\sin a]_0^a \Rightarrow F(a)=\frac{\sin a}{32}, F(b)=\frac{\sin b}{32}, F(c)=\frac{\sin c}{32}$$

$$LHS=F(a)F(b)F(c)[F(a)+F(b)+F(c)]$$

$$LHS=\frac{(\sin a)(\sin b)(\sin c)}{2^{15}}\left[\frac{\sin a+\sin b+\sin c}{2^5}\right]$$

$$\because \begin{matrix}\sin a\le a\\ \sin b\le c\\ \sin c\le c\end{matrix}\ \left\{\because a,b,c\in\left[0,\frac{\pi}{2}\right]\right\}$$

$$LHS\le\frac{abc}{2^{20}}(a+b+c)\le 2^{-20}abc(a+b+c)$$

$$LHS\le 2^{-20}abc(a+b+c)\le 2^{-20}(a^4+b^4+c^4)$$

Hence, we have to prove: $(a^4+b^4+c^4)\ge abc(a+b+c)$ ***(1)***

We have, $a^2+b^2+c^2\ge ab+bc+ac\Rightarrow a^4+b^4+c^4\ge a^2b^2+b^2c^2+a^2c^2$ ***(2)***

From (1) & (2), we need to prove only $\Rightarrow$

$$a^2b^2+b^2c^2+c^2a^2\ge abc(a+b+c)$$
$$a^2b^2+b^2c^2+c^2a^2\ge a^2bc+ab^2c+abc^2$$

Put $ab=u, bc=v, ac=w\Rightarrow u^2+v^2+w^2\ge uv+vw+uw\to$ ***true. Hence:***

$$a^4+b^4+c^4\ge abc(a^4+b^4+c^4)\Rightarrow 2^{-20}(a^4+b^4+c^4)\ge 2^{-20}abc(a+b+c)$$

$$\Rightarrow[F(a)+F(b)+F(c)]F(a)F(b)F(c)\le 2^{-20}(a^4+b^4+c^4)\ \textit{(proved)}$$

Solution 2 by Marian Ursarescu-Romania

$$F(a)=\int_0^a\frac{\cos^7 x}{\cos 6x+6\cos 4x+15\cos 2x+10}dx=$$

$$=\int_0^a\frac{\cos^7 x}{4\cos^3 2x-3\cos 2x+6(2\cos^2 2x-1)+15\cos 2x+10}dx=$$

$$=\int_0^a\frac{\cos^7 x}{4\cos^3 2x+12\cos^2 2x+12\cos 2x+4}dx=\frac{1}{4}\int_0^a\frac{\cos^7 x}{\cos^3 2x+3\cos^2 x+3\cos 2x+1}$$

$$=\frac{1}{4}\int_0^a \frac{\cos^7 x}{(1+\cos^2 x)^3}dx = \frac{1}{4}\int_0^a \frac{\cos^7 x}{(2\cos^2 x)^3}dx =$$

$$\frac{1}{2^4}\int_0^a \cos x\, dx = \frac{1}{2^5}\sin x\Big|_0^a = \frac{1}{2^5}\sin a \quad (1)$$

But $\sin a \le a, \forall a \in \left[0, \frac{\pi}{2}\right]$ *(2). From (1)+(2)*$\Rightarrow F(a) \le \frac{1}{2^5}a$ *(3).*

From (3) we must show: $\frac{1}{2^{15}}abc\left(\frac{a+b+c}{2^5}\right) \le \frac{1}{2^{20}}(a^4+b^4+c^4) \Leftrightarrow$

$$\Leftrightarrow abc(a+b+c) \le a^4+b^4+c^4 \quad (4)$$

But $a^4+b^4+c^4 \ge a^2b^2+b^2c^2+a^2c^2 \ge abc(a+b+c) \Rightarrow$ *(4) its true.*

349. **If $a,b,c \ge 1$ then:**

$$\int_1^a\int_1^b\int_1^c \left(\frac{x}{x^2+yz}+\frac{y}{y^2+zx}+\frac{z}{z^2+xy}\right)dx\,dy\,dz = \ln\sqrt{\prod a^{(b-1)(c-1)}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{x}{x^2+yz}+\frac{y}{y^2+zx}+\frac{z}{z^2+xy} \overset{A-G}{\le} \frac{x}{2\sqrt{x^2yz}}+\frac{y}{2\sqrt{y^2zx}}+\frac{z}{2\sqrt{z^2xy}}$$

$$=\frac{1}{2}\sum\frac{1}{\sqrt{xy}} \underset{(1)}{\overset{C-B-S}{\le}} \frac{1}{2}\sqrt{\sum\frac{1}{x}}\sqrt{\sum\frac{1}{y}} = \frac{1}{2}\left(\sum\frac{1}{x}\right)$$

$$(1) \Rightarrow LHS \le \frac{1}{2}\int_1^a\int_1^b\int_1^c\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)dxdydz = \frac{1}{2}\int_1^a\int_1^b\left(\int_1^c\frac{dx}{x}+\left(\frac{1}{y}+\frac{1}{z}\right)\right)dy\,dz =$$

$$=\frac{1}{2}\int_1^a\int_1^b\left(\ln c+\left(\frac{1}{y}+\frac{1}{z}\right)(c-1)\right)dydz == \frac{1}{2}\int_1^a\left[(\ln c)\int_1^b dy+\frac{1}{z}(c-1)\int_1^b dy+(c-1)\int_1^b\frac{dy}{y}\right]dz =$$

$$=\frac{1}{2}\int_1^a\left[(b-1)\ln c+\frac{(b-1)(c-1)}{z}+(c-1)\ln b\right]dz =$$

$$=\frac{1}{2}\left[(b-1)\ln c\int_1^a dz+(b-1)(c-1)\int_1^a\frac{dz}{z}+(c-1)\ln b\int_1^a dz\right]$$

$$=\frac{1}{2}[(a-1)(b-1)\ln c+(b-1)(c-1)\ln a+(c-1)(a-1)\ln b]$$

$$=\frac{1}{2}\left[\ln c^{(a-1)(b-1)}+\ln a^{(b-1)(c-1)}+\ln b^{(c-1)(a-1)}\right]=\frac{1}{2}\left[\ln\prod a^{(b-1)(c-1)}\right]=\ln\sqrt{\prod a^{(b-1)(c-1)}}$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

Since for $x,y,z>0$

1) $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\ge\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\Rightarrow\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{(\sqrt{x})^2}+\frac{1}{(\sqrt{y})^2}+\frac{1}{(\sqrt{z})^2}\ge\frac{1}{\sqrt{xy}}+\frac{1}{\sqrt{yz}}+\frac{1}{\sqrt{zx}}$

2) $\frac{x}{x^2+yz}+\frac{y}{y^2+zx}+\frac{z}{z^2+xy}=\frac{1}{x+\frac{yz}{x}}+\frac{1}{y+\frac{zx}{y}}+\frac{1}{z+\frac{xy}{z}}\le\frac{1}{2}\left(\frac{1}{\sqrt{yz}}+\frac{1}{\sqrt{zx}}+\frac{1}{\sqrt{xy}}\right)$

Hence $\int_1^a\int_1^b\int_1^c\left(\frac{x}{x^2+y^2}+\frac{y}{y^2+zx}+\frac{z}{z^2+xy}\right)dxdydz\le\int_1^a\int_1^b\int_1^c\frac{1}{2}\left(\frac{1}{\sqrt{yz}}+\frac{1}{\sqrt{zx}}+\frac{1}{\sqrt{xy}}\right)dxdydz\le$

$$\le\frac{1}{2}\int_1^a\int_1^b\int_1^c\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)dxdydz=\frac{1}{2}(yz\ln x+zx\ln y+xy\ln z)\Big|_1^a\Big|_1^b\Big|_1^c$$

$$=\frac{1}{2}\big((b-1)(c-1)(\ln a-\ln 1)\big)+(c-1)(a-1)(\ln b-\ln 1)+(a-1)(b-1)(\ln c-\ln 1)$$

$$=\frac{1}{2}\left(\ln a^{(b-1)(c-1)}+\ln b^{(c-1)(a-1)}+\ln c^{(a-1)(b-1)}\right)=\frac{1}{2}\ln a^{(b-1)(c-1)}b^{(c-1)(a-1)}c^{(a-1)(b-1)}$$

$$=\ln\left(a^{(b-1)(c-1)}b^{(c-1)(a-1)}e^{(a-1)(b-1)}\right)^{\frac{1}{2}}=\ln\sqrt{a^{(b-1)(c-1)}b^{(c-1)(a-1)}e^{(a-1)(b-1)}}$$

Therefore it is to be true.

350. If $0\le a<b\le 1, n\in\mathbb{N}, n\ge 2$

$$\Omega_1=\int_a^b\int_a^b\ldots\int_a^b(1+x_1^2)(1+x_2^2)\ldots(1+x_n^2)dx_1dx_2\ldots dx_n$$

$$\Omega_2=\int_a^b\int_a^b\ldots\int_a^b(1-x_1^2)(1-x_2^2)\ldots(1-x_n^2)dx_1dx_2\ldots dx_n$$

then: $\sqrt[n^2]{\Omega_1+\Omega_2}+\frac{1}{n}<1+\frac{2(b-a)}{n}$

Proposed by Daniel Sitaru – Romania

Solution by Chris Kyriazis-Athens-Greece

This solution is dedicated to: Lazaros (Kardasi) and Dan!

$$\Omega_1 = \left[\int_a^b (1+x^2)dx\right]^n = \left[b-a+\frac{b^3-a^3}{3}\right]^n$$

and

$$\Omega_2 = \left[\int_a^b (1-x^2)dx\right]^n = \left[b-a-\frac{b^3-a^3}{3}\right]^n$$

$$\Omega_2+\Omega_2 < \left[2(b-a)+\frac{b^3-a^3}{3}-\frac{b^3-a^3}{3}\right]^n = 2^n(b-a)^n$$

[Because of $x^n+y^n<(x+y)^n$ for positive x,y]

$$So,\ \Omega_1+\Omega_2 < [2(b-a)]^n = \left[\underbrace{1\cdot 1\cdot 1\cdot \dots \cdot 1}_{"n-1"\ times}\cdot 2(b-a)\right]^n \le$$

$$\overset{GM}{\underset{AM}{\le}} \left[\left[\frac{1+1+1+\dots+1+[2(b-a)]}{n}\right]^n\right]^n = \left[\frac{n-1+2(b-a)}{n}\right]^{n^2}$$

$$\Rightarrow \sqrt[n^2]{\Omega_1+\Omega_2} < \frac{n-1}{n}+\frac{2(b-a)}{n} \Rightarrow \sqrt[n^2]{\Omega_1+\Omega_2} < 1-\frac{1}{n}+\frac{2(b-a)}{n}$$

$$\Rightarrow \sqrt[n^2]{\Omega_1+\Omega_2}+\frac{1}{n} < 1+\frac{2(b-a)}{n}\ \textit{and we are done!}$$

351. If $a>0$ then:

$$\frac{1}{16}\int_0^a\int_0^a (x+y)^4 dx\,dy \le \int_0^{2a}\int_0^{2a}\int_0^1 (tx+(1-t)y)^4\,dt\,dy\,dx$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

$$RHS = \int_a^{2a}\int_a^{2a}\int_0^1 [tx+(1-t)y]^4\,dz\,dy\,dx = \frac{1}{5}\int_a^{2a}\int_a^{2a}\left[\frac{[tx+(1-t)y]^5}{x-y}\right]_0^1 dy\,dx$$

$$=\frac{1}{5}\int_a^{2a}\int_a^{2a}\frac{x^5-y^5}{x-y}dy\,dx=\frac{1}{5}\int_a^{2a}\int_a^{2a}[x^4+x^3y+x^2y^2+xy^3+y^4]\,dy\,dx$$

$$=\frac{1}{5}\left[\begin{array}{c}\left[\int_a^{2a}\int_a^{2a}x^4\,dy\,dx+\int_a^{2a}\int_a^{2a}x^3y\,dy\,dx+\int_a^{2a}\int_a^{2a}x^2y^2\,dy\,dx+\right.\\ +\int_a^{2a}\int_a^{2a}xy^3\,dy\,dx+\int_a^{2a}\int_a^{2a}y^4\,dy\,dx\end{array}\right]=$$

$$=\frac{1}{5}\left[\frac{x}{5}\left((2a)^5-a^5\right)a+\frac{7}{4\times 2}\left((2a)^4-a^4\right)(3a^2)+\frac{1}{3\times 3}\left((2a)^3-a^3\right)^2\right]$$

$$=\frac{1}{5}\left[\frac{62}{5}+\frac{45}{4}+\frac{4a}{a}\right]a^6=\frac{5237}{5\times 180}a^5\quad (1)$$

$$LHS=\frac{1}{16}\int_0^a\int_0^a(x+y)^4\,dy\,dx=\frac{1}{16}\int_0^a\left[\frac{1}{5}(x+y)^5\right]_0^a dx=$$

$$=\frac{1}{16}\times\frac{1}{5}\int_0^a\left[(x+a)^5-x^5\right]dx=\frac{1}{16\times 5\times 6}\left[(x+a)^6-x^6\right]_0^a=$$

$$=\frac{1}{96\times 5}\left[(2a)^6-a^6-a^6\right]=\frac{31}{240}a^6\quad (2)$$

We wish to show: $\frac{31}{240}a^6<\frac{5237}{5\times 180}a^6\Leftrightarrow\frac{31}{240}<\frac{5237}{900}$

It is clearly true as $LHS<1$ *and* $RHS>1$.

352. If $a\in(-1,1]$ then:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(-1)^k\cdot\cos(k\cos^{-1}a)}{k}\le\log\left(\sqrt{2e^a}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Omran Kouba-Damascus-Syria

For a complex z such that $|z|\le 1, z\neq -1$ *we have* $\log(1+z)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}z^k}{k}$

Hence for $\theta\in(-\pi,\pi)$ *we have* $\log\left(1+e^{i\theta}\right)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}e^{ik\theta}}{k}$

Taking real parts we get $\log|1+e^{i\theta}| = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}\cos(k\pi)}{k}$*. Finally, for* $\theta\in(-\pi,\pi)$

$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\cos(k\theta)}{k} = \frac{1}{2}\log(2+2\cos(\theta))$*. Or, for* $a\in(-1,1]$*, we have*

$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\cos(k\arccos a)}{k} = \log\sqrt{2+2a} \le \log\sqrt{2e^a}$*. Because* $1+a\le e^a$*. Done.*

353. Prove that:

$$\frac{\pi^{\frac{5}{2}}}{12\sqrt{2}} \le \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,(x)^{\frac{3}{2}}dx \le \frac{\pi^2}{8}$$

Proposed by Sagar Kumar-Kolkata-India

Solution by Soumitra Mandal-Chandar Nagore-India

Let $f(x)=\sqrt{\sin x}$ *for all* $x\in\left[0,\frac{\pi}{2}\right]$, $f'(x)=\frac{\cos x}{\sqrt{\sin x}}$; $f''(x) = -\sqrt{\sin x} - \frac{\cos^2 x}{2(\sin x)^{\frac{3}{2}}} =$

$= \frac{\sin^2 x+1}{2(\sin x)^{\frac{3}{2}}} \le 0$ *for all* $x\in\left[0,\frac{\pi}{2}\right]$*, hence f is concave, by Hermite – Hadamard Inequality*

$$\sqrt{\sin\left(\frac{\frac{\pi}{2}+0}{2}\right)}\cdot\frac{\pi}{2} \ge \int_0^{\frac{\pi}{2}}\sqrt{\sin x}\,dx \ge \frac{\pi}{2}\cdot\frac{\sqrt{\sin\frac{\pi}{2}}+\sqrt{\sin 0}}{2} = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}}\sqrt{\sin x}\cdot x^{\frac{3}{2}}dx \overset{\substack{CAUCHY\\SCHWARZ}}{\le} \sqrt{\left(\int_0^{\frac{\pi}{2}}\sin x\,dx\right)\left(\int_0^{\frac{\pi}{2}}x^3\,dx\right)} = \sqrt{\left[\frac{x^4}{4}\right]_{x=0}^{x=\frac{\pi}{2}}} = \frac{\pi^2}{8}$$

$$\int_0^{\frac{\pi}{2}} x^{\frac{3}{2}}\sqrt{\sin x}\,dx \overset{\substack{CHEBYSHEV'S\\INEQUALITY}}{\ge} \left(\frac{\pi}{2}-0\right)\int_0^{\frac{\pi}{2}} x^{\frac{3}{2}}\,dx\cdot\int_0^{\frac{\pi}{2}}\sqrt{\sin x}\,dx$$

$\ge \frac{\pi}{2}\cdot\frac{\pi^{\frac{5}{2}}}{10\sqrt{2}}\cdot\frac{\pi}{4}$*, we need to prove,* $\frac{\pi^2}{8}\cdot\frac{\pi^{\frac{5}{2}}}{10\sqrt{2}} \ge \frac{\pi^{\frac{5}{2}}}{12\sqrt{2}} \Leftrightarrow \pi^2 \ge \frac{20}{3}$*, which is true*

$$\therefore \frac{\pi^{\frac{5}{2}}}{12\sqrt{2}} \le \int_0^{\frac{\pi}{2}} \sqrt{\sin x} \cdot x^{\frac{3}{2}}\,dx \le \frac{\pi^2}{8}$$

354. If $a > 0$ then:

$$\int_a^{2a}\int_a^{2a}\int_a^{2a} \frac{\sqrt{a^2+4x^2}+\sqrt{a^2+4y^2}}{z\sqrt{a^2+(x+y)^2}}\,dx\,dy\,dz > 2a^2\log 2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Amit Dutta-Jamshedpur-India

We use a fundamental inequality: $\sqrt{x^2+y^2}+\sqrt{a^2+b^2} \ge \sqrt{(x+a)^2+(y+b)^2}$ *(1)*

Let's prove it. On Squaring both sides, we get:

$$x^2+y^2+a^2+b^2+2\sqrt{x^2+y^2}\sqrt{a^2+b^2} \ge (x+a)^2+(y+b)^2 \Rightarrow$$

$$\Rightarrow 2\sqrt{x^2+y^2}\sqrt{a^2+b^2} \ge 2ax+2by \Rightarrow \sqrt{x^2+y^2}\sqrt{a^2+b^2} \ge ax+by$$

Again, squaring both sides, we get: $(x^2+y^2)(a^2+b^2) \ge (ax+by)^2 \Rightarrow$

$\Rightarrow x^2a^2+b^2x^2+a^2y^2+b^2y^2 \ge a^2x^2+b^2y^2+2axby \Rightarrow (ax-by)^2 \ge 0$ *which is true and the inequality in equation (1) is true. Using the inequality in equation (1)*

$$\sqrt{a^2+4x^2}+\sqrt{a^2+4y^2} \ge \sqrt{(2a)^2+(2x+2y)^2} \ge 2\sqrt{a^2+(x+y)^2} \Rightarrow$$

$$\Rightarrow \frac{\sqrt{a^2+4x^2}+\sqrt{a^2+4y^2}}{\sqrt{a^2+(x+y)^2}} \ge 2$$

$$\int_a^{2a}\int_a^{2a}\int_a^{2a} \frac{\sqrt{a^2+4x^2}+\sqrt{a^2+4y^2}}{z\sqrt{a^2+(x+y)^2}}\,dx\,dy\,dz \ge \int_a^{2a}\int_a^{2a}\int_a^{2a} \frac{2}{z}\,dx\,dy\,dz \ge \int_a^{2a}\int_a^{2a} [2\ln z]_a^{2a}\,dx\,dy$$

$$\ge \int_a^{2a}\int_a^{2a} (2\log 2)\,dx\,dy \ge (2\log 2)\int_a^{2a} dx \int_a^{2a} dy \ge 2\log 2\,(a\cdot a) \ge 2a^2\log 2$$

$$\therefore \int_a^{2a}\int_a^{2a}\int_a^{2a} \frac{\sqrt{a^2+4x^2}+\sqrt{a^2+4y^2}}{z\sqrt{a^2+(x+y)^2}} \ge 2a^2\log 2$$

(proved)

Solution 2 by Serban George Florin-Romania

$$f(x)=\sqrt{a^2+4x^2}, f'(x)=\frac{4x}{\sqrt{a^2+4x^2}}; f''(x)=4\frac{\sqrt{a^2+4x^2}-x\cdot\frac{x}{\sqrt{a^2+4x^2}}}{a^2+4x^2}=4\cdot\frac{a^2+4x^2-4x^2}{(a^2+4x^2)^{\frac{3}{2}}}.$$

$$f''(x)=\frac{4a^2}{(a^2+4x^2)^{\frac{3}{2}}}>0; f\ covexe\Rightarrow f\left(\frac{x+y}{2}\right)\le\frac{f(x)+f(y)}{2}, 2\sqrt{a^2+4\left(\frac{x+b}{2}\right)^2}\le$$

$$\le\sqrt{a^2+4x^2}+\sqrt{a^2+4b^2}\ \ (Jensen)\Rightarrow\sqrt{a^2+4x^2}+\sqrt{a^2+4b^2}\ge 2\sqrt{a^2+(x+b)^2}$$

$$\frac{\sqrt{a^2+4x^2}+\sqrt{a^2+4b^2}}{\sqrt{a^2+(x+b)^2}}\ge 2$$

$$\int_a^{2a}\int_a^{2a}\int_a^{2a}\frac{\sqrt{a^2+4x^2}+\sqrt{a^2+4y^2}}{z\cdot\sqrt{a^2+(xyz)^2}}dx\,dy\,dz\ge\int_a^{2a}\int_a^{2a}\int_a^{2a}\frac{1}{z}dz\,dy\,dx=\int_a^{2a}\int_a^{2a}(\ln|z|)^2\,dx\,dy=$$

$$=\int_a^{2a}\int_a^{2a}\big((\ln 2a)-(\ln a)\big)\cdot 2dx\,dy=\int_a^{2a}\int_a^{2a}(2\ln 2)\,dx\,dy=2(\ln 2)\cdot 2\cdot a=2a^2\ln 2$$

355. **Prove:**

$$\int_0^{\frac{\pi}{2}}\left(\frac{9^{\sin x}}{4^{\sin x}+5^{\sin x}}+\frac{16^{\sin x}}{3^{\sin x}+5^{\sin x}}+\frac{25^{\sin x}}{3^{\sin x}+4^{\sin x}}\right)^3dx\ge\frac{1593}{8\ln 60}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Sagar Kumar-Kolkata-India

$$Applying\ Berstr\ddot{o}m: \int_0^{\frac{\pi}{2}}\left(\frac{9^{\sin x}}{4^{\sin x}+5^{\sin x}}+\frac{16^{\sin x}}{3^{\sin x}+5^{\sin x}}+\frac{25^{\sin x}}{3^{\sin x}+4^{\sin x}}\right)^3dx\ge$$

$$\ge\int_0^{\frac{\pi}{2}}\frac{(3^{\sin x}+4^{\sin x}+5^{\sin x})^3}{8}dx\overset{AM\ge GM}{\ge}\int_0^{\frac{\pi}{2}}\frac{27}{8}(60^{\sin x})dx\ge\int_0^{\frac{\pi}{2}}\frac{27}{8}(60)^{\sin x}\cos x\,dx$$

$$\ge\frac{27}{8}\left(\frac{(60)^{\sin x}}{\ln 60}\right)\Bigg|_0^{\frac{\pi}{2}}\ge\frac{27\times 60}{8\ln 60}\ge\frac{27\times 51}{8\ln 60}\ge\frac{405}{2\ln 60}\ge\frac{1593}{8\ln 60}\ \ (proved)$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\int_0^{\frac{\pi}{2}}\left(\sum_{cyc}\frac{9^{\sin x}}{4^{\sin x}+5^{\sin x}}\right)^3 dx \overset{BERGSTROM}{\geq} \int_0^{\frac{\pi}{2}}\left(\frac{\left(\sum_{cyc}3^{\sin x}\right)^2}{2\sum_{cyc}3^{\sin x}}\right)^3 =$$

$$=\frac{1}{8}\int_0^{\frac{\pi}{2}}(3^{\sin x}+4^{\sin x}+5^{\sin x})^3\,dx \geq \frac{27}{8}\int_0^{\frac{\pi}{2}}60^{\sin x}\,dx \geq \frac{27}{8}\int_0^{\frac{\pi}{2}}60^{\sin x}\cdot\cos x\,dx =$$

$$=\frac{27}{8}\left[\frac{60^{\sin x}}{\log 60}\right]_{x=0}^{x=\frac{\pi}{2}} = \frac{27\cdot 59}{8\log 60} = \frac{1593}{8\log 60}\ \textit{(proved)}$$

356. If $a > 0$ then:

$$a^2\cdot\sqrt[4]{e^{9a^2}} < \int_a^{2a}\int_a^{2a}\sqrt[16]{e^{(x+3y)^2}}\,dx\,dy < a\cdot\int_a^{2a} e^{x^2}\,dx$$

Proposed by Daniel Sitaru – Romania

Solution by Artan Ajredini-Presheva-Serbie

Since e^x is a convex function we apply the Hermite – Hadamard inequality for double integral: $\frac{1}{a^2}\int_a^{2a}\int_a^{2a}\sqrt[16]{c^{(x+3y)^2}}\,dx\,dy \geq e^{\left(\frac{\frac{3a}{2}+\frac{9a}{2}}{4}\right)^2} = e^{\left(\frac{3a}{2}\right)^2} = \sqrt[4]{e^{9a^2}} \Rightarrow$

$$\Rightarrow \int_a^{2a}\int_a^{2a}\sqrt[16]{e^{(x+3y)^2}}\,dx\,dy \geq a^2\sqrt[4]{e^{9a^2}}$$

Also, since e^x is convex we have: $e^{\left(\frac{x+3y}{4}\right)^2} \leq \frac{1}{4}e^{x^2}+\frac{3}{4}e^{y^2}$. Consequently:

$$\int_a^{2a}\int_a^{2a}\sqrt[16]{e^{(x+3y)^2}}\,dx\,dy \leq \frac{1}{4}\int_a^{2a}\int_a^{2a}e^{x^2}\,dx + \frac{3}{4}\int_a^{2a}\int_a^{2a}e^{y^2}\,dy =$$

$$=\frac{a}{4}\int_a^{2a}e^{x^2}\,dx + \frac{3a}{4}\int_a^{2a}e^{y^2}\,dy = \frac{a}{4}\int_a^{2a}e^{x^2}\,dx + \frac{3a}{4}\int_a^{2a}e^{x^2}\,dx = a\int_a^{2a}e^{x^2}\,dx$$

357. For $n \in \mathbb{N}^* \wedge n \geq 2$. Prove:

$$\int_0^1 \left(\sum_{k=1}^{n} e^{x^k}\right) dx > n + \frac{n}{\left((n+1)!\right)^{\frac{1}{n}}}.$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Lazaros Zachariadis-Thessaloniki-Greece

$$\int_0^1 \left(e^x + e^{x^2} + \cdots + e^{x^n}\right) dx \overset{Jensen}{\geq} \int_0^1 n \cdot e^{\frac{x+x^3+\cdots+x^n}{n}} dx \overset{e^x \geq x+1}{\geq}$$

$$\geq \int_0^1 n \cdot \left(\frac{x + x^2 + \cdots + x^n}{n} + 1\right) dx = \int_0^1 n \cdot \frac{x + \cdots + x^n + n}{n} dx =$$

$$= \int_0^1 (x + x^2 + \cdots + x^n + n)\, dx = \left[\frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^{n+1}}{n+1} + nx\right]_0^1 =$$

$$= \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1} + n \geq n \cdot \sqrt[n]{\frac{1}{2} \cdot \frac{1}{3} \cdot \ldots \cdot \frac{1}{n+1}} + n = n \cdot \frac{1}{[(n+1)!]^{\frac{1}{n}}} + n$$

Solution 2 by Sagar Kumar-Kolkata-India

$$\int_0^1 \left(\sum_{k=1}^{n} e^{x^k}\right) dx \geq \int_0^1 \sum_{k=1}^{n} (1 + x^k)\, dx \geq \sum_{k=1}^{n} \left(1 + \frac{1}{k+1}\right) \geq n + \sum_{k=1}^{n} \frac{1}{k+1}$$

$$Now,\ \sum_{k=1}^{n} \frac{1}{k+1} \geq n \left(\frac{1}{(n+1)!}\right)^{\frac{1}{n}}$$

$$\int_0^1 \left(\sum_{k=1}^{n} e^{x}\right) dx \geq n + n\left(\frac{1}{(n+1)!}\right)^{\frac{1}{n}}$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$e^m \geq 1 + m \text{ for all } m \geq 0, \text{ now, } \sum_{k=1}^{n} e^{x^k} \geq \sum_{k=1}^{n} (1 + x^k) = n + \sum_{k=1}^{n} x^k$$

$$\int_0^1 \left(\sum_{k=1}^{n} e^{x^k}\right) dx \geq n \int_0^1 dx + \sum_{k=1}^{n} \int_0^1 x^k\, dx = n + \sum_{k=1}^{n} \frac{1}{k+1} \overset{A.M \geq G.M}{\geq}$$

$$\geq n+n\sqrt[n]{\prod_{k=1}^{n}\left(\frac{1}{k+1}\right)}=n+\frac{n}{\sqrt[n]{(n+1)!}}$$

358.

$$-1<a,b,c<1,\Omega(a)=\int_0^{\pi}\frac{\log(1+a\cos x)}{\cos x}dx$$

Prove that:

$$\frac{1}{\pi^2}\left(\Omega^2(a)+\Omega^2(b)+\Omega^2(c)\right)\geq\sum(\sin^{-1}a\cdot\sin^{-1}b)$$

Proposed by Daniel Sitaru – Romania

Solution by Marian Ursărescu – Romania

Let $f(a)=\frac{\ln(1+a\cos x)}{\cos x}$ *is a continuous function in* $a\Rightarrow\Omega'(a)=\int_0^{\pi}\frac{1}{1+a\cos x}dx$

$$\left.\begin{array}{r}Let\ \tan\frac{x}{2}=t\Rightarrow x=2\arctan t\Rightarrow dx=\frac{2}{1+t^2}dt\\ x=0\Rightarrow t=0;x=\pi\Rightarrow t=\infty\end{array}\right\}\Rightarrow\Omega'(a)$$

$$=\int_0^{\infty}\frac{1}{1+a\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}dt$$

$$=2\int_0^{\infty}\frac{1}{1+t^2+a-at^2}dt=2\int_0^{\infty}\frac{1}{(1-a)t^2+1+a}dt=\frac{2}{1-a}\int_0^{\infty}\frac{1}{t^2+\left(\sqrt{\frac{1+a}{1-a}}\right)^2}dt=$$

$$=\frac{2}{1-a}\cdot\frac{1}{\sqrt{\frac{1+a}{1-a}}}\arctan\frac{t}{\sqrt{\frac{1+a}{1-a}}}\Bigg|_0^{\infty}=\frac{\pi}{\sqrt{1-a^2}}\Rightarrow$$

$$\left.\begin{array}{r}\Omega(a)=\pi\int\frac{1}{\sqrt{1-a^2}}da=\pi\arcsin a+c\\ But\ \Omega(a)=0\Rightarrow c=0\end{array}\right\}\Rightarrow\Omega(a)=\pi\arcsin a\Rightarrow we\ must\ show:$$

$\sum(\arcsin a)^2\geq\sum\arcsin a\cdot\arcsin b$, *which its true because* $\sum x^2\geq\sum xy$

359. If $a, b, c \in \mathbb{N}^*$,

$$\Omega(a) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)\cdot \ldots \cdot (n+a)}$$

then:

$$\left(\frac{b\Omega(a) + c\Omega(b) + a\Omega(c)}{a+b+c}\right)^{a+b+c} \geq \frac{1}{a^b \cdot b^c \cdot c^a \cdot (a!)^b \cdot (b!)^c \cdot (c!)^a}$$

Proposed by Daniel Sitaru – Romania

Solution by Shivam Sharma-New Delhi-India

$$Let,\ S_a = \sum_{n=1}^{\infty}\left(\frac{1}{n(n+1)(n+2)\cdot \ldots \cdot (n+a)}\right) \Rightarrow \frac{1}{a}\left(\frac{1}{n(n+1)\ldots(n+a-1)} - \frac{1}{(n+1)\ldots(n+a)}\right)$$

$$\Rightarrow \frac{1}{a}\left(S_{a-1} - \sum_{n=1}^{\infty}\left(\frac{1}{(n+1)\ldots(n+a)}\right)\right).\ Replace,\ n+1 = k \Rightarrow$$

$$\Rightarrow \frac{1}{a}\left(S_{a-1} - \left(\sum_{k=1}^{\infty}\left(\frac{1}{k(k+1)-(k+a-1)}\right) - \frac{1}{a!}\right)\right) \Rightarrow \frac{1}{a}\left(S_{a-1} - S_{a-1} + \frac{1}{a!}\right)$$

$$(OR)\ S_a = \frac{1}{a\cdot a!}.\ Now,\ \Omega(a) = S_a = \frac{1}{a\cdot a!}$$

Now, applying this initial problem, and applying A.M.-G.M., we get,

$$\frac{b\Omega(a) + c\Omega(b) + a\Omega(c)}{a+b+c} = \left(\frac{b\left(\frac{1}{a\cdot a!}\right) + c\left(\frac{1}{b\cdot b!}\right) + a\left(\frac{1}{c\cdot c!}\right)}{a+b+c}\right)^{a+b+c} \overset{AM-GM}{\geq}$$

$$\geq \frac{1}{a^b b^c c^a (a!)^b (b!)^c (c!)^a}.\ Hence,$$

$$\left(\frac{b\Omega(a) + c\Omega(b) + a\Omega(c)}{a+b+c}\right)^{a+b+c} \geq \frac{1}{a^b b^c c^a (a!)^b (b!)^c (c!)^a}$$

(proved)

360.

$$\Omega(n)=\int_{-1}^{1}x\log(1+n^{3x})\,dx,n\in\mathbb{N}^*$$

Prove that:

$$9\left(1+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}\right)^2>4e^{2\Omega(n)}\left(1+e^{\Omega(n)}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursarescu-Romania

$$\left.\begin{array}{r}\Omega(n)=\int_{-1}^{1}x\ln(1+n^{3x})\,dx\\ x=-t|'\Rightarrow dx=-dt\;x=-1\Rightarrow t=1;\;x=1\Rightarrow t=-1\end{array}\right\}\Rightarrow$$

$$\Omega(n)=\int_{1}^{-1}-t\ln(1+n^{-3t})\,(-dt)=-\int_{-1}^{1}t\ln\left(1+\frac{1}{n^{3t}}\right)dt=-\int_{-1}^{1}t\ln\left(\frac{n^{3t}+1}{n^{3t}}\right)dt=$$

$$=-\int_{-1}^{1}t\ln(1+n^{3t})\,dt+\int_{-1}^{1}t\ln n^{3t}\,dt\Rightarrow 2\Omega(n)=\int_{-1}^{1}3t^2\ln dt\Rightarrow 2\Omega(n)=t^3\ln n\Big|_{-1}^{1}\Rightarrow$$

$\Rightarrow\Omega(n)=\ln n$. ***We must show this:*** $9\left(1+\sqrt{2}+\cdots+\sqrt{n}\right)^2>4n^2(n+1)\Leftrightarrow$

$$\Leftrightarrow 1+\sqrt{2}+\cdots+\sqrt{n}>\frac{2n\sqrt{n+1}}{3},\forall n\geq 1$$

$$P(1):1>\frac{2\sqrt{2}}{3}\Leftrightarrow 3>2\sqrt{2}\ \textit{true.}$$

$$\textit{Now: }P(k):1+\sqrt{2}+\cdots+\sqrt{k}>\frac{2k\sqrt{k+1}}{3}$$

$$P(k+1):1+\sqrt{2}+\cdots+\sqrt{k+1}>\frac{2(k+1)\sqrt{k+2}}{3}$$

$$\textit{From P(k)}\Rightarrow 1+\sqrt{2}+\cdots+\sqrt{k}+\sqrt{k+1}>\frac{2k\sqrt{k+1}}{3}+\sqrt{k+1}$$

$$\textit{We must show this: }\frac{2k\sqrt{k+1}}{3}+\sqrt{k+1}>\frac{2(k+1)\sqrt{k+2}}{3}$$

$$2k+3>2\sqrt{(k+1)(k+2)}\Leftrightarrow 4k^2+12k+9>4k^2+12k+8\Leftrightarrow 9>8\ \textit{true.}$$

Solution 2 by Amit Dutta-Jamshedpur-India

$$\Omega(n)=\int_{-1}^{1} x\log(1+n^{3x})\,dx$$

Using $\int_0^a F(x)\,dx=\int_0^a F(a-x)\,dx \Rightarrow \Omega(n)=\int_{-1}^{1} -x\log(1+n^{-3x})\,dx$

$$\Omega(n)=\int_{-1}^{1} -x[\log(1+n^{3x})-\log(n^{3x})]\,dx$$

$$\Omega(n)=-\int_{-1}^{1} x\log(1+n^{3x})\,dx+\int_{-1}^{1} x\log(n^{3x})\,dx$$

$$\Omega(n)=-\Omega(n)+\int_{-1}^{1}(3x^2)\log(n)\,dx;\ 2\Omega(n)=3\int_{-1}^{1} x^2(\log n)\,dx$$

$$2\Omega(n)=(3\log n)\int_{-1}^{1} x^2\,dx;\ 2\Omega(n)=(3\log n)\times 2\int_{-1}^{1} x^2\,dx$$

$$\Omega(n)=3\log n\times\frac{1}{3}\Rightarrow\Omega(n)=\log_e n$$

The inequality is: $9\left(1+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}\right)^2>4e^{2\Omega(n)}\left(1+e^{\Omega(n)}\right)$*. i.e., we have to prove* $9\left(1+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}\right)^2>4e^{2\ln n}\left(1+e^{\ln n}\right)$

$$9\left(1+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}\right)^2>4n^2(n+1)$$

or $\left(1+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}\right)>\left(\frac{2n\sqrt{n+1}}{3}\right)$ *(1)*

Using, AM of m^{th} power $\geq$ m^{th} power of AM

$$\frac{a_1^m+a_2^m+\cdots+a_n^m}{n}\leq\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)^m\ \forall m\in(0,1)$$

Using this $\left(\frac{1+2+3+\cdots+n}{n}\right)^{\frac{1}{n}}\geq\left(\frac{1+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}}{n}\right)\Rightarrow\left[\frac{n(n+1)}{2n}\right]^{\frac{1}{2}}\geq\left(\frac{1+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}}{n}\right)>\frac{2\sqrt{n+1}}{3}$

$\Rightarrow\left[\frac{n(n+1)}{2n}\right]^{\frac{1}{2}}>\frac{2\sqrt{n+1}}{3}\Rightarrow\frac{\sqrt{n+1}}{\sqrt{2}}>\frac{2\sqrt{n+1}}{3}\Rightarrow 3>2\sqrt{2}\rightarrow$ *which is true. Hence the inequality in (1) is true.*

361. If $f:\mathbb{R}\to(0,\infty)$, f continuous then:

$$\int_0^a\int_0^a\int_0^a\left(\frac{f^4(x)+f^4(y)+f^4(z)}{f^3(x)+f^3(y)+f^3(z)}\right)^5 dx\,dy\,dz \ge a^2\int_0^a f^5(x)\,dx$$

Proposed by Daniel Sitaru – Romania

Solution by Omran Kouba-Damascus-Syria

Let $P(x)=5(x^4-x^3)-x^5+1$. Clearly we have $P(1)=P'(1)=0$, so $P(x)$ is divisible by $(x-1)^2$. An easy calculation shows that:

$P(x)=(x-1)^2(x^2(3-x)+2x+1)$. Thus, for

$x\in[0,3]$ we have $P(x)\ge 0$. Consider, positive numbers t,u,v and define

$$x=\frac{\sqrt[5]{3}t}{\sqrt[5]{t^5+u^5+v^5}},\ y=\frac{\sqrt[5]{3}u}{\sqrt[5]{t^5+u^5+v^5}},\ z=\frac{\sqrt[5]{3}v}{\sqrt[5]{t^5+u^5+v^5}}$$

These numbers belong to $[0,3]$. From $P(x)+P(y)+P(z)\ge 0$ we conclude that

$3^{\frac{4}{5}}\cdot\frac{t^4+u^4+v^4}{\left(\sqrt[5]{t^5+u^5+v^5}\right)^4}\ge 3^{\frac{3}{5}}\cdot\frac{t^3+u^3+v^3}{\left(\sqrt[5]{t^5+u^5+v^5}\right)^3}$. *Equivalently* $\left(\frac{t^4+u^4+v^4}{t^3+u^3+v^3}\right)^5\ge\frac{1}{3}\left(t^5+u^5+v^5\right)$

It follows that for $f:[0,a]\to(0,+\infty)$ we have

$$\int_0^a\int_0^a\int_0^a\left(\frac{f^4(z)+f^4(y)+f^4(z)}{f^3(x)+f^3(y)+f^3(z)}\right)^5 dx\,dy\,dz\ge\int_0^a\int_0^a\int_0^a\frac{1}{3}\left(f^5(x)+f^5(y)+f^5(x)\right)dx\,dy\,dz$$

$$=a^2\int_0^a f^5(x)\,dx$$

362. If $a,b,c>0$ then:

$$2\cdot\int_a^{2a}\int_a^{2b}\int_a^{2c}\frac{(2x+y)(2y+z)(2z+x)}{(x+y+z)^2}dx\,dy\,dz\le 3abc(a+b+c)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Rovsen Pirguliyev-Sumgait-Azerbaidian

$$\textit{It is known that } (2x+y)(2y+z) \le (x+y+z)^2 \quad \textit{(1)}$$

$$\textit{Similarly } (2y+z)(2z+x) \le (x+y+z)^2 \ \textit{(2)}$$

$$(2x+y)(x+2z) \le (x+y+z)^2$$

$$\textit{(1)} \times \textit{(2)} \times \textit{(3)} \Rightarrow (2x+y)(2y+z)(2z+x) \le (x+y+z)^3 \textit{ then we have}$$

$$LHS \le 2\int_a^{2a}\int_a^{2b}\int_a^{2c}(x+y+z)\,dx\,dy\,dz = 2\left(\frac{3a^2bc}{2}+\frac{3ab^2c}{2}+\frac{3abc^2}{2}\right) =$$

$$= 3abc(a+b+c)$$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$2\int_a^{2c}\int_a^{2b}\int_a^{2a}\frac{(2x+y)(2y+z)(2z+x)}{(x+y+z)^2}dx \overset{MG\le MA}{\le} 2\int\int\int\frac{\left(\frac{3(x+y+z)}{3}\right)^3}{(x+y+z)^2}dx\,dy\,dz =$$

$$= 2\int\int\int(x+y+z)\,dx\,dy\,dz = 2\cdot\int\int\left(\frac{x^2}{2}+(y+z)x\right)\Bigg|_a^{2a}dy\,dz =$$

$$= 2\cdot\int\int_b^{2b}\left(\frac{3a^2}{2}+a(y+z)\right)dy\,dz = 2\int\int_b^{2b}\left(ay+\frac{3a^2}{2}+az\right)dy\,dz =$$

$$= 2\cdot\int_c^{2c}\left(\frac{ay^2}{2}+\left(\frac{3a^2}{2}+az\right)y\right)\Bigg|_b^{2b}dz = 2\cdot\int_c^{2c}\left(\frac{3ab^2}{2}+\left(\frac{3a^2}{2}+az\right)\cdot b\right)dz =$$

$$= 2\cdot\int_c^{2c}\left(abz+\frac{3ab^2}{2}+\frac{3a^2b}{2}\right)dz = 2\left(\frac{abz^2}{2}+\left(\frac{3a^2b}{2}+\frac{3ab^2}{2}\right)z\right)\Bigg|_c^{2c} =$$

$$= 2\left(\frac{3abc^2}{2}+\left(\frac{3a^2b}{2}+\frac{3ab^2}{2}\right)c\right) = 3abc(a+b+c)$$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\textit{Since } (2x+y)(2y+z)(2z+x) \le (x+y+z)^3\textit{. Hence}$$

$$2\int_a^{2a}\int_a^{2b}\int_a^{2c}\frac{(2x+y)(2y+z)(2z+x)}{(x+y+z)^2}dx\,dy\,dz \le 2\int_a^{2a}\int_a^{2b}\int_a^{2c}\frac{(x+y+z)^3}{(x+y+z)^2}dx\,dy\,dz =$$

$$=\int_a^{2a}\int_a^{2b}\int_a^{2c} 2(x+y+z)\,dx\,dy\,dz = 2\left(\frac{x^2yz}{2}+\frac{xy^2z}{2}+\frac{xyz^3}{2}\right)=$$

$$=x^2yz+xy^2z+xyz^2\Big|_a^{2a}\Big|_b^{2b}\Big|_c^{2c}=$$

$$=((2a)^2-a^2)\big((2b)-b\big)\big((2c)-c\big)+\big((2a)-a\big)((2b)^2-(b)^2)\big((2c)-c\big)+$$

$$+2(2a-a)(2b-b)((2c)^2-c^2)=3a^2bc+3ab^2c+3abc^2=3abc(a+b+c)$$

Therefore it is to be true.

363. If $0<a<\frac{\pi}{4}$ then:

$$a^2\cot\left(\frac{3a}{2}\right)\le\int_a^{2a}\int_a^{2a}\cot\left(\frac{4x+3y}{7}\right)dx\,dy\le\log(2\cos a)^a$$

Proposed by Daniel Sitaru – Romania

Soumitra Mandal-Chandar Nagore-India

Let $f(x)=\cot x$ *for all* $x\in\left[0,\frac{\pi}{4}\right]$ *then* $f'(x)=-\csc^2 x\,;f''(x)=2\csc^2 x\cot x>0$

for all $x\in\left[0,\frac{\pi}{4}\right]$, *hence* f *is convex* $\cot\left(\frac{4x+3y}{7}\right)\le\frac{4}{7}\cot x+\frac{3}{7}\cot y\Rightarrow$

$$\int_a^{2a}\int_a^{2a}\cot\left(\frac{4x+3y}{7}\right)dx\,dy\le$$

$$\le\frac{4}{7}\int_a^{2a}\int_a^{2a}\cot x\,dx\,dy+\frac{3}{7}\int_a^{2a}\int_a^{2a}\cot y\,dx\,dy=\frac{4a}{7}\int_a^{2a}\cot x\,dx+\frac{3a}{7}\int_a^{2a}\cot y\,dy=$$

$$=\frac{4a}{7}|\log(\sin x)|_{x=a}^{x=2a}+\frac{3a}{7}|\log(\sin y)|_{y=a}^{y=2a}=a\log\left(\frac{\sin 2a}{\sin a}\right)=a\log(2\cos x)=$$

$=\log(2\cos x)^a$. *Applying Hermite – Hadamard for Double Integral;*

$$\frac{1}{(2a-a)(2a-a)}\int_a^{2a}\int_a^{2a}\cot\left(\frac{4x+3y}{7}\right)dx\,dy\ge\cot\left(\frac{4\cdot\left(\frac{2a+a}{2}\right)+3\cdot\left(\frac{2a+a}{2}\right)}{7}\right)\Rightarrow$$

$$\Rightarrow\int_a^{2a}\int_a^{2a}\cot\left(\frac{4x+3y}{7}\right)dx\,dy\ge a^2\cot\left(\frac{3a}{2}\right)$$

$$\therefore a^2\cot\left(\frac{3a}{2}\right) \le \int_a^{2a}\int_a^{2a}\cot\left(\frac{4x+3y}{7}\right)dx\,dy \le \log(2\cos a)^a$$

(proved)

364. If $a, b, c > 0$ then:

$$3\int_a^{2a}\int_b^{2b}\int_c^{2c}\left(\frac{x+y+z}{x^2+y^2+z^2}\right)dx\,dy\,dz \le (ab+bc+ca)\ln 2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

Since $(xy+yz+zx)\left(\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}\right) = (x+y+z) + \left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\right) + \left(\frac{x^2}{z}+\frac{z^2}{y}+\frac{y^2}{x}\right) \ge$

$\ge 3(x+y+z)$. *Hence* $\frac{3(x+y+z)}{\frac{x}{y^2}+\frac{y}{zx}+\frac{z}{xy}} \le xy+yz+zx \Rightarrow \frac{3(xyz)(x+y+z)}{x^2+y^2+z^2} \le xy+yz+zx \Rightarrow$

$\Rightarrow \frac{3(x+y+z)}{x^2+y^2+z^2} \le \frac{xy+yz+zx}{xyz} = \frac{1}{x}+\frac{1}{y}+\frac{1}{z}$. *Hence*

$$3\int_a^{2a}\int_b^{2b}\int_c^{2c}\frac{x+y+z}{x^2+y^2+z^2}dx\,dy\,dz = \int_a^{2a}\int_b^{2b}\int_c^{2c}\frac{3(x+y+z)}{(x^2+y^2+z^2)}dx\,dy\,dz \le$$

$$\le \int_a^{2a}\int_b^{2b}\int_c^{2c}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)dx\,dy\,dz = xy\ln z + yz\ln x + zx\ln y\Big|_a^{2a}\Big|_b^{2b}\Big|_c^{2c} =$$

$$= [ab\ln(2c) + bc\ln(2a) + ca\ln(2b)] - [ab\ln c + bc\ln a + ca\ln b] =$$

$$= [ab\ln 2 + bc\ln 2 + ca\ln 2 + ab\ln c + bc\ln a + ca\ln b]$$

$$-[ab\ln c + bc\ln a + ca\ln b]$$

$= ab\ln 2 + bc\ln 2 + ca\ln 2 = (ab+bc+ca)\ln 2$. *Therefore it is to be true.*

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$3\sum_{cyc}x^2 \ge (x+y+z)^2, \int_a^{2a}\int_b^{2b}\int_c^{2c}\frac{x+y+z}{x^2+y^2+z^2}dx\,dy\,dz \le$$

$$\leq \int_c^{2c}\int_b^{2b}\int_a^{2a} \frac{9}{x+y+z}dx\,dy\,dz \leq \sum_{cyc}\int_c^{2c}\int_b^{2a}\int_a^{2a}\frac{dx}{x} = \ln 2\sum_{cyc} ab$$

(proved)

Solution 3 by Nguyen Van Nho-Nghe An-Vietnam

We have: $3\sum x^2 \geq (\sum x)^2 \rightarrow \frac{\sum x}{\sum x^2} \leq \frac{3}{\sum x} \overset{Cauchy}{\leq} (xyz)^{-\frac{1}{3}}$*. So:*

$$LHS \leq 3\int_a^{2a}\int_b^{2b}\int_c^{2c}(xyz)^{-\frac{1}{3}}dx\,dy\,dz = 3\prod\left(\frac{3}{2}x^{\frac{2}{3}}\Big|_a^{2a}\right) =$$

$$= \frac{81}{8}\left(2^{\frac{2}{3}}-1\right)^3(abc)^{\frac{2}{3}} \overset{Cauchy}{<} (ab+bc+ca)\ln 2 \quad \textit{(Done)}$$

365. If $f:[0,1]\rightarrow(0,\infty)$, f – continuous, $\int_0^1 f(x)\,dx = 1$ then:

$$\int_0^1\int_0^1 f(x)f(y)\,dx\,dy \leq \int_0^1 f^{10}(x)\,dx$$

Proposed by Daniel Sitaru – Romania

Solution by Chris Kyriazis-Athens-Greece

Using Hölder's inequality, for $p=10, q=\frac{10}{9}$*, we have that:*

$$1 = \int_0^1 f(x)\,dx \leq \left(\int_0^1 f^{10}(x)dx\right)^{\frac{1}{10}}\cdot\left(\int_0^1 1^{\frac{10}{9}}\,dx\right)^{\frac{9}{10}} = \left(\int_0^1 f^{10}(x)dx\right)^{\frac{1}{10}} \Rightarrow$$

$$\Rightarrow \int_0^1 f^{10}(x)\,dx \geq 1 \quad (1)$$

So,

$$\int_0^1\int_0^1 f(x)f(y)dx\,dy = \int_0^1 f(x)\,dx\int_0^1 f(y)\,dy = 1 \overset{(1)}{\leq} \int_0^1 f^{10}(x)\,dx$$

366. Let be $\Omega:\left(0,\frac{\pi}{2}\right)\to\mathbb{R}$,

$$\Omega(x)=\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_{\varepsilon}^{x}\frac{\sin t}{t}dt$$

Prove that:

$$(a+b+c)\Omega\left(\frac{a^2+b^2+c^2}{a+b+c}\right)\ge a\Omega(a)+b\Omega(b)+c\Omega(c)$$

Proposed by Daniel Sitaru – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$\Omega=\lim_{\substack{x\to\infty\\ \varepsilon>0}}\int_{\varepsilon}^{x}\frac{\sin t}{t}dt \text{ then } \Omega'(x)=\frac{\sin x}{x},\Omega''(x)=\frac{\cos x}{x^2}(x-\tan x)\le 0$$

since, $\sin x\le x\le\tan x$ for all $x\ge 0$, hence Ω is concave so, from the definition of concave function

$$\sum_{cyc}\frac{a}{a+b+c}\Omega(a)\le\Omega\left(\sum_{cyc}\left(\frac{a}{a+b+c}\right)\cdot a\right)$$

$$\sum_{cyc}a\Omega(a)\le(a+b+c)\Omega\left(\frac{a^2+b^2+c^2}{a+b+c}\right)$$

367. If $0<a\le b<\frac{\pi}{4}$ then:

$$\int_a^b\int_a^b\int_a^b\left(\frac{(x+y)^2}{(y+z)\sin(z+x)}+\frac{(y+z)^2}{(z+x)\sin(x+y)}+\frac{(z+x)^2}{(x+y)\sin(y+z)}\right)dxdydz\ge 3(b-a)^3$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Daniel Sitaru-Romania

$$u=x+y,v=y+z,w=z+x$$

$$\sum_{cyc(x,y,z)}\frac{(x+y)^2}{(y+z)\sin(z+x)}=\sum_{cyc(u,v,w)}\frac{u^2}{v\cdot\sin w}\overset{\sin w<w}{\ge}$$

$$> \sum_{cyc(u,v,w)} \frac{u^2}{vw} \overset{BERGSTROM}{\geq} \frac{(u+v+w)^2}{uv+vw+wu} \geq 3$$

$$(u+v+w)^2 \geq 3(uv+vw+wu)$$

$$\int_a^b\int_a^b\int_a^b \sum_{cyc(x,y,z)} \frac{(x+y)^2}{(y+z)\sin(z+x)} \geq \int_a^b\int_a^b\int_a^b 3dxdydz = 3(b-a)^3$$

Solution 2 by Serban George Florin-Romania

$$\sum \frac{(x+y)^2}{(y+z)\sin(z+x)} \geq \sum \frac{(x+y)^2}{(y+z)(z+x)}, \qquad (sint \leq t)$$

$$\sum \frac{(x+y)^2}{(y+z)(z+x)} \geq \sqrt[3]{\frac{(x+y)^2(y+z)^2(z+x)^2}{(x+y)^2(y+z)^2(z+x)^2}} = 3$$

$$\int_a^b\int_a^b\int_a^b \sum \frac{(x+y)^2}{(y+z)\sin(z+x)} dxdydz \geq 3\int_a^b\int_a^b\int_a^b dxdydz \geq 3(b-a)^3$$

368. If $0 \leq a,b,c,d,e,f,x,y,z \leq 1$ then:

$$108\int_0^1\int_0^1\int_0^1 \frac{(3abcdefxyz-1)dxdydz}{(a^2+b^2+x^2+3)(c^2+d^2+y^2+3)(e^2+f^2+z^2+3)} \leq 1$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\because a,b,c,d,e,f,x,y,z \leq 1 \therefore 1 \geq a^4,b^4,c^4,d^4,e^4,f^4,x^4,y^4,z^4$$

$$\therefore a^2+b^2+x^2+3 = a^2+b^2+x^2+1+1+1 \geq a^2+b^2+x^2+a^4+b^4+x^4 \geq$$

$$\overset{A-G}{\geq} 6abx \Rightarrow a^2+b^2+x^2+3 \overset{(1)}{\geq} 6abx. \textit{ Again,}$$

$$c^2+d^2+y^2+3 = c^2+d^2+y^2+1+1+1 \geq c^2+d^2+y^2+c^4+d^4+y^4 \overset{A-G}{\geq}$$

$$\geq 6cdy \Rightarrow c^2+d^2+y^2+3 \overset{(2)}{\geq} 6cdy. \textit{ Also,}$$

$$e^2+f^2+z^2+3 = e^2+f^2+z^2+1+1+1 \geq e^2+f^2+z^2+e^4+f^4+z^4 \overset{A-G}{\geq}$$

$$\geq 6efz \Rightarrow e^2+f^2+z^2+3 \overset{(3)}{\geq} 6efz$$

$$(1).(2).(3) \Rightarrow \left(a^2+b^2+x^2+3\right)\left(c^2+d^2+y^2+3\right)\left(e^2+f^2+z^2+3\right) \overset{(a)}{\geq} 216abcdefxyz$$

Case 1: $3abcdefxyz-1 \leq 0$

Then, $LHS \leq \int_0^1\int_0^1\int_0^1 0\,dx\,dy\,dz = 0 < 1 \Rightarrow$ *given inequality is true*

Case 2: $3abcdefxyz-1 > 0$

$$(a) \Rightarrow LHS \overset{(i)}{\leq} \frac{1}{2}\int_0^1\int_0^1\int_0^1 \frac{3m-1}{m}dx\,dy\,dz, \text{ where } m = abcdefxyz$$

$\because m \leq 1 \therefore \frac{3m-1}{m} = 3-\frac{1}{m} \overset{(ii)}{\leq} 2 \therefore$ *(i), (ii)* $\Rightarrow LHS \leq \int_0^1\int_0^1\int_0^1 dx\,dy\,dz = 1 \Rightarrow$ *given inequality is true (Hence proved)*

369. Find:

$$L = \lim_{n\to\infty}\sqrt[n]{p!+\frac{(p+1)!}{1!}+\frac{(p+2)!}{2!}+\cdots+\frac{(p+n)!}{n!}}, p \in \mathbb{N}^*, p - \text{fixed}$$

Proposed by Marian Ursarescu-Romania

Solution by Ravi Prakash-New Delhi-India

$$p!+\frac{(p+1)!}{1!}+\frac{(p+2)!}{2!}+\cdots+\frac{(p+n)!}{n!} = p!\left[1+\frac{(p+1)!}{p!\,1!}+\frac{(p+2)!}{p!\,2!}+\cdots+\frac{(p+n)!}{p!\,n!}\right]$$

$$= p!\left[{}^{p+1}C_0+{}^{p+1}C_1+{}^{p+2}C_2+{}^{p+3}C_3+\cdots+{}^{p+n}C_n\right] = p!\left[{}^{p+2}C_1+{}^{p+2}C_2+\cdots+{}^{p+n}C_n\right]$$

$$= p!\left[{}^{p+3}C_2+{}^{p+3}C_3+\cdots+{}^{p+n}C_n\right] = p!\left[{}^{p+4}C_3+\cdots+{}^{p+n}C_n\right] = \cdots$$

$$= p!\,{}^{p+n+1}C_n = \frac{(p+n+1)!}{(p+1)!\,n!}p!$$

$$\frac{(p+n+1)(p+n)\dots(n+1)}{p+1} = \frac{n^{n+1}}{(p+1)}\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\dots\left(1+\frac{p}{n}\right)\left(1+\frac{p+1}{n}\right)$$

$$L = \lim_{n\to\infty}\frac{\left(n^{\frac{1}{n}}\right)^{p+1}}{(p+1)^{\frac{1}{n}}}\left(1+\frac{1}{n}\right)^{\frac{1}{n}}\left(1+\frac{2}{n}\right)^{\frac{1}{n}}\dots\left(1+\frac{p}{n}\right)^{\frac{1}{n}}\left(1+\frac{p+1}{n}\right)^{\frac{1}{n}}$$

$$= \frac{1}{1}e\cdot e^2\cdot e^3\cdot\ldots\cdot e^{p+1} = e^{\frac{(p+1)(p+2)}{2}}$$

370. $(x_n)_{n\geq 0}, x_0 > 0, x_{n+1} = x_n + \frac{1}{x_n^p}, p \in \mathbb{N}^*, p -$ **fixed**

Find:

$$\Omega = \lim_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n \sqrt[p+1]{n}}$$

Proposed by Marian Ursarescu-Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$\text{We know, } \lim_{u\to 0} \frac{(1+u)^r - 1}{u} = r \text{ now } x_{n+1} - x_n = \frac{1}{x_n^p} > 0 \text{ for all } n \in \mathbb{N}$$

Hence the sequence is increasing, implying its bounded

$$\text{then let } \lim_{n\to\infty} x_n = l \Rightarrow l = l + \frac{1}{l^p} \Rightarrow l \to \infty, \text{ which is a contradiction}$$

$$\therefore \lim_{n\to\infty} x_n = \infty \text{ let } L = \lim_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n \sqrt[p+1]{n}} = \left(\lim_{n\to\infty} \frac{\sum_{k=1}^n x_k}{n}\right)\left(\lim_{n\to\infty} \frac{1}{\sqrt[p+1]{n}}\right)$$

$$\overset{\substack{\text{Caesaro}\\ \text{Stolz}}}{\cong} \left(\lim_{n\to\infty} \frac{\sum_{k=1}^{n+1} x_k - \sum_{k=1}^{n} x_k}{n+1-n}\right)\left(\lim_{n\to\infty} \frac{1}{\sqrt[p+1]{n}}\right) = \lim_{n\to\infty} \frac{x_{n+1}}{\sqrt[p+1]{n}}$$

$$\Rightarrow L^{p+1} = \lim_{n\to\infty} \frac{x_{n+1}^{p+1}}{n} \overset{\substack{\text{Caesaro}\\ \text{Stolz}}}{\cong} \lim_{n\to\infty} \frac{x_{n+2}^{p+1} - x_{n+1}^{p+1}}{n+1-n} = \lim_{n\to\infty} \left(x_{n+2}^{p+1} - x_{n+1}^{p+1}\right)$$

$$= \lim_{n\to\infty} \left\{\left(x_{n+1} + \frac{1}{x_{n+1}^p}\right)^{p+1} - x_{n+1}^{p+1}\right\} = \lim_{n\to\infty} \frac{\left(1 + \frac{1}{x_{n+1}^{p+1}}\right)^{p+1} - 1}{\frac{1}{x_{n+1}^{p+1}}}$$

$$= \lim_{x_{n+1}\to\infty} \frac{\left(1 + \frac{1}{x_{n+1}^{p+1}}\right)^{p+1} - 1}{\frac{1}{x_{n+1}^{p+1}}} = p+1 \Rightarrow L = \sqrt[p+1]{p+1}$$

371. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{2}+\sum_{k=1}^{n}\frac{k^p}{k^{4p}+k^{2p}+1}\right)^{n^{3p-1}}, p \in \mathbb{N}^*$$

Proposed by Marian Ursarescu-Romania

Solution by Ravi Prakash-New Delhi-India

$$k^{4p}+k^{2p}+1=\left(k^{2p}+1\right)^2-k^{2p}=\left(k^{2p}-k^p+1\right)\left(k^{2p}+k^p+1\right)$$

$$\therefore \frac{k^p}{k^{4p}+k^{2p}+1}=\frac{1}{2}\left[\frac{1}{k^{2p}-k^p+1}-\frac{1}{k^{2p}+k^p+1}\right]$$

$$\Rightarrow \sum_{k=1}^{n}\frac{k^p}{k^{4p}+k^{2p}+1}=\frac{1}{2}\sum_{k=1}^{n}\left[\frac{1}{k^{2p}-k^p+1}-\frac{1}{k^{2p}+k^p+1}\right]$$

$$=\frac{1}{2}\left[1-\frac{1}{n^p+n^p+1}\right]$$

$$\Rightarrow \left(\frac{1}{2}+\sum_{k=1}^{n}\frac{k^p}{k^{4p}+k^{2p}+1}\right)^{n^{3p-1}}=\left(1-\frac{1}{2\left(n^{2p}+n^p+1\right)}\right)^{n^{3p-1}}$$

$$=\left[\left(1-\frac{1}{2(n^{2p}+n^p+1)}\right)^{-(2n^{2p}+2n^p+1)}\right]^m \text{ where } m=-\left(\frac{n^{3p-1}}{2n^{2p}+2n^p+1}\right)$$

$$As\ n\to\infty, m\to-\infty \therefore \Omega=e^{-\infty}=0$$

372. $x_0>0, x_{n+1}=x_n+\frac{a}{x_n}, a>0, n\in\mathbb{N}^*$

Find:

$$\Omega=\lim_{n\to\infty}\frac{\sum_{i,j=1}^{n}x_ix_j}{n^3}$$

Proposed by Marian Ursarescu-Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$We\ know\ \lim_{u\to 0}\frac{(1+u)^r-1}{u}=r\ and\ x_{n+1}-x_n=\frac{a}{x_n}>0\ for\ all\ n\in\mathbb{N}$$

hence the sequence is increasing, implying is bounded let

$$\lim_{n\to\infty} x_n = l \text{ then } l = l + \frac{a}{l} \Rightarrow l \to \infty \text{ which is a contradiction}$$

$$\therefore \lim_{n\to\infty} x_n = \infty \ \ now, \ \Omega = \lim_{n\to\infty} \frac{\sum_{i,j=1}^{n} x_i x_j}{n^3}$$

$$= \frac{1}{2}\left\{\lim_{n\to\infty} \frac{(x_1 + x_2 + \cdots + x_n)^2}{n^3} - \lim_{n\to\infty} \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n^3}\right\} = \frac{L_1 - L_2}{2}$$

$$L_1 = \lim_{n\to\infty} \frac{(x_1 + x_2 + \cdots + x_n)^2}{n^3} \Rightarrow \sqrt{L_1} = \lim_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n^{\frac{3}{2}}}$$

$$\overset{\substack{CAESARO\\STOLZ}}{\cong} \lim_{n\to\infty} \frac{x_{n+1}}{(n+1)^{\frac{3}{2}} - n^{\frac{3}{2}}} = \left(\lim_{n\to\infty} \frac{x_{n+1}}{\sqrt{n}}\right)\left(\lim_{n\to\infty} \frac{1}{\frac{\left(1+\frac{1}{n}\right)^{\frac{3}{2}} - 1}{\frac{1}{n}}}\right) = \frac{2}{3}\lim_{n\to\infty} \frac{x_{n+1}}{\sqrt{n}}$$

$$= \frac{2}{3}\sqrt{\lim_{n\to\infty} \frac{x_{n+1}^2}{n}} \overset{\substack{CAESARO\\STOLZ}}{\cong} \frac{2}{3}\sqrt{\lim_{n\to\infty}\left(x_{n+2}^2 - x_{n+1}^2\right)} = \frac{2}{3}\sqrt{\lim_{n\to\infty}\left\{\left(x_{n+1} + \frac{a}{x_{n+1}}\right)^2 - x_{n+1}^2\right\}}$$

$$= \frac{2}{3}\sqrt{\lim_{x_n\to\infty} \frac{\left(1+\frac{a}{x_n^2}\right)^2 - 1}{\frac{1}{x_n^2}}} = \frac{2\sqrt{2a}}{3} \Rightarrow L_1 = \frac{8a}{9}$$

$$L_2 = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} x_k^2}{n^3} \overset{\substack{CAESARO\\STOLZ}}{\cong} \lim_{n\to\infty} \frac{x_{n+1}^2}{(n+1)^3 - n^3} = \left(\lim_{n\to\infty} \frac{x_{n+1}^2}{n^2}\right)\left(\lim_{n\to\infty} \frac{1}{\frac{\left(1+\frac{1}{n}\right)^3 - 1}{\frac{1}{n}}}\right)$$

$$\overset{\substack{CAESARO\\STOLZ}}{\cong} \frac{1}{3}\lim_{n\to\infty} \frac{x_{n+2}^2 - x_{n+1}^2}{n} \cdot \lim_{n\to\infty} \frac{1}{\frac{\left(1+\frac{1}{n}\right)^2 - 1}{\frac{1}{n}}}$$

$$\overset{\substack{CAESARO\\STOLZ}}{\triangleq}\frac{1}{6}\lim_{n\to\infty}\{(x_{n+3}^2-x_{n+2}^2)-(x_{n+2}^2-x_{n+1}^2)\}=0$$

$$\therefore L_2=0 \text{ then } \lim_{n\to\infty}\frac{\sum_{i,j=1}^{n}x_ix_j}{n^3}=\frac{4a}{9}\quad (Ans:)$$

373. Find:

$$\Omega=\lim_{n\to\infty}(n+1)!\left(e-\sum_{k=0}^{n}\frac{1}{k!}\right)$$

Proposed by D.M.Batinetu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\text{For } k\geq 2, \frac{(n+1)!}{(n+k)!}=\frac{1}{(n+k)(n+k-1)\dots(n+2)}\leq\frac{1}{(n+2)^{k-1}}$$

$$\therefore e-\sum_{k=0}^{n}\frac{1}{k!}=\sum_{k=n+1}^{\infty}\frac{1}{k!}=\frac{1}{(n+1)!}+\sum_{k=2}^{\infty}\frac{1}{(n+k)!}=\frac{1}{(n+1)!}\left[1+\sum_{k=2}^{\infty}\frac{(n+1)!}{(n+k)!}\right]$$

$$<\frac{1}{(n+1)!}\left[1+\sum_{k=2}^{\infty}\frac{1}{(n+2)^{k-1}}\right]\Rightarrow(n+1)!\left[e-\sum_{k=0}^{\infty}\frac{1}{k!}\right]\leq 1+\frac{\frac{1}{(n+2)}}{1-\frac{1}{(n+2)}}=1+\frac{1}{n+1}$$

$$\text{Also, } e-\sum_{k=0}^{\infty}\frac{1}{k!}>\frac{1}{(n+1)!}\Rightarrow 1<(n+1)!\left[e-\sum_{k=0}^{\infty}\frac{1}{k!}\right]<1+\frac{1}{n+1}$$

$$\text{Taking limit and using sandwich theorem, we get } \lim_{n\to\infty}(n+1)!\left[e-\sum_{k=0}^{\infty}\frac{1}{k!}\right]=1$$

Solution 2 by Marian Ursarescu-Romania

$$\text{Using Cesaro – Stolz from } \frac{0}{0}: \Omega=\lim_{n\to\infty}\frac{e-\left(1+\frac{1}{1!}+\dots+\frac{1}{n!}\right)}{\frac{1}{(n+1)!}}$$

$$\text{Let } a_n=e-\left(1+\frac{1}{1!}+\dots+\frac{1}{n!}\right), b_n=\frac{1}{(n+1)!}. \text{ Then:}$$

$$a)\ \lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n=0$$

$$b)\ b_n \text{ is strict decreasing}$$

$$c)\ \lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}=\lim_{n\to\infty}\frac{e-\left(1+\frac{1}{1!}+\dots+\frac{1}{(n+1)!}\right)-e+\left(1+\frac{1}{1!}+\dots+\frac{1}{n!}\right)}{\frac{1}{(n+2)!}-\frac{1}{(n+1)!}}=$$

$$=\lim_{n\to\infty}\frac{-\frac{1}{(n+1)!}}{\frac{(n+1)!-(n+2)!}{(n+1)!\,(n+2)!}}=\lim_{n\to\infty}\frac{-(n+2)!}{(n+1)!\,(1-n-2)}=\lim_{n\to\infty}\frac{-(n+2)}{-(n+1)}=1$$

From Cesaro – Stolz $\Rightarrow \Omega = 1.$

374.

$$a_n = 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}$$

Find:

$$\Omega = \lim_{n\to\infty}\sqrt{n}\prod_{k=1}^{n}\left(1-\frac{1}{a_{k+1}\sqrt{k+1}}\right)$$

Proposed by Marian Ursarescu-Romania

Solution 1 by Omran Kouba-Damascus-Syria

Let $a_n = 1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}$. *Find:* $\Omega = \lim_{n\to\infty}\sqrt{n}\prod_{k=1}^{n}\left(1-\frac{1}{a_{k+1}\sqrt{k+1}}\right)$

The answer is $\Omega=\frac{1}{2}$. *First note that*

$$\frac{1}{\sqrt{n}}-2\left(\sqrt{n+1}-\sqrt{n}\right)=\frac{1}{\sqrt{n}}-\frac{2}{\sqrt{n+1}+\sqrt{n}}=\frac{1}{\left(\sqrt{n+1}+\sqrt{n}\right)^2\sqrt{n}}=\mathcal{O}\left(\frac{1}{n^{\frac{3}{2}}}\right)$$

and since the series $\sum\frac{1}{n^{\frac{3}{2}}}$ *is convergent we conlude that there exists a real number* ℓ

such that $\lim_{n\to\infty}\left(a_n-2\sqrt{n+1}\right)=\ell$. *In particular,*

$$\lim_{n\to\infty}\frac{a_n}{\sqrt{n}}=2 \qquad (1)$$

On the other hand, $1-\frac{1}{a_{k+1}\sqrt{k+1}}=\frac{1}{a_{k+1}}\left(a_{k+1}-\frac{1}{\sqrt{k+1}}\right)=\frac{a_k}{a_{k+1}}$

Thus

$$\sqrt{n}\prod_{k=1}^{n}\left(1-\frac{1}{a_{k+1}\sqrt{k+1}}\right)=\sqrt{n}\prod_{k=1}^{n}\frac{a_k}{a_{k+1}}=\frac{a_1\sqrt{n}}{a_{n+1}} \qquad (2)$$

Combining (1) and (2) we get $\Omega=\lim_{n\to\infty}\sqrt{n}\prod_{k=1}^{n}\left(1-\frac{1}{a_{k+1}\sqrt{k+1}}\right)=\frac{1}{2}$.

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$a_{k+1}=1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{k+1}}\Rightarrow a_{k+1}-\frac{1}{\sqrt{k+1}}=\sum_{i=1}^{k}\frac{1}{\sqrt{i}}=a_k\Rightarrow$$

$$\Rightarrow 1-\frac{1}{a_{k+1}\sqrt{k+1}}=\frac{a_k}{a_{k+1}}\therefore\Omega=\lim_{n\to\infty}\sqrt{n}\prod_{k=1}^{n}\left(1-\frac{1}{a_{k+1}\sqrt{k+1}}\right)=\lim_{n\to\infty}\sqrt{n}\prod_{k=1}^{n}\frac{a_k}{a_{k+1}}$$

$$=\lim_{n\to\infty}\frac{\sqrt{n}a_1}{a_{n+1}}=\lim_{n\to\infty}\frac{\sqrt{n}}{a_{n+1}}\overset{\substack{CESARO\\STOLZ}}{=}\lim_{n\to\infty}\frac{\sqrt{n+1}-\sqrt{n}}{a_{n+2}-a_{n+1}}=\lim_{n\to\infty}\frac{\sqrt{n+1}-\sqrt{n}}{\frac{1}{\sqrt{n+2}}}$$

$$=\lim_{n\to\infty}\frac{\sqrt{n+2}}{\sqrt{n+1}+\sqrt{n}}=\lim_{n\to\infty}\frac{\sqrt{1+\frac{2}{n}}}{\sqrt{1+\frac{1}{n}}+1}=\frac{1}{2}\ \ \textit{(Ans:)}$$

375. **Compute:**

$$\lim_{n\to\infty}\sqrt[4]{n!}\prod_{k=1}^{n}\left((k+1)^{\frac{3}{4}}-k^{\frac{3}{4}}\right)$$

Proposed by Mihály Bencze-Romania

Solution by Marian Ursarescu-Romania

$$a_n=\sqrt[4]{1\cdot 2\cdot\ldots\cdot n}\prod_{k=1}^{n}\left(\sqrt[4]{(k+1)^3}-\sqrt[4]{k^3}\right)=\prod_{k=1}^{n}\left[\sqrt[k]{k}\left(\sqrt[4]{(k+1)^3}-\sqrt[k]{k^3}\right)\right]\ \textit{(1)}$$

$$\textit{Let } f:[k,k+1]\to\mathbb{R}, f(x)=\sqrt[4]{x^3}$$

$$\textit{From Lagrange Theorem}\Rightarrow\exists c_k\in(k,k+1)\textit{ such that}$$

$$\frac{f(k+1)-f(k)}{k+1-k}=f'(c_k)\Rightarrow\sqrt[4]{(k+1)^3}-\sqrt[4]{k^3}=\frac{3}{4}\cdot\frac{1}{\sqrt[4]{c}}\ \ \textit{(2)}$$

$$\textit{But } c\in(k,k+1)\Rightarrow k<c<k+1\Rightarrow\sqrt[4]{k}<\sqrt[4]{c}<\sqrt[4]{k+1}\Rightarrow\sqrt[4]{\frac{k}{c}}<1\ \ \textit{(3)}$$

$$\textit{From (2)}\Rightarrow\sqrt[4]{k}\left(\sqrt[4]{(k+1)^3}\cdot\sqrt[4]{k^3}\right)=\frac{3}{4}\sqrt[4]{\frac{k}{c}}\overset{(3)}{<}\frac{3}{4}\ \textit{(4)}$$

$$\textit{From (1)+(4)}\Rightarrow 0<a_n<\left(\frac{3}{4}\right)^n\Rightarrow\left(\lim_{n\to\infty}\left(\frac{3}{4}\right)^n=0\right),\lim_{n\to\infty}a_n=0$$

376. Find:

$$\Omega = \lim_{n\to\infty}\frac{1}{n(n+1)}\sum_{k=1}^{n} k\sin\left(\frac{k^2+k}{n^2+n}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$\int_a^b f(x)dx = \lim_{|x_r - x_{r-1}|\to 0}\sum_{r=1}^{n} f(\xi_r)\,(x_r - x_{r-1}) \text{ where } \xi_r \in (x_{r-1}, x_r)$$

$$\text{Let } S_n = 1+2+\cdots+n = \frac{n(n+1)}{2}, \text{ considering a partition}$$

$$P = \left(\frac{S_1}{S_n}, \frac{S_2}{S_n}, \dots, 1\right) \text{ now } \left|\frac{S_r}{S_n} - \frac{S_{r-1}}{S_n}\right| = \frac{2r}{n(n+1)} \to 0 \text{ as } n\to\infty$$

$$\lim_{n\to\infty}\frac{1}{n(n+1)}\sum_{k=1}^{n} k\sin\left(\frac{k^2+k}{n^2+n}\right)$$

$$= \frac{1}{2}\lim_{\left|\frac{S_r}{S_n} - \frac{S_{r-1}}{S_n}\right|\to 0}\sum_{r=1}^{n}\left(\frac{S_r}{S_n} - \frac{S_{r-1}}{S_n}\right)\sin\left(\frac{S_r}{S_n}\right) = \frac{1}{2}\int_0^1 \sin x\,dx = \frac{1-\cos 1}{2}$$

377. Find:

$$\Omega = \lim_{n\to\infty}\frac{1}{n(n+1)(2n+1)}\sum_{k=1}^{n} k^2 \tan^{-1}\left(\frac{k(k+1)(2k+1)}{n(n+1)(2n+1)}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Omran Kouba-Damascus-Syria

$$\text{Let } S_n = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}. \text{ Define } x_k^{(n)} = \frac{S_k}{S_n} \text{ for } 1\le k\le n, \text{ and consider the}$$

$$\text{subdivision } \sigma = \left(0, x_1^{(n)}, x_2^{(n)}, \dots, x_n^{(n)}\right) \text{ of the interval } [0,1] \text{ with step}$$

$$h_n = \max_{1\le k\le n}\left(x_k^{(n)} - x_{k-1}^{(n)}\right) = \frac{1}{S_n}\max_{1\le k\le n} k^2 = \frac{6n}{(n+1)(2n+1)} \underset{n\to\infty}{\to} 0.$$

It follows that

$$R(\arctan,\sigma_n)=\sum_{k=1}^{n}\left(x_k^{(n)}-x_{k-1}^{(n)}\right)\arctan\left(x_k^{(n)}\right)$$

Is the Riemann sum of the function arctan corresponding to the subdivision σ_n of $[0,1]$. The fact that the step of this subdivision tends to zero implies that

$$\lim_{n\to\infty}R(\arctan,\sigma_n)=\int_0^1\arctan x\,dx=[\chi\arctan\chi]_0^1-\int_0^1\frac{xdx}{1+x^2}=\frac{\pi}{4}-\frac{1}{2}\ln 2$$

But

$$R(\arctan,\sigma_n)=\frac{6}{n(n+1)(2n+1)}\sum_{k=1}^{n}k^2\arctan\left(\frac{k(k+1)(2k+1)}{n(n+1)(2n+1)}\right)$$

Thus

$$\Omega=\lim_{n\to\infty}\frac{1}{n(n+1)(2n+1)}\sum_{k=1}^{n}k^2\arctan\left(\frac{k(k+1)(2k+1)}{n(n+1)(2n+1)}\right)=\frac{\pi}{24}-\frac{1}{12}\ln 2.$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

We know $\int_a^b f(x)\,dx=\lim_{|x_r-x_{r-1}|\to 0}\sum_{r=1}^{n}f(\xi_r)\,(x_r-x_{r-1})$ *where* $\xi_r\in(x_{r-1},x_r)$

Let $S_n=\sum_{r=1}^{n}r^2=\frac{n(n+1)(2n+1)}{6}$*, considering a partition*

$P=\left(\frac{S_1}{S_n},\frac{S_2}{S_n},\dots,1\right)$ *now* $\left|\frac{S_r}{S_n}-\frac{S_{r-1}}{S_n}\right|=\frac{6r^2}{n(n+1)(2n+1)}\to 0$ *as* $n\to\infty$

$$\lim_{n\to\infty}\frac{1}{n(n+1)(2n+1)}\sum_{k=1}^{n}\tan^{-1}\left(\frac{k(k+1)(2k+1)}{n(n+1)(2n+1)}\right)$$

$$=\frac{1}{6}\lim_{\left|\frac{S_r}{S_n}-\frac{S_{r-1}}{S_n}\right|\to 0}\sum_{r=1}^{n}\left(\frac{S_r}{S_n}-\frac{S_{r-1}}{S_n}\right)\tan^{-1}\left(\frac{S_r}{S_n}\right)$$

$$=\frac{1}{6}\int_0^1\tan^{-1}x\,dx=\frac{1}{6}[x\tan^{-1}x]_{x=0}^{x=1}-\frac{1}{6}\int_0^1\frac{xdx}{1+x^2}=\frac{\pi}{24}-\frac{\ln 2}{12}$$

378.

$$\Omega_n = \int_0^{\frac{\pi}{4}} (4x^2 - \pi x + 4n^2)\ln(1+\tan x)\,dx, n \in \mathbb{N}^*$$

Find:

$$\Omega = \lim_{n\to\infty} \frac{\Omega_n}{1+2+3+\cdots+n}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Lazaros Zachariadis-Thessaloniki-Greece

$$0 \le x \le \frac{\pi}{4} \Leftrightarrow -\frac{\pi^2}{4} \le -\pi x \le 0$$

$$0 \le x \le \frac{\pi}{4} \Leftrightarrow 0 \le 4x^2 \le \frac{\pi^2}{4} \quad (+)$$

$$-\frac{\pi^2}{4} \le 4x^2 - \pi x \le \frac{\pi^2}{4} \quad (1)$$

$$0 \le x \le \frac{\pi}{4} \underset{\nearrow}{\overset{\tan x}{\Leftrightarrow}} 0 \le \tan x \le 1 \Leftrightarrow 1 \le \tan x + 1 \le 2$$

$$\underset{\nearrow}{\overset{\ln x}{\Leftrightarrow}} 0 \le \ln(1+\tan x) \le \ln 2 \quad (2)$$

$$\underset{+4n^2}{\overset{(1)}{\rightarrow}} 4n^2 - \frac{n^2}{4} \le 4x^2 - \pi x + 4n^2 \le 4n^2 + \frac{\pi^2}{4}$$

$$\underset{(2)}{\Leftrightarrow} \left(4n^2 - \frac{\pi^2}{4}\right)\ln(1+\tan x) \le (4x^2 - \pi x + 4n^2)\ln(1+\tan x) \le \left(4n^2 + \frac{\pi^2}{4}\right)\ln(1+\tan x)$$

$$\Rightarrow \left(4n^2 - \frac{\pi^2}{4}\right)\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx \le \int_0^{\frac{\pi}{4}} f(x)\,dx \le \left(4n^2 + \frac{\pi^2}{4}\right)\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx$$

$$\underset{easy\ integrate\ by\ parts}{\overset{\int_0^{\frac{\pi}{4}} \ln(1+\tan x)dx = \frac{\pi}{8}\ln 2}{\Rightarrow}} \left(4n^2 - \frac{\pi^2}{4}\right)\cdot\frac{\pi\ln 2}{8} \le \underline{0}_n \le \left(4n^2 + \frac{\pi^2}{4}\right)\cdot\frac{\pi}{8}\ln 2$$

$$\Rightarrow \frac{\frac{n^2\pi\ln 2}{2}-\frac{\pi^3}{32}}{1+\cdots+n} \le \frac{0_n}{1+\cdots n} \le \frac{\frac{n^2\pi\ln 2}{2}-\frac{\pi^3}{32}}{1+\cdots+n}$$

$$\Rightarrow \lim_{n\to\infty}\frac{n^2\cdot\frac{\pi\ln 2}{2}-\frac{\pi^3}{32}}{\frac{(n^2+n)}{2}} \le \lim_{n\to\infty}\frac{0_n}{1+\cdots+n} \le \lim_{n\to\infty}\frac{n^2\cdot\frac{\pi\ln 2}{2}+\frac{\pi^3}{32}}{\frac{n^2+n}{2}}$$

$$\Rightarrow \lim_{n\to\infty}\frac{\frac{n^2\cdot\pi\ln 2}{2}}{\frac{n^2}{2}} \le \lim(\dots) \le \lim_{n\to\infty}\frac{\frac{n^2\cdot\pi\cdot\ln 2}{2}}{\frac{n^2}{2}}$$

$$\pi\ln 2 \le \lim(\dots) \le \pi\ln 2 \Rightarrow \lim_{n\to\infty}\frac{0_n}{1+\cdots+n} = \pi\ln 2$$

Solution 2 by Shivam Sharma-New Delhi-India

As we know, the following lemma, If $f(x)$ is a continuous function on $[0,a]$

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

Using the following above lemma, we get,

$$I_n \Rightarrow \int_0^{\frac{\pi}{4}}\left(4\left(\frac{\pi}{4}-x\right)^2-\pi\left(\frac{\pi}{4}-x\right)+4n^2\right)\ln\left(1+\tan\left(\frac{\pi}{4}-2\right)\right)dx$$

$$\Rightarrow \int_0^{\frac{\pi}{4}}\left(\left(4\left(\frac{\pi^2}{16}+x^2-2\left(\frac{\pi}{x}\right)\right)\right)-\frac{\pi^2}{4}+\pi x+4n^2\right)\ln\left(1+\tan\left(\frac{\pi}{4}-2\right)\right)dx$$

$$\Rightarrow \int_0^{\frac{\pi}{4}}(4x^2-\pi x+4n^2)\ln\left(1+\frac{\tan\left(\frac{\pi}{4}\right)-\tan(x)}{1+\tan\left(\frac{\pi}{4}\right)\tan(x)}\right)dx$$

$$\Rightarrow \int_0^{\frac{\pi}{4}}(4x^2-\pi x+4n^2)\ln\left(\frac{2}{1+\tan(x)}\right)dx \Rightarrow \ln(2)\int_0^{\frac{\pi}{4}}(4x^2-\pi x+4n^2)\,dx - I_n$$

$$2\Omega_n = \int_0^{\frac{\pi}{4}}\ln(2)\,(4x^2-\pi x+4n^2)dx$$

$$\Omega_n = \frac{\ln(2)}{2}\int_0^{\frac{\pi}{4}}(4x^2-\pi x+4n^2)\,dx$$

$$\Omega_n = \frac{\ln(2)}{2}\left[\frac{4x^3}{3}-\frac{\pi x^2}{2}+4n^2x\right]_0^{\frac{\pi}{4}} \Rightarrow \frac{\ln(2)}{2}\left[\frac{4}{3}\left(\frac{\pi^3}{64}\right)-\frac{\pi}{2}\left(\frac{\pi^2}{16}\right)+4n^2\left(\frac{\pi}{4}\right)\right]$$

$$\Rightarrow \frac{\ln(2)}{2}\left[\frac{\pi^3}{48}-\frac{\pi^3}{32}+4n^2\left(\frac{\pi}{4}\right)\right] \Rightarrow \frac{\ln(2)}{2}\left[\frac{\pi^3}{48}-\frac{\pi^3}{32}+\pi n^2\right]$$

$$\Rightarrow \frac{\ln(2)}{2}\left[-\frac{\pi^3}{96}+\pi n^2\right]\text{(OR)}\,\Omega_n = \frac{\ln(2)}{2}\left[\pi n^2-\frac{\pi^3}{96}\right]$$

$$\text{Now, }\Omega = \lim_{n\to\infty}\left(\frac{\Omega_n}{1+2+3+\cdots+n}\right) \Rightarrow \lim_{n\to\infty}\left(\frac{\frac{\ln(2)}{2}\left[\pi n^2-\frac{\pi^3}{96}\right]}{\sum_{k=1}^{n}(k)}\right) \Rightarrow \lim_{n\to\infty}\left(\frac{\frac{\ln(2)}{2}\left[\pi n^2-\frac{\pi^3}{96}\right]}{\frac{n(n+1)}{2}}\right) \Rightarrow$$

$$\Rightarrow \ln(2)\lim_{n\to\infty}\left(\frac{\pi n^2-\frac{\pi^3}{96}}{n^2+n}\right) \Rightarrow \ln(2)\lim_{n\to\infty}\left(\frac{\pi-\frac{1}{n^2}\left(\frac{\pi^3}{96}\right)}{1+\frac{1}{n}}\right)$$

$$\text{(OR) }\Omega = \pi\ln(2)\text{ (Answer)}$$

Solution 3 by Marian Ursarescu-Romania

$$\left.\begin{matrix}\Omega_n = \displaystyle\int_0^{\frac{\pi}{4}}(4x^2-\pi x+4n^2)\ln(1+\tan x)\,dx \\ x=\frac{\pi}{4}-y \Rightarrow dx=-dy;\, x=0 \Rightarrow y=\frac{\pi}{4} \wedge x=\frac{\pi}{4} \Rightarrow y=0\end{matrix}\right\} \Rightarrow$$

$$\Omega_n = \int_{\frac{\pi}{4}}^{0}\left[4\left(\frac{\pi}{4}-y\right)^2-\pi\left(\frac{\pi}{4}-y\right)+4n^2\right]\ln\left(1+\tan\left(\frac{\pi}{4}-y\right)\right)(-dy)$$

$$= \int_0^{\frac{\pi}{4}}\left(\frac{\pi^2}{4}-2\pi y-4y^2-\frac{\pi^2}{4}+\pi y+4n^2\right)\ln\left(1+\frac{1-\tan y}{1+\tan y}\right)dy$$

$$= \int_0^{\frac{\pi}{4}}(4y^2-\pi y+4n^2)\left(\ln 2-\ln(1+\tan y)\right)dy =$$

$$= \ln 2 \int_0^{\frac{\pi}{4}} (4y^2 - \pi y + 4n^2)\, dy - \int_0^{\frac{\pi}{4}} (4y^2 - \pi y + 4n^2) \ln(1 + \tan y)\, dy \Rightarrow$$

$$\Omega = \ln 2 \left(4\frac{y^3}{3}\Big|_0^{\frac{\pi}{4}} - \pi \frac{y^2}{2}\Big|_0^{\frac{\pi}{4}} + 4n^2 y\Big|_0^{\frac{\pi}{4}} \right) - \Omega_n \Rightarrow$$

$$2\Omega_n = \ln 2 \left(\frac{4}{3} \cdot \frac{\pi^3}{64} - \frac{\pi}{2} \cdot \frac{\pi^2}{16} + 4n^2 \cdot \frac{\pi}{n} \right) \Rightarrow$$

$$\Omega_n = \frac{\ln 2}{2} \left(\frac{\pi^3}{48} - \frac{\pi^3}{32} + n^2 \pi \right) \Rightarrow \Omega_n = \frac{\ln 2}{2} \left(n^2 \pi - \frac{\pi^3}{96} \right)$$

$$\Omega = \lim_{n\to\infty} \frac{\frac{\ln 2}{2}\left(n^2 \pi - \frac{\pi^3}{96} \right)}{\frac{n(n+1)}{2}} = \pi \ln 2$$

379. If α, β and γ are three distinct real values such that

$$\frac{\sin\alpha + \sin\beta + \sin\gamma}{\sin(\alpha+\beta+\gamma)} = \frac{\cos\alpha + \cos\beta + \cos\gamma}{\cos(\alpha+\beta+\gamma)} = 2 \text{ and}$$

$\cos(\alpha+\beta) + \cos(\beta+\gamma) + \cos(\alpha+\gamma) = a$. Then find the value of

$$\lim_{x\to a} \frac{\sqrt{x^2 - a^2}}{\sqrt{x-a} + \sqrt{x} - \sqrt{a}}$$

Proposed by Amit Dutta – Jamshedpur – Jharkhand – India

Solution by Marian Ursărescu – Romania

$$\begin{matrix} \sin\alpha + \sin\beta + \sin\gamma = 2\sin(\alpha+\beta+\gamma) \\ \cos\alpha + \cos\beta + \cos\gamma = 2\cos(\alpha+\beta+\gamma) \end{matrix} \quad (1)$$

Let $z_1 = \cos\alpha + i\sin\alpha, z_2 = \cos\beta + i\sin\beta, z_3 = \cos\gamma + i\sin\gamma, z_1, z_2, z_3 \in \mathbb{C}$ *with*

$$|z_1| = |z_2| = |z_3| = 1$$

From (1) $\Rightarrow z_1 + z_2 + z_3 = 2(\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma))$ *(1)*

But $z_1 \cdot z_2 \cdot z_3 = \cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)$ *(2)*

From (1)+(2)$\Rightarrow z_1 + z_2 + z_3 = 2z_1 \cdot z_2 \cdot z_3$ *(3)*

But $|z_1| = 1 \Rightarrow |z_1|^2 = 1 \Rightarrow z_1 \cdot \overline{z_1} = 1, z_2\overline{z_2} = 1, z_3\overline{z_3} = 1$

$$\textit{From (3)} \Rightarrow \overline{z_1+z_2+z_3} = 2\overline{z_1z_2z_3} \Rightarrow \overline{z_1}+\overline{z_2}+\overline{z_3} = 2\overline{z_1z_2z_3} \Rightarrow$$

$$\Rightarrow \frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3} = \frac{2}{z_1z_2z_3} \Rightarrow z_1z_2+z_2z_3+z_1z_3 = 2 \Rightarrow$$

$$\cos(\alpha+\beta)+\cos(\beta+\gamma)+\cos(\gamma+\alpha)+i(\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)) = 2$$

$$\left.\begin{array}{r}\Rightarrow \cos(\alpha+\beta)+\cos(\beta+\gamma)+\cos(\gamma+\alpha) = 2\\ \textit{But} \cos(\alpha+\beta)+\cos(\beta+\gamma)+\cos(\gamma+\alpha) = a\end{array}\right\} \Rightarrow a = 2$$

$$\lim_{x\to a}\frac{\sqrt{x^2-a^2}}{\sqrt{x-a}+\sqrt{x}-\sqrt{a}} = \lim_{x\to 2}\frac{\sqrt{x^2-4}}{\sqrt{x-2}+\sqrt{x}-\sqrt{2}}$$

$$\sqrt{x-2} = t, t \ge 0 \Rightarrow x-2 = t^2 \Rightarrow x = t^2+2, t \to 0 \textit{ because } x \to 2$$

$$= \lim_{\substack{t\to 0\\ t>0}}\frac{\sqrt{(t^2+2)^2-4}}{t+\sqrt{t^2+2}-\sqrt{2}} = \lim_{\substack{t\to 0\\ t>0}}\frac{\sqrt{t^4+4t^2+4-4}\left(t+\sqrt{t^2+2}+\sqrt{2}\right)}{t^2+2+\sqrt{t^2+2}+t^2+2-2}$$

$$= \lim_{\substack{t\to 0\\ t>0}}\frac{\sqrt{t^4+4+2}\left(t+\sqrt{t^2+2}+\sqrt{2}\right)}{2t\left(t+\sqrt{t^2+2}\right)} = \lim_{\substack{t\to 0\\ t>0}}\frac{t\sqrt{t^4+4}\left(t+\sqrt{t^2+2}+\sqrt{2}\right)}{2t\left(t+\sqrt{t^2+2}\right)} = \frac{2\cdot\sqrt{2}}{2\cdot\sqrt{2}} = 2$$

380. Find:

$$\Omega = \lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le i<j\le n}\sqrt[n]{e^{2na+(i+j)b}}, a, b \ge 0$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Abdallah Almalih-Damascus-Syria

We know that:

$$\sum_{1\le i<j\le n} f(i,j) = \sum_{1\le i<j\le n} f(i,j) + \sum_{i=1}^{n} f(i,i) + \sum_{n\ge i>j\ge 1} f(i,j)$$

But if $f(i,j) = f(j,i)$ ***(symmetric) then***

$$\sum_{1\le i,j\le n} f(i,j) = 2\sum_{1\le i<j\le n} f(i,j) + \sum_{i=1}^{n} f(i,j)$$

$$\textit{So, } \sum_{1\le i<j\le n} f(i,j) = \frac{1}{2}\left\{\sum_{1\le i,j\le n} f(i,j) - \sum_{i=1}^{n} f(i,j)\right\}$$

$$\textit{Now, let } f(i,j) = \sqrt[5]{e^{2na+(i+j)b}}$$

$$f(i,j)=\sqrt[n]{e^{2an}\cdot e^{(i+j)b}}=e^{2a}\cdot e^{\frac{i+j}{n}b}=e^{2a}\cdot e^{\frac{1}{n}b}\cdot e^{\frac{j}{n}b}, \text{ so } f(i,j)=f(j,i) \text{ and}$$

$$\sum_{1\le i,j\le n} f(i,j)=e^{2a}\sum_{1\le i,j\le n} e^{\frac{i}{n}b}\cdot e^{\frac{j}{n}b}=e^{2a}\left(\sum_{i=1}^{n} e^{\frac{i}{n}b}\right)^2$$

$$=e^{2a}\left(\sum_{i=1}^{n}\left[e^{\frac{b}{n}}\right]^i\right)^2=e^{2n}\frac{\left(e^{\frac{b(n+1)}{n}}-e^{\frac{b}{n}}\right)^2}{e^{\frac{b}{n}}-1}=e^{2a}\left(\frac{e^b\cdot e^{\frac{b}{n}}-e^{\frac{b}{n}}}{e^{\frac{b}{n}-1}}\right)^2=e^{2a}e^{\frac{2b}{n}}\left(\frac{e^b-1}{e^{\frac{b}{n}}-1}\right)^2$$

and $\sum_{i=1}^{n} f(i,j)=\sum_{i=1}^{n} e^{2a}e^{\frac{2i}{n}b}=e^{2a}\sum_{i=1}^{n}\left(e^{\frac{2b}{n}}\right)^i=e^{2a}\left[\frac{e^{\frac{2b}{n}(n+1)}-e^{2\frac{b}{n}}}{e^{\frac{2b}{n}}-1}\right]=e^{2a}\left(\frac{e^{2b}\cdot e^{\frac{2b}{n}}-e^{\frac{2b}{n}}}{e^{\frac{2b}{n}}-1}\right)$

$$=e^{2a}e^{\frac{2b}{n}}\left(\frac{e^{2b}-1}{e^{\frac{2b}{n}}-1}\right)$$

$$\sum_{1\le i<j\le n} f(i,j)=e^{2a}e^{\frac{2b}{n}}\left(\frac{e^b-1}{e^{\frac{b}{n}}-1}\right)^2;\ \sum_{i=1}^{n} f(i,i)=e^{2a}e^{\frac{2b}{n}}\left(\frac{e^{2b}-1}{e^{\frac{2b}{n}}-1}\right)$$

$$\sum_{1\le i<j\le n} f(i,j)=\frac{1}{2}\left(\sum_{1\le i<j\le n} f(i,j)-\sum_{i=1}^{n} f(i,i)\right)$$

$$=\frac{1}{2}e^{2a}e^{\frac{2b}{n}}\left[\left(\frac{e^b-1}{\frac{e^b}{n}-1}\right)^2-\left(\frac{e^{2b}-1}{e^{2b}-1}\right)\right]$$

Let $x=e^b, y=e^{\frac{b}{n}}$ *for more easy writing then* $\left(\frac{x-1}{y-1}\right)^2-\left(\frac{x^2-1}{y^2-1}\right)=\left(\frac{x-1}{y-1}\right)\left[\frac{x-1}{y-1}-\frac{x+1}{y+1}\right]$

$$=\frac{x-1}{y-1}\left[\frac{(x-1)(y+1)-(x+1)(y-1)}{y^2-1}\right]=\frac{x-1}{y-1}\left(\frac{2x-2y}{y^2-1}\right)$$

$$=\frac{x-1}{y-1}\left(\frac{x-y}{y^2-1}\right)\cdot 2=\frac{e^b-1}{e^{\frac{b}{n}}-1}\left(\frac{e^b-e^{\frac{b}{n}}}{e^{\frac{2b}{n}}-1}\right)\cdot 2, \text{ so}$$

$$\sum_{1\le i<j\le n} f(i,j)=\frac{e^{2a}}{2}e^{\frac{2b}{n}}\left(\frac{e^b-1}{e^{\frac{b}{n}}-1}\right)\left(\frac{e^b-e^{\frac{b}{n}}}{e^{\frac{2b}{n}}-1}\right)\cdot 2=\left(e^b-1\right)\cdot\frac{e^b-e^{\frac{b}{n}}}{\left(e^{\frac{b}{n}}-1\right)\left(e^{\frac{2b}{n}}-1\right)}e^{2a}e^{\frac{2b}{n}}$$

So, the limit which we need to calculate it is

$$\lim_{n\to\infty}\frac{1}{n^2}\left(e^b-1\right)e^{2a}e^{\frac{2b}{n}}\cdot\frac{e^b-e^{\frac{b}{n}}}{\left(\frac{e^b}{n}-1\right)\left(e^{\frac{2b}{n}}-1\right)}=$$

$$=\lim_{n\to\infty}\left(e^b-1\right)e^{2a}\cdot\frac{e^{\frac{2b}{n}}\left(e^b-e^{\frac{b}{n}}\right)}{\left(e^{\frac{b}{n}}+1\right)}\cdot\frac{1}{n^2\left(e^{\frac{b}{n}}-1\right)^2}$$

Let's calculate $\lim_{n\to\infty}\frac{e^1}{n^2\left(e^{\frac{b}{n}}-1\right)^2}$. *Put* $x=\frac{1}{n}$ *so* $x\to 0$ *as* $n\to\infty$

$$\lim_{x\to 0}\frac{x^2}{(e^{bx}-1)^2}=\lim_{x\to 0}\frac{1}{b^2\left(\frac{e^{bx}-1}{bx}\right)^2}=\frac{1}{b^2}$$

$$So,\ \lim_{n\to\infty}\frac{(e^b-1)e^{2n}e^{\frac{2b}{n}}\left(e^b-e^{\frac{b}{n}}\right)}{e^{\frac{b}{n}}+1}\cdot\frac{1}{n^2\left(e^{\frac{b}{n}}-1\right)^2}=\frac{(e^b-1)e^{2a}e^0(e^b-1)}{2}\cdot\frac{1}{b^2}=\frac{\left(e^b-1\right)^2e^{2a}}{2b^2}$$

Solution 2 by Ravi Prakash-New Delhi-India

Case 1. When $b=0$,

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le i<j\le n}\left(e^{2na+(i+j)b}\right)^{\frac{1}{n}}=\lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le i<j\le n}e^{2a}=\lim_{n\to\infty}\frac{1}{n^2}e^{2a}\cdot\frac{n(n-1)}{2}=\frac{1}{2}e^{2a}$$

Case 2. $b>0$

$$\sum_{1\le i<j\le n}\left(e^{2na+(i+j)b}\right)^{\frac{1}{n}}=e^{2a}\sum_{1\le i<j\le n}\left(e^{\frac{b}{n}}\right)^{i+j}=e^{2a}\sum_{1\le i<j\le n}\alpha^{i+j}$$

where $\alpha=e^{\frac{b}{n}}$. *But*

$$\sum_{1\le i<j\le n}\alpha^{i+j}=\sum_{i=1}^{n-1}\alpha^i\sum_{j=i+1}^{n}\alpha^j=\sum_{j=1}^{n-1}\alpha^i\left\{\frac{\alpha^{i+1}(\alpha^{n-1}-1)}{\alpha-1}\right\}$$

$$=\sum_{i=1}^{n-1}\alpha^i\left(\frac{\alpha^{n+1}-\alpha^{i+1}}{\alpha-1}\right)=\frac{1}{\alpha-1}\sum_{i=1}^{n-1}\left[\alpha^{n+1}\alpha^i-\alpha^{2i+1}\right]=$$

$$=\frac{\alpha^{n+1}}{\alpha-1}\left[\frac{\alpha(\alpha^{n-1}-1)}{\alpha-1}-\frac{\alpha^3(\alpha^{2n-2}-1)}{\alpha^2-1}\right]=\frac{1}{(\alpha-1)^2}\left[\alpha^{2n+1}-\alpha^{n+2}-\frac{(\alpha^{2n+1}-\alpha^3)}{\alpha+1}\right]$$

$$=\frac{1}{\left(e^{\frac{b}{n}}-1\right)^2}\left[\left(e^{\frac{b}{n}}\right)^{2n+1}-\left(e^{\frac{b}{n}}\right)^{n+1}\right]-\frac{1}{e^{\frac{b}{n}+1}}\left\{\left(e^{\frac{b}{n}}\right)^{2n+1}-e^{\frac{3b}{n}}\right\}$$

$$=\frac{1}{\left(e^{\frac{b}{n}}-1\right)^2}\left[e^{2b}e^{\frac{b}{n}}-e^{b}e^{\frac{b}{n}}-\frac{1}{e^{\frac{b}{n}}+1}\left\{e^{2b}\cdot e^{\frac{b}{n}}-e^{\frac{3b}{n}}\right\}\right]$$

Thus,

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le i<j\le n}\left(e^{2na+(i+j)b}\right)^{\frac{1}{n}}=$$

$$=e^{2a}\lim_{n\to\infty}\left[\frac{1}{b^2}\cdot\left(\frac{\frac{b}{n}}{e^{\frac{b}{n}}-1}\right)^2\left\{e^{2b}e^{\frac{b}{n}}-e^{b}\cdot e^{\frac{b}{n}}-\frac{1}{e^{\frac{b}{n}}+1}\left\{e^{2b}e^{\frac{b}{n}}-e^{\frac{3b}{n}}\right\}\right\}\right]$$

$$=\frac{e^{2a}}{b^2}\left[(1)^2\left(e^{2b}-e^{b}-\frac{1}{2}\left(e^{2b}-1\right)\right)\right]=\frac{e^{2a}}{2b^2}\left[2e^{2b}-2e^{b}-e^{2b}+1\right]$$

$$=\frac{e^{2a}\left(e^{b}-1\right)^2}{2b^2}$$

Solution 3 by Marian Ursarescu-Romania

$$\textit{If } a=0 \textit{ its easy} \Rightarrow \Omega=e^{2a}$$

$$\Omega=\lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le i<j\le n}e^{\frac{2na+(i+j)b}{n}}=\lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le i<j\le n}e^{2a}\cdot e^{\frac{(i+j)b}{n}}=e^{2a}\lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le i<j\le n}e^{\frac{bi}{n}}\cdot e^{\frac{bi}{n}}\quad (1)$$

$$\textit{But } \lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le i<j\le n}e^{\frac{bi}{n}}\cdot e^{\frac{bj}{n}}=\frac{1}{2}\left(\lim_{n\to\infty}\frac{1}{n^2}\left(\sum_{k=1}^{n}e^{\frac{bk}{n}}\right)\right)^2-\frac{1}{n^2}\sum_{k=1}^{n}e^{\frac{2bj}{n}}\quad (2)$$

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}e^{\frac{bk}{n}}\right)^2=\left(\int_0^1 e^{bx}\,dx\right)^2=\left(\frac{e^{bx}}{b}\Big|_0^1\right)^2=\left(\frac{e^{b}-1}{b}\right)^2\quad (3)$$

$$\lim_{n\to\infty}\frac{1}{n^2}\cdot\sum_{k=1}^{n}e^{\frac{2bi}{n}}=\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{n}\sum_{k=1}^{n}e^{\frac{2bi}{n}}\right)=0\quad (4)$$

$$\textit{Because } \frac{1}{n}\sum_{k=1}^{n}e^{\frac{2bi}{4}} \textit{ its an convergent sequence.}$$

$$\textit{Form } (1)+(2)+(3)+(4)\Rightarrow\Omega=e^{2a}\cdot\frac{1}{2}\cdot\left(\frac{e^{b}-1}{b}\right)^2$$

381.

$$x_n=\left(1+\frac{1}{n}\right)^n, y_n=1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}, z_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n,$$

$$n\in\mathbb{N}^*$$

Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{2x_n+3y_n+5z_n}{5(e+\gamma)}\right)^n$$

Proposed by Daniel Sitaru – Romania

Solution by Marian Ursărescu – Romania

$$\Omega=\lim_{n\to\infty}\left(\frac{2x_n+3y_n+5z_n}{5(e+\gamma)}\right)^n=$$

$$=\lim_{n\to\infty}\left(1+\frac{2x_n+3y_n+5z_n-5(e+\gamma)}{5(e+\gamma)}\right)^n=e^{\frac{1}{5(e+\gamma)}\lim\limits_{n\to\infty}n[2(x_n-e)+3(y_n-e)+5(z_n-\gamma)]}\quad (1)$$

$$\lim_{n\to\infty}n[2(x_n-e)]=2\lim_{n\to\infty}\frac{x_n-e}{\frac{1}{n}}=2\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^n-e}{\frac{1}{n}}\overset{(Heine)}{=}2\lim_{n\to\infty}\frac{\left(1+\frac{1}{x}\right)^x-e}{\frac{1}{x}}=$$

$$=2\lim_{n\to\infty}\frac{(1+t)^{\frac{1}{t}}-e}{t}=2\lim_{n\to\infty}\left(1+\frac{1}{t}\right)^{\frac{1}{t}}\left[-\frac{1}{t^2}\cdot\ln(1+t)+\frac{1}{t}\cdot\frac{1}{t+1}\right]$$

$$=2\lim_{n\to\infty}\left(1+\frac{1}{t}\right)^t\left[\frac{-(t+1)\ln(t+1)+t}{t^3+t^2}\right]=2\lim_{n\to\infty}\left(1+\frac{1}{t}\right)^t\left[-\frac{\ln(t+1)-1+1}{3t^2+2t}\right]=$$

$$=2\lim_{n\to\infty}\left(1+\frac{1}{t}\right)^t\cdot\frac{-\ln(t+1)}{t(2+3t)}=2\cdot e\cdot\left(-\frac{1}{2}\right)=-e\quad (2)$$

Now, using Cesaro – Stolz for $\frac{0}{0}$

$$\lim_{n\to\infty}n[3(yn-e)]=3\lim_{n\to\infty}\frac{y_n-e}{\frac{1}{n}}=3\lim_{n\to\infty}\frac{y_{n+1}-e-y_n+e}{\frac{1}{n+1}-\frac{1}{n}}=3\lim_{n\to\infty}\frac{\frac{1}{(n+1)!}}{-\frac{1}{n(n+1)}}=$$

$$=3\lim_{n\to\infty}-\frac{1}{(n-1)!}=0\quad (2)$$

$$\lim_{n\to\infty} n[5(z_n-\gamma)] = 5\lim_{n\to\infty}\frac{z_n-\gamma}{\frac{1}{n}} = 5\lim_{n\to\infty}\frac{z_{n+1}-\gamma-z_n+\gamma}{\frac{1}{n+1}-\frac{1}{n}} =$$

$$= 5\lim_{n\to\infty}\frac{\frac{1}{n+1}-\ln(n+1)+\ln n}{-\frac{1}{n(n+1)}}$$

$$= 5\lim_{n\to\infty}\frac{\frac{1}{n+1}-\ln\left(\frac{n+1}{n}\right)}{-\frac{1}{n(n+1)}} = 5\lim_{n\to\infty}\frac{1-(n+1)\ln\left(1+\frac{1}{n}\right)}{-\frac{1}{n}}$$

$$= 5\lim_{n\to\infty}\frac{1-(x+1)\ln\left(1+\frac{1}{x}\right)}{-\frac{1}{x}} = 5\lim_{n\to\infty}\frac{1-\left(\frac{1}{t}+1\right)\ln(1+t)}{-t}$$

$$= 5\lim_{t\to 0}\frac{t-(1+t)\ln(1+t)}{-t^2} = 5\lim_{t\to 0}\frac{1-\ln(1+t)-1}{-2t}$$

$$= 5\lim_{n\to\infty}\frac{\ln(1+t)}{2t} = \frac{5}{2} \quad (3)$$

$$\text{From (1)+(2)+(3)} \Rightarrow \Omega = e^{\frac{1}{5(e+\gamma)}\left(-e+\frac{5}{2}\right)}$$

382. If $a, b \in \mathbb{N}$ then:

$$\left(\sqrt{a}+\sqrt{b}\right)\sqrt[4]{ab} \le \lim_{n\to\infty}\frac{2}{n}\sum_{k=1}^{n}\sqrt[n]{a^k b^{n-k}} \le \left(\sqrt{a}+\sqrt{b}\right)^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$\text{Let } a > b \ge 1 \text{ now}$$

$$\left(\sqrt{a}+\sqrt{b}\right)\sqrt[4]{ab} \le \lim_{n\to\infty}\frac{2}{n}\sum_{k=1}^{n}\sqrt[n]{a^k b^{n-k}} \le \left(\sqrt{a}+\sqrt{b}\right)^2$$

$$\Leftrightarrow \left(1+\sqrt{\frac{a}{b}}\right)\sqrt[4]{\frac{a}{b}} \le \lim_{n\to\infty}\frac{2}{n}\sum_{k=1}^{n}\left(\frac{a}{b}\right)^{\frac{k}{n}} \le \left(1+\sqrt{\frac{a}{b}}\right)^2$$

$$\Leftrightarrow\left(1+\sqrt{\frac{a}{b}}\right)\sqrt[4]{\frac{a}{b}}\le 2\int_0^1\left(\frac{a}{b}\right)^x dx\le\left(1+\sqrt{\frac{a}{b}}\right)^2\Leftrightarrow\left(1+\sqrt{\frac{a}{b}}\right)\sqrt[4]{\frac{a}{b}}\le 2\frac{\frac{a}{b}-1}{\log\left(\frac{a}{b}\right)}\le\left(1+\sqrt{\frac{a}{b}}\right)^2$$

$$\Leftrightarrow (1+m)\sqrt{m}\le\frac{m^2-1}{\log m}\le(1+m)^2\ \ where\ m=\sqrt{\frac{a}{b}}$$

$$Let\ f(m)=\sqrt{m}-\frac{1}{\sqrt{m}}-\log m\ for\ all\ m\ge 1$$

$$f'(m)=\frac{1}{2\sqrt{m}}+\frac{1}{2m^{\frac{3}{2}}}-\frac{1}{m}=\frac{m-2\sqrt{m}+1}{2m^{\frac{3}{2}}}=\frac{\left(\sqrt{m}-1\right)^2}{2m^{\frac{3}{2}}}\ge 0$$

$$\therefore f(m)\ge f(1)=0\Rightarrow(1+m)\sqrt{m}\le\frac{m^2-1}{\log m}$$

$$Let\ \varphi(m)=\log m-\frac{m-1}{m+1} for\ all\ m\ge 1,\varphi'(m)=\frac{1}{m}-\frac{2}{(m+1)^2}=\frac{m^2+1}{m(m+1)^2}>0$$

$$\therefore \varphi(m)\ge\varphi(1)=0\Rightarrow\frac{m^2-1}{\log m}\le(1+m)^2$$

(Hence proved)

Solution 2 by Sagar Kumar-Kolkata-India

$$S=\lim_{n\to\infty}\frac{2}{n}\sum_{k=1}^{n}\left(\frac{a}{b}\right)^{\frac{k}{n}}b$$

$$S=2b\int_0^1\left(\frac{a}{b}\right)^{\gamma}d\gamma=\frac{2b\left(\frac{a}{b}\right)|_0^1}{\ln(a)-\ln b},\ S=\frac{2b\left(\frac{a}{b}-1\right)}{\ln(a)-\ln(b)}=\frac{2(a-b)}{\ln(a)-\ln(b)},\ S=\frac{2(a-b)}{f(a)-f(b)}where\ f(x)=\ln x$$

By LMVT

$$S=2c\ \ \ (1)\ \ \ -c\in(a,b),S\le 2b$$

RHS

$$\left(\sqrt{a}+\sqrt{b}\right)^2=\left(\left(\sqrt{a}-\sqrt{b}\right)^2+4\sqrt{ab}\right)$$

$$Clearly\ it\ is\ minimum\ when\ \sqrt{a}=\sqrt{b}\Rightarrow\left(\sqrt{a}+\sqrt{b}\right)^2\ge 4\sqrt{ab}$$

$$\left(\sqrt{a}+\sqrt{b}\right)^2_{\min}=4b\Rightarrow S\le 4b$$

LHS

$$\left(\sqrt{a}+\sqrt{b}\right)(ab)^{\frac{1}{4}};\ \frac{\sqrt{a}+\sqrt{b}}{2}\geq(ab)^{\frac{1}{4}}\ (AM\geq GM)$$

$$\Rightarrow\underbrace{\left(\sqrt{a}+\sqrt{b}\right)(ab)^{\frac{1}{4}}}_{\substack{\max\ =2a\\ when\ \sqrt{a}=\sqrt{b}}}\leq\frac{\left(\sqrt{a}+\sqrt{b}\right)^2}{2}\leq\frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2}+2\sqrt{ab}$$

$$Hence\ LHS\leq S\leq RHS\Rightarrow\left(\sqrt{a}+\sqrt{b}\right)\sqrt[4]{ab}\leq\lim_{n\to\infty}\frac{2}{n}\sum_{k=1}^{n}\sqrt[n]{a^k\cdot b^{n-k}}\leq\left(\sqrt{a}+\sqrt{b}\right)^2$$

(proved)

383.

$$\lim_{n\to\infty}\frac{a_n\cdot a_{n+2}^6\cdot a_{n+4}}{a_{n+1}^4\cdot a_{n+3}^4}=10, a_n>0, n\in\mathbb{N}, n\geq 2$$

Find:

$$\Omega=\lim_{n\to\infty}\sqrt[n^4]{a_n}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursarescu-Romania

We use Cesaró – Stolz ⇒

$$\ln\Omega=\ln\sqrt[n^4]{a_n}=\frac{\ln a_n}{n^4}\Rightarrow\lim_{n\to\infty}\ln\Omega=\lim_{n\to\infty}\frac{\ln a_n}{n^4}=\lim_{n\to\infty}\frac{\ln a_{n+1}-\ln a_n}{(n+1)^4-n^4}=$$

$$=\lim_{n\to\infty}\frac{\ln\frac{a_{n+1}}{a_n}}{4n^3+6n^2+4n+1}=\lim_{n\to\infty}\frac{\ln\frac{a_{n+1}}{a_n}}{n^3}\cdot\frac{n^3}{4n^3+6n^2+4n+1}\quad (1)$$

$$\lim_{n\to\infty}\frac{\ln\frac{a_{n+1}}{a_n}}{n^3}=\lim_{n\to\infty}\frac{\ln\frac{a_{n+2}}{a_{n+1}}-\ln\frac{a_{n+1}}{a_n}}{(n+1)^3-n^3}=\lim_{n\to\infty}\frac{\ln\frac{a_n\cdot a_{n+2}}{a_{n+1}^2}}{3n^2+3n+1}=\lim_{n\to\infty}\frac{\ln\frac{a_n\cdot a_{n+2}}{a_{n+1}^2}}{n^2}\cdot\frac{n^2}{3n^2+3n+1}\quad (2)$$

$$\lim_{n\to\infty}\frac{\ln\frac{a_n\cdot a_{n+2}}{a_{n+1}^2}}{n^2}=\lim_{n\to\infty}\frac{\ln\frac{a_{n+1}\cdot a_{n+3}}{a_{n+2}^2}\cdot\ln\frac{a_n\cdot a_{n+2}}{a_{n+1}^2}}{(n+1)^2-n^2}=\lim_{n\to\infty}\frac{\ln\frac{a_{n+1}^3\cdot a_{n+3}}{a_n\cdot a_{n+2}^3}}{2n+1}=$$

$$=\lim_{n\to\infty}\frac{\ln\frac{a_{n+2}^3\cdot a_{n+4}}{a_{n+1}\cdot a_{n+3}}-\ln\frac{a_{n+1}^3\cdot a_{n+3}}{a_n\cdot a_{n+2}^3}}{(2n+3)-(2n+1)}=\lim_{n\to\infty}\frac{\ln\frac{a_{n+2}^6\cdot a_n\cdot a_{n+4}}{a_{n+3}^4\cdot a_{n+1}^4}}{2}=\frac{10}{2}=5\quad (3)$$

$$From\ (1)+(2)+(3)\Rightarrow\lim_{n\to\infty}\ln\Omega=\frac{5}{12}\Rightarrow\lim_{n\to\infty}\Omega=e^{\frac{5}{12}}=\sqrt[12]{e^5}$$

Solution 2 by Soumitra Mandal-Chandar Nagore -India

$$\Omega=\lim_{n\to\infty}\sqrt[n^4]{a_n}\Rightarrow\ln\Omega=\lim_{n\to\infty}\frac{\ln a_n}{n^4}\overset{\substack{CESARO\\ STOLZ\ LEMMA}}{\cong}\lim_{n\to\infty}\frac{\ln a_{n+1}-\ln a_n}{(n+1)^4-n^4}$$

$$=\left(\lim_{n\to\infty}\frac{\ln a_{n+1}-\ln a_n}{n^3}\right)\left(\lim_{n\to\infty}\frac{1}{\frac{\left(1+\frac{1}{n}\right)^4-1}{\frac{1}{n}}}\right)=\frac{1}{4}\lim_{n\to\infty}\frac{\ln a_{n+1}-\ln a_n}{n^3}$$

$$\overset{\substack{CESARO\ STOLZ\\ LEMMA}}{\cong}\frac{1}{4}\lim_{n\to\infty}\frac{(\ln a_{n+2}-\ln a_{n+1})-(\ln a_{n+1}-\ln a_n)}{(1+n)^3-n^3}$$

$$=\frac{1}{4}\lim_{n\to\infty}\frac{\ln a_{n+2}-\ln a_{n+1}^2+\ln a_n}{n^2}\cdot\lim_{n\to\infty}\frac{1}{\frac{\left(1+\frac{1}{n}\right)^3-1}{\frac{1}{n}}}$$

$$\overset{\substack{CESARO\ STOLZ\\ LEMMA}}{\cong}\frac{1}{12}\lim_{n\to\infty}\frac{\left(\ln a_{n+3}-\ln a_{n+2}^2+\ln a_{n+1}\right)-\left(\ln a_{n+2}-\ln a_{n+1}^2+\ln a_n\right)}{(n+1)^2-n^2}$$

$$=\frac{1}{12}\lim_{n\to\infty}\frac{\ln a_{n+3}-\ln a_{n+2}^3+\ln a_{n+1}^3-\ln a_n}{n}\cdot\lim_{n\to\infty}\frac{1}{\frac{\left(1+\frac{1}{n}\right)^2-1}{\frac{1}{n}}}$$

$$\overset{\substack{CESARO\ STOLZ\\ LEMMA}}{\cong}\frac{1}{24}\lim_{n\to\infty}\frac{\left(\ln a_{n+4}-\ln a_{n+3}^3+\ln a_{n+2}^3-\ln a_{n+1}\right)-\left(\ln a_{n+3}-\ln a_{n+2}^3+\ln a_{n+1}^3-\ln a_n\right)}{n+1-n}$$

$$=\frac{1}{24}\lim_{n\to\infty}\ln\left(\frac{a_{n+4}a_{n+2}^6a_n}{a_{n+3}^4a_{n+1}^4}\right)=\frac{\ln 10}{24}$$

$$\Omega=\sqrt[24]{e^{10}}=\sqrt[12]{e^5}\ \textit{(Ans :)}$$

384. **Find:**

$$\Omega=\lim_{n\to\infty}\frac{1}{n^3}\sum_{1\le i<j<k\le n}\sin\left(2+\frac{3i}{n}\right)\sin\left(2+\frac{3j}{n}\right)\sin\left(2+\frac{3k}{n}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Marian Ursarescu-Romania

We use this:

$$\left(\sum_{k=1}^{n} a_k\right)^3 = \sum_{k=1}^{n} a_k^3 + 3\sum_{1\le i<j\le n} a_i^2 a_j + 3\sum_{1\le i<j\le n} a_i a_j^2 + 6\sum_{1\le i<j<k\le n} a_i a_j a_k \quad (1)$$

$$\text{From (1)} \Rightarrow \lim_{n\to\infty}\frac{1}{n^3}\sum_{1\le i<j<k\le n}\sin\left(2+\frac{3i}{n}\right)\sin\left(2+\frac{3j}{n}\right)\sin\left(2+\frac{3k}{n}\right)$$

$$=\lim_{n\to\infty}\frac{1}{6}\begin{bmatrix}\frac{1}{n^3}\left(\sum_{k=1}^{n}\sin\left(2+\frac{3k}{n}\right)\right)^3-\frac{1}{n^3}\sum_{k=1}^{n}\sin^3\left(2+\frac{3k}{n}\right)-\\ -\frac{3}{n^3}\sum_{1\le i<j\le n}\sin^2\left(2+\frac{3i}{n}\right)\sin\left(2+\frac{3i}{n}\right)+\frac{3}{n^2}\sum_{1\le i<j\le n}\sin\left(2+\frac{3i}{n}\right)\sin^2\left(2+\frac{3i}{n}\right)\end{bmatrix} \quad (2)$$

$$\text{New: } \lim_{n\to\infty}\frac{1}{n^3}\left(\sum_{k=1}^{n}\sin\left(2+\frac{3k}{n}\right)\right)^3=$$

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\sin\left(2+\frac{3k}{n}\right)\right)^3=\left(\int_0^1\sin(2+3x)\,dx\right)^3=$$

$$=\left(-\frac{\cos(2+3x)}{3}\Big|_0^1\right)^3=\left(-\frac{\cos 5+\cos 2}{3}\right)^3 \quad (3)$$

$$\lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{n}\sin^3\left(2+\frac{3k}{n}\right)=\lim_{n\to\infty}\frac{1}{n^2}\left(\frac{1}{n}\cdot\sum_{k=1}^{n}\sin^3\left(2+\frac{3k}{n}\right)\right)$$

$$\text{But } \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sin^3\left(2+\frac{3k}{n}\right)=\int_0^1\sin^3(2+3x)\,dx\in\mathbb{R}^*$$

$$\Rightarrow \lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{n}\sin^3\left(2+\frac{3k}{n}\right)=0 \quad (4)$$

$$\lim_{n\to\infty}\frac{1}{n^3}\sum_{1\le i<j\le n}\sin^2\left(2+\frac{3i}{n}\right)\cdot\sin\left(2+\frac{3i}{n}\right)=0 \quad \text{(5) because}$$

$$\frac{1}{n^3}\left|\sum_{1\le i<j\le n}\sin^2\left(2+\frac{3i}{n}\right)\sin\left(2+\frac{3i}{n}\right)\right|\le\frac{1}{n^3}\sum_{1\le i<j\le n}\left|\sin^2\left(2+\frac{3i}{n}\right)\right|\left|\sin\left(2+\frac{3i}{n}\right)\right|\le$$

$$\le\frac{1}{n^3}\sum_{k=2}^{n}\left(\sin\left(2+\frac{3k}{n}\right)\right) \quad (6)$$

$$\left.\begin{array}{r}\lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{n}\sin\left(2+\frac{3k}{n}\right)=\lim_{n\to\infty}\frac{1}{n^2}\left(\frac{1}{n}\sum_{k=1}^{n}\sin\left(2+\frac{3k}{n}\right)\right)\\ \text{but } \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sin\left(2+\frac{3k}{n}\right)=\int_0^1\sin(2+3x)\,dx\end{array}\right\}\Rightarrow$$

$$\lim_{n\to\infty}\frac{1}{n^3}\sum_{k=2}^{n}\left|\sin\left(2+\frac{3k}{n}\right)\right|=0 \quad (7)$$

From (6) + (7) ⇒ (5) its true. Similarly,

$$\lim_{n\to\infty}\frac{1}{n^3}\sum_{1\le i<j\le n}\sin\left(2+\frac{3i}{n}\right)\cdot\sin^2\left(3+\frac{3i}{n}\right)=0 \quad (8)$$

From (2) + (3) + (4) + (5) + (8) $\Rightarrow \Omega=\frac{1}{6}\left(\frac{\cos 2-\cos 5}{3}\right)^3$

385. Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{2^{2n}(n+1)}\sum_{k=0}^{n-1}(n-k)\binom{2n+1}{2n-2k}$$

Proposed by Daniel Sitaru – Romania

Solution by Marian Ursărescu – Romania

$$\sum_{k=0}^{n-1}(n-k)C_{2n+1}^{2n-2k}=\sum_{k=0}^{n-1}(n-k)\frac{(2n+1)!}{(2n-2k)!\,(2k+1)!}=$$

$$=\sum_{k=0}^{n-1}(n-k)\cdot\frac{(2n+1)(2n)!}{(2n-2k)(2n-2k-1)!\,(2k+1)!}=$$

$$=\frac{2n+1}{2}\sum_{k=0}^{n-1}C_{2n}^{2k+1}=\frac{2n+1}{2}\left(C_{2n}^{1}+C_{2n}^{3}+\dots+C_{2n}^{2n-1}\right)=$$

$$=\frac{2n+1}{2}\cdot 2^{2n-1}=(2n+1)\cdot 2^{2n-2} \quad (1)$$

From (1) $\Rightarrow \Omega=\lim_{n\to\infty}\frac{(2n+1)\cdot 2^{2n-2}}{2^{2n}(n+1)}=2\cdot 2^{-2}=\frac{1}{2}$

386. Find:

$$\Omega=\lim_{n\to\infty}\frac{(n-1)\cdot\frac{1}{n}+(n-2)\cdot\left(\frac{1}{n}+\frac{1}{n-1}\right)+\dots+1\cdot\left(\frac{1}{n}+\frac{1}{n-1}+\dots+\frac{1}{2}\right)}{(n+1)^3-n^3}$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

$$Numerator=(n-1)\frac{1}{n}+(n-2)\left(\frac{1}{n}+\frac{1}{n-1}\right)+\dots+\left(\frac{1}{n}+\frac{1}{n-1}+\dots+\frac{1}{2}\right)=$$

$$=\frac{1}{n}[(n-1)+(n-2)+\cdots+1]+\frac{1}{n-1}[(n-2)+(n-3)+\cdots+1]+$$

$$+\frac{1}{n-2}[(n-3)+\cdots+1]+\cdots+\frac{1}{2}\ \textit{(1)}$$

$$=\frac{1}{n}\cdot\frac{n(n-1)}{2}+\frac{1}{n-1}\cdot\frac{(n-1)(n-2)}{2}+\frac{1}{n-2}\cdot\frac{(n-2)(n-3)}{2}+\cdots+\frac{1}{3}\cdot\frac{3\times 2}{2}+\frac{1}{2}$$

$$=\frac{1}{2}(n-1)+\frac{1}{2}(n-2)+\frac{1}{2}(n-3)+\cdots+\frac{1}{2}(2)+\frac{1}{2}=$$

$$=\frac{1}{2}[(n-1)+(n-2)+\cdots+1]=\frac{n(n-1)}{4}.\ \textit{Also, } (n+1)^3-n^3=3n^2+3n+1$$

$$\therefore\ \Omega=\lim_{n\to\infty}\frac{\frac{1}{4}n(n-1)}{3n^2+3n+1}=\lim_{n\to\infty}\frac{1}{4}\left[\frac{\left(1-\frac{1}{n}\right)}{3+3\left(\frac{1}{n}\right)+\frac{1}{n^2}}\right]=\frac{1}{4}\cdot\frac{1-0}{3+0+0}=\frac{1}{12}$$

387. Find:

$$\Omega=\lim_{n\to\infty}\left((n+1)\cdot\sqrt[5n+5]{(5n+5)\cos\frac{\pi}{n+1}}-n\cdot\sqrt[5n]{5n\cos\frac{\pi}{n}}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Marian Ursărescu – Romania

$$\textit{Let } f:[n,n+1]\to\mathbb{R},\ f(x)=x\left(5x\cos\frac{\pi}{x}\right)^{\frac{1}{5x}}.$$

$$\textit{From Lagrange theorem} \Rightarrow \exists c\in(n,n+1)\ \textit{then: } f(n+1)-f(n)=f'(c)\Rightarrow$$

$$(n+1)^{5n+5}\sqrt{(5n+5)\cos\frac{\pi}{a+1}}-n^{5n}\sqrt{5n\cos\frac{\pi}{4}}=f'(c)\Rightarrow$$

$$\lim_{n\to\infty}\left[(n+1)^{5n+5}\sqrt{(5n+5)\cos\frac{\pi}{n+1}}-n^{5n}\sqrt{5n\cos\frac{\pi}{4}}\right]=\lim_{n\to\infty}f'(c)\quad \textit{(1)}$$

$$\textit{Because } c\in(n,n+1)\ \textit{and } n\to\infty\ \textit{we can assume, WLOG,}$$

$$\lim_{n\to\infty}f'(c)=\lim_{n\to\infty}f'(n)\quad \textit{(2)}$$

$$f'(x)=\left(5x\cos\frac{\pi}{x}\right)^{\frac{1}{5x}}+x\left(5x\cos\frac{\pi}{x}\right)^{\frac{1}{5x}}\left[-\frac{1}{5x^2}\ln\left(5x\cos\frac{\pi}{x}\right)+\frac{1}{5x}\cdot\frac{5\cos\frac{\pi}{x}+5x\sin\frac{\pi}{x}\cdot\frac{\pi}{x^2}}{5x\cos\frac{\pi}{x}}\right]$$

$$\Rightarrow f'(x)=\left(5x\cos\frac{\pi}{x}\right)^{\frac{1}{5x}}\left[1-\frac{\ln\left(5x\cos\frac{\pi}{x}\right)}{5x}+\frac{1}{5}\cdot\frac{5\cos\frac{\pi}{x}+5\sin\frac{\pi}{x}\cdot\frac{\pi}{x}}{5x\cos\frac{\pi}{x}}\right]\quad (3)$$

From (1)+(2)+(3) $\Rightarrow$

$$\Omega=\lim_{n\to\infty}\left(5n\cos\frac{\pi}{n}\right)^{\frac{1}{5n}}\left[1-\frac{\ln\left(5n\cos\frac{\pi}{5n}\right)}{5n}+\frac{1}{5}\cdot\frac{5\cos\frac{\pi}{n}+5\sin\frac{\pi}{n}\cdot\frac{\pi}{n}}{5n\cos\frac{\pi}{n}}\right]\quad (4)$$

$$\lim_{n\to\infty}\left(5n\cos\frac{\pi}{n}\right)^{\frac{1}{5n}}=\sqrt[5]{\lim_{n\to\infty}\sqrt[n]{5n\cos\frac{\pi}{n}}}=\sqrt[5]{\lim_{n\to\infty}\frac{(5n+5)\cos\frac{\pi}{n+1}}{5n\cos\frac{\pi}{n}}}=1\quad (5)$$

$$\lim_{n\to\infty}\frac{\ln\left(5n\cos\frac{\pi}{5n}\right)}{5n}=\lim_{n\to\infty}\frac{\ln\left((5n+5)\cos\frac{\pi}{5n+5}\right)-\ln\left(5n\cos\frac{\pi}{5n}\right)}{5n+5-5n}=$$

$$=\lim_{n\to\infty}\ln\left(\frac{(5n+5)\cos\frac{\pi}{n+1}}{5n\cos\frac{\pi}{n}}\right)=\ln 1=0\quad (6)$$

$$\lim_{n\to\infty}\frac{5\cos\frac{\pi}{n}+5\cos\frac{\pi}{n}\cdot\frac{\pi}{n}}{5n\cos\frac{\pi}{n}}=0\quad (7)$$

From (4)+(5)+(6)+(7)$\Rightarrow$ $\Omega=1.$

388.

$$\Omega(x)=\frac{1}{x^{n+2}}\left(\prod_{k=1}^{n}\tan^{-1}(nx)-\prod_{k=1}^{n}\sin(nx)\right),n\in\mathbb{N}^*$$

Find:

$$\Omega=\lim_{x\to 0}\Omega(x)$$

Proposed by Marian Ursărescu – Romania

Solution by Ravi Prakash-New Delhi-India

For $x\neq 0, |x|<\frac{1}{n}, \tan^{-1}x=x-\frac{1}{3}x^3+\frac{1}{5}x^5-\cdots; \sin x=x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\cdots$

$\therefore$ *For* $x\neq 0, |x|<\frac{1}{n}, 1\le k\le n; \tan^{-1}(kx)=kx-\frac{1}{3}k^3x^3+\frac{1}{5}k^5x^5-\cdots$

$$\sin(kx)=kx-\frac{1}{6}k^3x^3+\frac{1}{120}k^5x^5-\cdots$$

$$\Rightarrow \prod_{k=1}^{n} \tan^{-1}(kx) = \prod_{k=1}^{n} \left[kx - \frac{1}{3}k^3x^3 + \frac{1}{5}k^5x^5 \dots\right]$$

$$= n!\, x^n \prod_{k=1}^{n} \left[1 - \frac{1}{3}k^2x^2 + \frac{1}{5}k^4x^4 - \cdots\right] = n!\, x^n \left[1 - \frac{1}{3}x^2 \sum_{k=1}^{n} k^2 + 0(x^4)\right] =$$

$$= n!\, x^m \left[1 - \frac{1}{3}x^2 \cdot \frac{n(n+1)(2n+1)}{6} + 0(x^4)\right] =$$

$$= n!\, x^n \left[1 - \tfrac{1}{18}x^2 n(n+1)(2n+1) + 0(x^4)\right].$$ *Similarly,*

$$\prod_{k=1}^{n} \sin(kx) = n!\, x^n \left[1 - \frac{1}{36}x^2 n(n+1)(2n+1) + 0(x^4)\right]$$

Thus,

$$\prod_{k=1}^{n} \tan^{-1}(kx) - \prod_{k=1}^{n} \sin(kx) = n!\, x^n \left[-\frac{1}{36}x^2 n(n+1)(2n+1) + 0(x^4)\right]$$

$\therefore$ *For* $x \neq 0, |x| < \frac{1}{n}; \frac{1}{x^{n+2}}\left[\prod_{k=1}^{n} \tan^{-1}(kx) - \prod_{k=1}^{n} \sin(kx)\right] =$

$= -\frac{1}{36} n!\, n(n+1)(2n+1) + 0(x^2)$. *Taking limit as* $x \to 0^+$, *we get*

$$\Omega = -\frac{1}{36}(n+1)!\,(2n^2+n)$$

389.

$$\Omega_n = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} x^{n-1} \sin^n dx + \int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} x^{n-1} \arcsin^n x\, dx; n \in \mathbb{N}, n \geq 1$$

Find:

$$\lim_{n\to\infty} (n^2 \Omega_n)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Amit Dutta-Jamshedpur-India

$$\Omega_n = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} x^{n-1}\sin^n x\,dx + \int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} x^{n-1}\arcsin^n x\,dx, n \in \mathbb{N}; n \geq 1$$

$$\textit{Find: } \lim_{n\to\infty}(n^2\Omega_n)$$

$$\Omega_n = I_1 + I_2 \textit{ where } I_1 = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} x^{n-1}\sin^n x\,dx, I = \int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} x^{n-1}\arcsin(\sin^n x)\,dx$$

$$I_2 = \int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} x^{n-1}(\sin^{-1} x)^n dx$$

$$\textit{Put } \sin^{-1} x = t \Rightarrow x = \sin t\,; dx = (\cos t)dt$$

$$I_2 = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\sin t)^{n-1}\,t^n(\cos t)dt; I_2 = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\sin x)^{n-1}\,x^n(\cos x)dx$$

$$\Omega_n = I_1 + I_2$$

$$\Omega_n = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \{x^{n-1}\sin^n x + (\sin x)^{n-1}x^n\cos x\}\,dx$$

$$\Omega_n = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} x^{n-1}\sin^{n-1} x\,\{x\cos x + \sin x\}dx$$

$$\textit{Put } x\sin x = u; (x\cos x + \sin x)dx = du$$

$$\Omega_n = \int_{\frac{\pi}{12}}^{\frac{\pi}{4\sqrt{2}}} u^{n-1}\,du = \left[\frac{u^n}{n}\right]_{\frac{\pi}{12}}^{\frac{\pi}{4\sqrt{2}}}$$

$$\Omega_n = \frac{1}{n}\left\{\left(\frac{\pi}{4\sqrt{2}}\right)^n - \left(\frac{\pi}{12}\right)^n\right\}$$

$$\lim_{n\to\infty}(n^2\Omega_n)=\lim_{n\to\infty}n\left\{\left(\frac{\pi}{4\sqrt{2}}\right)^n-\left(\frac{\pi}{12}\right)^n\right\}=\lim_{n\to\infty}\left\{\frac{n}{\left(\frac{4\sqrt{2}}{\pi}\right)^n}-\frac{n}{\left(\frac{12}{\pi}\right)^n}\right\}=$$

$$=\lim_{n\to\infty}\left\{\frac{n}{a^n}-n/b^n\right\},a,b>1;a=\frac{4\sqrt{2}}{\pi},b=\frac{12}{\pi}$$

$$=\lim_{n\to\infty}\left[\left(\frac{n^{\frac{1}{n}}}{a}\right)^n-\left(\frac{n^{\frac{1}{n}}}{b}\right)^n\right]$$

$$\textit{Let } l=\lim_{n\to\infty}n^{\frac{1}{n}}$$

$$\ln l=\lim_{n\to\infty}\frac{\ln(n)}{n}=\lim_{n\to\infty}\left(\frac{1}{n}\right)=0\Rightarrow l=1\Rightarrow\lim_{n\to\infty}\left(\frac{n^{\frac{1}{n}}}{a}\right)^n=\lim_{n\to\infty}\left(\frac{1}{a}\right)^n=0;a>1$$

$$\textit{Similarly, }\lim_{n\to\infty}\left(\frac{n^{\frac{1}{n}}}{b}\right)^n=\lim_{n\to\infty}\left(\frac{1}{b}\right)^n=0\,,b>1$$

$$\therefore\lim_{n\to\infty}(n^2\Omega_n)=\lim_{n\to\infty}\left\{\left(\frac{n^{\frac{1}{n}}}{a}\right)^n-\left(\frac{n^{\frac{1}{n}}}{b}\right)^n\right\}$$

$$\lim_{n\to\infty}(n^2\Omega_n)=0$$

Solution 2 by Marian Ursarescu-Romania

$$\int_1^{\frac{\sqrt{2}}{2}}x^{n-1}\arcsin^n x\,dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\sin^{n-1}t\cdot t^n\cdot\cos t\,dt=\arcsin x=t\Rightarrow x=\sin t\Rightarrow$$

$$\Rightarrow dx=\cos t\,dt;x=\frac{1}{2}\Rightarrow t=\frac{\pi}{6}\wedge x=\frac{\sqrt{2}}{2}\Rightarrow t=\frac{\pi}{4}$$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\left(\frac{\sin^n t}{n}\right)'t^n dt=\frac{1}{n}\cdot\sin^n t\cdot t^n\Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{4}}-\int_{\frac{\pi}{6}}^{\frac{\pi}{6}}\frac{\sin^n t}{n}\cdot nt^{n-1}dt$$

$$=\frac{1}{n}\left(\left(\frac{\sqrt{2}}{2}\cdot\frac{\pi}{4}\right)^n-\left(\frac{1}{2}\cdot\frac{\pi}{6}\right)^n\right)-\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\sin^n t\cdot t^{n-1}dt\quad (1)$$

$$\textit{From (1)} \Rightarrow \Omega_n = \frac{1}{n}\left[\left(\frac{\sqrt{2\pi}}{8}\right)^n - \left(\frac{\pi}{12}\right)^n\right] \Rightarrow \lim_{n\to\infty} n^2\,\Omega_n = \lim_{n\to\infty} n\left[\left(\frac{\sqrt{2}n}{8}\right)^n - \left(\frac{\pi}{12}\right)^n\right] = 0$$

$$\textit{because} \lim_{n\to\infty} n^\alpha a^n = 0, \alpha > 1, n \in \mathbb{N}^*; a \in (-1,1)$$

390. Find:

$$\Omega = \lim_{n\to\infty}\frac{1}{n}\left(\tan^{-1} n + \frac{1}{2}\tan^{-1}(n-1) + \frac{1}{3}\tan^{-1}(n-2) + \cdots + \frac{1}{n}\tan^{-1} 1\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Marian Ursărescu – Romania

$$\textit{Let } a_n = \frac{1}{n}\left(\arctan n + \frac{1}{2}\arctan(n-1) + \cdots + \frac{1}{n}\arctan 1\right)$$

$$|a_n| = \frac{1}{n}\left|\arctan n + \frac{1}{2}\arctan(n-1) + \cdots + \frac{1}{n}\arctan 1\right| \le$$

$$\le \frac{1}{n}|\arctan n| + \frac{1}{2}|\arctan(n-1)| + \cdots + \frac{1}{n}|\arctan 1| \le \frac{\pi}{2}\frac{\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)}{2} \quad \textit{(1)}$$

$$\textit{But} \lim_{n\to\infty}\frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{n} \overset{C.S}{=} \lim_{n\to\infty}\frac{1}{n+1} = 0 \quad \textit{(2)}$$

$$\textit{From (1)} + \textit{(2)} \Rightarrow \Omega = 0.$$

391. Find:

$$\Omega = \lim_{n\to\infty}\prod_{k=1}^{n}\left(1 + \frac{k^4}{n^5}\right)^{1+\frac{k^4}{n^5}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\textit{Let } a_n = \prod_{k=1}^{n}\left(1 + \frac{k^4}{n^5}\right)^{1+\frac{k^4}{n^5}}$$

$$\log a_n = \sum_{k=1}^{n}\left(1+\frac{k^4}{n^5}\right)\log\left(1+\frac{k^4}{n^5}\right) = \sum_{k=1}^{n}\left(1+\frac{k^4}{n^5}\right)\left[\frac{k^4}{n^5} - \left(\frac{k^4}{n^5}\right)^2\frac{1}{2} + \cdots\right] =$$

$$=\sum_{k=1}^{n}\frac{k^4}{n^5}+\frac{1}{2}\sum_{k=1}^{n}\frac{k^8}{n^{10}}+\cdots$$

We know $\lim_{n\to\infty}\frac{1}{n^5}\sum_{k=1}^{n}k^4=\frac{1}{5}$ *and if* $r\geq 2$, *then*

$$\lim_{n\to\infty}\frac{1}{n^{5r}}\sum_{k=1}^{n}(k^4)^r=\lim_{n\to\infty}\frac{1}{n^{r-1}}\cdot\sum_{k=1}^{n}\frac{k^{4r}}{n^{4r+1}}=(0)\left(\frac{1}{4r+1}\right)=0$$

Thus, $\lim_{n\to\infty}\log a_n=\frac{1}{5}+0=\frac{1}{5}\Rightarrow\lim_{n\to\infty}a_n=e^{\frac{1}{5}}$

Solution 2 by Marian Ursarescu-Romania

Let $a_n=\prod_k^n\left(1+\frac{k^4}{n^5}\right)^{1+\frac{k^4}{n^5}}\Rightarrow\ln a_n=\sum_{k=1}^{n}\left(1+\frac{k^4}{n^5}\right)\ln\left(1+\frac{k^4}{n^5}\right)$. *Now we show:*

$$x\leq(1+x)\ln(1+x)\leq x+\frac{x^2}{2},\forall x\geq 0\quad (1)$$

$$f(x)=(1+x)\ln(1+x)-x-\frac{x^2}{2};f:[0,+\infty]\to\mathbb{R};f'(x)=\ln(1+x)-x,$$

$f''(x)=\frac{-x}{1+x}\leq 0,f'(0)=0,f(0)=0\Rightarrow f(x)\leq 0,\forall x\geq 0$. *Similarly for left side. From*

$$(1)\Rightarrow\frac{k^4}{n^5}\leq\left(1+\frac{k^4}{n^5}\right)\ln\left(1+\frac{k^4}{n^5}\right)\leq\frac{k^4}{n^5}+\frac{k^8}{2n^{10}}\Rightarrow$$

$$\sum_{k=1}^{n}\frac{k^4}{n^5}\leq\ln a_n\leq\sum_{k=1}^{n}\frac{k^4}{n^5}+\frac{1}{2}\sum_{k=1}^{n}\frac{k^8}{n^{10}}\quad (2)$$

$$\text{But }\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^4}{n^5}=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^4=\int_0^1 x^4dx=\frac{x^5}{5}\bigg|_0^1=\frac{1}{5}\quad (3)$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^8}{n^{10}}\overset{C.S.}{=}\lim_{n\to\infty}\frac{(n+1)^8}{(n+1)^{10}-n^{10}}=\lim_{n\to\infty}\frac{(n+1)^8}{n^9+\cdots+1}=0\quad (4)$$

From (2)+(3)+(4)$\Rightarrow\lim_{n\to\infty}\ln a_n=\frac{1}{5}\Rightarrow\Omega=\lim_{n\to\infty}a_n=e^{\frac{1}{5}}$

392. Find:

$$\Omega=\lim_{n\to\infty}\sqrt[n]{\frac{1}{n+1}+\sum_{k=0}^{n}\left(\frac{1}{k+1}\binom{n}{k}\sum_{i=1}^{k}\binom{i}{k}\right)}$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\sum_{i=i}^{k}\binom{i}{k}=\binom{1}{k}+\binom{2}{k}+\cdots+\binom{k-1}{k}+\binom{k}{k}=1\ \textit{[rest are zeros]}$$

$$\therefore\sum_{k=0}^{k}\frac{1}{k+1}\binom{n}{k}\left(\sum_{i=1}^{k}\binom{i}{k}\right)=\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}=\frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1}=$$

$$=\frac{1}{n+1}(2^{n+1}-1)\Rightarrow\frac{1}{n+1}+\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}\left(\sum_{i=1}^{k}\binom{i}{k}\right)=2^{n+1}\Rightarrow$$

$$\Rightarrow\Omega=\lim_{n\to\infty}\left[\frac{1}{n+1}+\sum_{k=0}^{n}\frac{1}{k+1}\left(\sum_{i=1}^{k}\binom{i}{k}\binom{n}{k}\right)\right]^{\frac{1}{n}}=\lim_{n\to\infty}2^{1+\frac{1}{n}}=2$$

Solution 2 by Sagar Kumar-Kolkata-India

$$as\ n_{C_\gamma}=\begin{cases}0 & for\ n<\gamma\\ 1 & for\ n=\gamma\end{cases}\Rightarrow\sum_{i=1}^{k}i_{C_k}=1\Rightarrow$$

$$\Rightarrow\Omega=\lim_{n\to\infty}\left(\frac{1}{n+1}+\sum_{k=0}^{n}\left(\frac{1}{k+1}n_{C_K}\sum_{i=1}^{n}i_{C_k}\right)\right)^{\frac{1}{n}}$$

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{n+1}+\sum_{k=0}^{n}\frac{n_{C_k}}{k+1}\right)^{\frac{1}{n}}\quad (1)$$

We know that $(1+x)^n=\sum_{k=0}^{n}n_{C_k}(x)^k$

Integrating both sides from 0 *to* 1

$$\frac{2^{n+1}-1}{n+1}=\sum_{k=0}^{n}\frac{n_{C_k}}{k+1}\quad (2)$$

from (2)

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{2^{n+1}-1}{n+1}\right)^{\frac{1}{n}};\ \Omega=\lim_{n\to\infty}\left(\frac{2^{n+1}}{n+1}\right)^{\frac{1}{n}};\ \Omega=\lim_{n\to\infty}2^{\frac{n+1}{n}}\left(\frac{1}{n+1}\right)^{\frac{1}{n}}$$

$$\lim_{n\to\infty}\left(\frac{1}{n+1}\right)^{\frac{1}{n}}=S;\ \log(S)=\lim_{n\to\infty}\frac{-\log(n+1)}{n}$$

$$\log(S)=0;\ S=1;\ \Omega=\lim_{n\to\infty}2^{1+\frac{1}{n}}\left(\frac{1}{n+1}\right)^{\frac{1}{n}}=2$$

393. Find:

$$\Omega = \lim_{n\to\infty}\left(n!\cdot\prod_{k=1}^{n}(\tan^{-1}(k+1)-\tan^{-1}k)\right)$$

Proposed by Marian Ursărescu – Romania

Solution by Ravi Prakash-New Delhi-India

$$For\ 1\le k\le n; \tan^{-1}(k+1)-\tan^{-1}(k)=\tan^{-1}\left(\frac{k+1-k}{1+(k+1)k}\right)=$$

$$=\tan^{-1}\left(\frac{1}{1+k+k^2}\right)<\frac{1}{1+k+k^2}<\frac{1}{k^2}$$

$$\therefore \prod_{k=1}^{n}[\tan^{-1}(k-1)-\tan^{-1}(k)]<\prod_{k=1}^{n}\frac{1}{k^2}=\left(\frac{1}{n!}\right)^2\Rightarrow$$

$$\Rightarrow 0<n!\prod_{k}^{n}[\tan^{-1}(k+1)-\tan^{-1}(k)]<\frac{1}{n!}$$

As $\lim_{n\to\infty}\frac{1}{n!}=0$*, by the sandwich theorem, we get:*

$$\lim_{n\to\infty}\left[n!\prod_{k=1}^{n}(\tan^{-1}(k+1)-\tan^{-1}(k))\right]=0$$

394. Find:

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(k+n)^5}{7+\tan^{-1}(k+n)+(k+n)^6}$$

Proposed by Daniel Sitaru – Romania

Solution by Marian Ursărescu – Romania

Let $a_n=\sum_{k=1}^{n}\frac{(k+n)^5}{7+\arctan(k+n)+(k+n)^6}$. *Because* $-\frac{\pi}{2}<\arctan x<\frac{\pi}{2},\forall x\in\mathbb{R}\Rightarrow$

$$\Rightarrow\sum_{k=1}^{n}\frac{(k+n)^5}{7+\frac{\pi}{2}+(k+n)^6}<a_n<\sum_{k=1}^{n}\frac{(k+n)^5}{7-\frac{\pi}{2}+(k+n)^6}\quad (1)$$

Now we want to show this: $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(k+n)^5}{\alpha+(k+n)^6}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(k+n)}$ *(2),* $\forall\alpha>0$

$$(2) \Leftrightarrow \lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{(k+n)^5}{\alpha+(k+n)^6}-\frac{1}{(k+n)}\right]=0 \Leftrightarrow \lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{(k+n)^6-\alpha-(k+n)^6}{(\alpha+(k+n)^6)(k+n)}\right]=0 \Leftrightarrow$$

$$\Leftrightarrow \lim_{n\to\infty}-\alpha\sum_{k=1}^{n}\frac{1}{(\alpha+(k+1)^6)(k+n)}=0, \textit{ which, obvious its true. But}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k+n}=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}}=\int_0^1\frac{1}{1+x}dx=\ln(1+x)\Big|_0^1=\ln 2 \quad (3)$$

From (1)+(2)+(3) $\Rightarrow \Omega=\ln 2$.

395. Find:

$$\Omega_n=\lim_{x\to 0}\frac{\ln(1+\sinh^n x)-\ln^n(1+\sinh x)}{x^{n+1}}, n\in\mathbb{N}, n\geq 2$$

Proposed by Marian Ursărescu – Romania

Solution by Ravi Prakash-New Delhi-India

$$\textit{For } |x|<1, \ln(1+x)=x-\frac{x^2}{2}+\cdots. \textit{ Also,}$$

$$\sinh x=\frac{1}{2}(e^x-e^{-x}) \Rightarrow \ln(1+(\sinh x)^n)=\left(\frac{1}{2}(e^x-e^{-x})\right)^n-\frac{1}{2}\left(\frac{1}{2}(e^x-e^{-x})\right)^{2n}+\cdots$$

$$=\left(\frac{1}{2}(e^x-e^{-x})\right)^n\left[1-\frac{1}{2}\left(\frac{1}{2}(e^x-e^{-x})\right)^n+\cdots\right] \textit{ and } \ln(1+\sinh x)=\frac{1}{2}(e^x-e^{-x})-$$

$$-\frac{1}{2}\left[\frac{1}{2}(e^x-e^{-x})\right]^2+\cdots=\left(\frac{1}{2}(e^x-e^{-x})\right)\left[1-\frac{1}{2}\left(\frac{1}{2}(e^x-e^{-x})\right)+\cdots\right]\Rightarrow$$

$$\Rightarrow (\ln(1+\sinh x))^n=\left(\frac{1}{2}(e^x-e^{-x})\right)^n\left[1-\frac{1}{2}\left(\frac{1}{2}(e^x-e^{-x})\right)+\cdots\right]^n=$$

$$=\frac{1}{2^n}(e^x-e^{-x})^n\left[1-\frac{n}{4}(e^x-e^{-x})+\cdots\right]$$

For, sufficiently small x

$$\therefore \ln(1+(\sinh x)^n)-(\ln(1+\sinh x))^n=-\frac{1}{2^{2n+1}}(e^x-e^{-x})^{2n}+$$

$$+(e^x-e^{-x})^{3n} \textit{ and higher power} +\frac{n}{2^{n+2}}(e^x-e^{-x})^{n+1}+(e^x-e^{-x})^{n+2} \textit{ and higher}$$

$$\textit{power} \Rightarrow \therefore \textit{ for } x\neq 0, x \textit{ sufficiently small, } \frac{\ln(1+(\sinh x)^n)-(\ln(1+\sinh x))^n}{x^{n+1}}=$$

$$= -\frac{1}{2^{2n+1}}\left(\frac{e^x-e^{-x}}{x}\right)^{2n} x^{n-1} + \frac{n}{2^{n+2}}\left(\frac{e^x-e^{-x}}{x}\right)^{n+1} + \left(\frac{e^x-e^{-x}}{x}\right)^{n+2} x \text{ and similar expressions.}$$

$$\text{But } \lim_{x\to 0}\frac{e^x-e^{-x}}{x} = \lim_{x\to 0}\left[\frac{e^x-1}{x} + \frac{e^{-x}-1}{(-x)}\right] = 1+1 = 2$$

$$\therefore \lim_{x\to 0}\frac{\ln[1+(\sinh x)^n]-(\ln(1+\sinh x))^n}{x^{n+1}} = 0 + \frac{n}{2^{n+2}}\cdot 2^{n+1} = \frac{n}{2} \text{ if } n > 1.$$

396. Find:

$$\Omega = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sin\left(\frac{k}{n}\right)}{\sqrt[7]{k+n^7}}$$

Proposed by Daniel Sitaru – Romania

Solution by Marian Ursărescu – Romania

$$\frac{1}{\sqrt[7]{n^7+n}} \le \frac{1}{\sqrt[7]{n^7+k}} \le \frac{1}{\sqrt[7]{n^7+1}}, \forall k = \overline{1,n} \Rightarrow \frac{\sin\left(\frac{k}{n}\right)}{\sqrt[7]{n^7+n}} \le \frac{\sin\left(\frac{k}{n}\right)}{\sqrt[7]{n^7+k}} \le \frac{\sin\left(\frac{k}{n}\right)}{\sqrt[7]{n^7+1}} \Rightarrow$$

$$\Rightarrow \sum_{k=1}^{n}\frac{\sin\left(\frac{k}{n}\right)}{\sqrt[7]{n^7+n}} \le \sum_{k=1}^{n}\frac{\sin\left(\frac{k}{n}\right)}{\sqrt[7]{n^7+k}} \le \sum_{k=1}^{n}\frac{\sin\left(\frac{k}{n}\right)}{\sqrt[7]{n^7+1}} \quad (1)$$

$$\text{But } \lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sin\frac{k}{n}}{\sqrt[7]{n^7+n}} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{\sqrt[7]{n^7+n}}\cdot\frac{1}{n}\cdot\sin\frac{k}{n} = \lim_{n\to\infty}\frac{n}{\sqrt[7]{n^7+n}}\cdot\frac{1}{n}\sum_{k=1}^{n}\sin\left(\frac{k}{n}\right) =$$

$$= \int_0^1 \sin x\,dx = -\cos\Big|_0^1 = 1-\cos 1 \quad (2)$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sin\left(\frac{k}{n}\right)}{\sqrt[7]{n^7+1}} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{\sqrt[7]{n^7+1}}\cdot\frac{1}{n}\cdot\sin\frac{k}{n} = \lim_{n\to\infty}\frac{n}{\sqrt[7]{n^7+1}}\cdot\frac{1}{n}\sum_{k=1}^{n}\sin\frac{k}{n} =$$

$$= \int_0^1 \sin x\,dx = 1-\cos 1 \quad (3)$$

$$\text{From } (1)+(2)+(3) \Rightarrow \Omega = 1-\cos 1.$$

397. Find:

$$\Omega = \lim_{n\to\infty}\frac{\sin n}{n}\left(\left(1+\frac{\sqrt[n]{n}}{n}\right)^{\frac{1}{n}} + \left(1-\frac{\sqrt[n]{n}}{n}\right)^{\frac{1}{n}}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$1<\left(1+\frac{n^{\frac{1}{n}}}{n}\right)=1+\frac{1}{n^{1-\frac{1}{n}}}\le 2 \text{ and}\left(1-\frac{n^{\frac{1}{n}}}{n}\right)\le 1$$

$$1<\left(1+\frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}+\left(1-\frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}\le 3$$

$$\text{Also, }\sin n\in[-1,1]\ \therefore\ -3<(\sin n)\left[\left(1+\frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}+\left(1-\frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}\right]<3$$

$$\Rightarrow -\frac{3}{n}<\frac{\sin n}{n}\left[\left(1+\frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}+\left(1-\frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}\right]<\frac{3}{n}$$

$$\text{As }\lim_{n\to\infty}\left(-\frac{3}{n}\right)=\lim_{n\to\infty}\frac{3}{n}=0.\text{ By the Sandwich theorem}$$

$$\lim_{n\to\infty}\frac{\sin n}{n}\left[\left(1+\frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}+\left(1-\frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}\right]=0$$

Solution 2 by Rovsen Pirguliyev-Sumgait-Azerbaidian

$$\text{We prove that: }\left(n+\sqrt[n]{n}\right)^{\frac{1}{n}}+\left(n-\sqrt[n]{n}\right)^{\frac{1}{n}}\le 2\cdot n^{\frac{1}{n}}\quad (*)$$

$$\text{Prove: }\left(n+\sqrt[n]{n}\right)^{\frac{1}{n}}=k,\left(n-\sqrt[n]{n}\right)^{\frac{1}{n}}=m\Rightarrow n+\sqrt[n]{n}=k^n, n-\sqrt[n]{n}=m^n\Rightarrow$$

$$\Rightarrow k^n+m^n=2n\Rightarrow\frac{k^n+m^n}{2}=n\Rightarrow\left(\frac{k^n+m^n}{2}\right)^{\frac{1}{n}}=n^{\frac{1}{n}}$$

$$\text{Then }(*)\Rightarrow\frac{k+m}{2}\le\sqrt[n]{\frac{k^n+m^n}{2}}\text{ the function }y=x^n\text{ is convex (geometric interpretation)}$$

$$\text{by induction: }\left(\frac{k+m}{2}\right)^{n+1}\le\frac{k^n+m^n}{2}\cdot\frac{k+m}{2},$$

$$\frac{k^n+m^n}{2}\cdot\frac{k+m}{2}\le\frac{k^{n+1}+m^{n+1}}{2}\Rightarrow(k^n-m^n)(k-m)>0;\ \Omega\le 2\lim_{n\to\infty}\frac{\sin n}{n}=0$$

Solution 3 by Sagar Kumar-Kolkata-India

$$\Omega=\lim_{n\to\infty}\frac{\sin(n)}{n}\left(\left(1+\frac{(n)^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}+\left(1-\frac{(n)^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}\right)$$

$$AM\geq GM$$

$$\Rightarrow\left(1+\frac{(n)^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}+\left(1-\frac{(n)^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}\geq 2\left(1-\frac{n^{\frac{2}{n}}}{n^2}\right)^{\frac{1}{n}}$$

$$\Rightarrow\lim_{n\to\infty}\left(\left(1+\frac{(n)^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}+\left(1-\frac{(n)^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}\right)\frac{\sin(n)}{n}\geq$$

$$\geq\lim_{n\to\infty}\frac{2\sin n}{n}\left(1-\frac{n^{\frac{2}{n}}}{n^2}\right)^{\frac{1}{n}}\geq 0\quad (1)$$

Also, by using m^{th} *power theorem:* $\left(1+\frac{(n)^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}+\left(1-\frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}\leq(2)^{\frac{1}{n}}$

$$as\ \frac{1}{n}=m\ as\ m\to\infty\ as\ n\to\infty\Rightarrow m\in(0,1)$$

$$\lim_{n\to\infty}\left(\left(1+\frac{(n)^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}+\left(1-\frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}\right)\frac{\sin n}{n}\leq\lim_{n\to\infty}\frac{\sin(n)}{n}2^{\frac{1}{n}}\leq 0\quad (2)$$

From (1) and (2) $\Omega=0$ *(Answer)*

398. Find:

$$S=\sum_{n=0}^{\infty}\frac{1}{16^n(2n+1)^3}\binom{2n}{n}$$

Proposed by Zaharia Burghelea-Romania

Solution by Shafiqur Rahman-Bangladesh

$$\sum_{n=0}^{\infty}\frac{1}{16^n(2n+1)^3}\binom{2n}{n}=\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{16^n}\binom{2n}{n}\left(\int_0^1 x^{2n}\ln^2 x\,dx\right)=$$

$$=\frac{1}{2}\ln^2 x\left[\sum_{n=0}^{\infty}\frac{1}{4^n}\binom{2n}{n}\left(\frac{x^2}{4}\right)^n\right]dx=\frac{1}{2}\int_0^1\frac{\ln^2 x}{\sqrt{1-\frac{x^2}{4}}}dx\left[x\to 2\sin\frac{x}{2}\right]=\frac{1}{2}\int_0^{\frac{\pi}{3}}\ln^2\left(2\sin\frac{x}{2}\right)dx$$

$$\therefore\sum_{n=0}^{\infty}\frac{1}{16^n(2n+1)^3}\binom{2n}{n}=\frac{7\pi^3}{216}$$

399. If $a,b,c,d>0$, are different in pairs, $\Delta=\begin{vmatrix}1 & a & a^3 & a^4\\ 1 & b & b^3 & b^4\\ 1 & c & c^3 & c^4\\ 1 & d & d^3 & d^4\end{vmatrix}$ then:

$$\frac{8\Delta}{(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)}<3(a+b+c+d)^2$$

Proposed by Daniel Sitaru – Romania

Solution by Radu Butelca – Romania

$$\Delta=\begin{vmatrix}0 & a-b & a^3-b^3 & a^4-b^4\\ 0 & b-c & b^3-c^3 & b^4-c^4\\ 0 & c-d & c^3-d^3 & c^4-d^4\\ 1 & d & d^3 & d^4\end{vmatrix}=-\begin{vmatrix}a-b & a^3-b^3 & a^4-b^4\\ b-c & b^3-c^3 & b^4-c^4\\ c-d & c^3-d^3 & c^4-d^4\end{vmatrix}=$$

$$=-\begin{vmatrix}a-b & (a-b)(a^2+ab+b^2) & (a-b)(a^2+b^2)(a+b)\\ b-c & (b-c)(b^2+bc+c^2) & (b-c)(b^2+c^2)(b+c)\\ c-d & (c-d)(c^2+cd+d^2) & (c-d)(c^2+d^2)(c+d)\end{vmatrix}=$$

$$=-(a-b)(b-c)(c-d)\cdot\begin{vmatrix}1 & a^2+ab+b^2 & (a^2+b^2)(a+b)\\ 1 & b^2+bc+c^2 & (b^2+c^2)(b+c)\\ 1 & c^2+cd+d^2 & (c^2+d^2)(c+d)\end{vmatrix}\quad (1)$$

$$\Delta'=\begin{vmatrix}1 & a^2+ab+b^2 & a^3+b^2a+a^2b+b^3\\ 1 & b^2+bc+c^2 & b^3+b^2c+bc^2+c^3\\ 1 & c^2+cd+d^2 & c^3+c^2d+d^2c+d^3\end{vmatrix}=$$

$$=\begin{vmatrix}0 & a^2+ab-bc-c^2 & a^3+b^2a+a^2b-bc-bc^2-c^3\\ 0 & b^2+bc-cd-a^2 & b^3+b^2c+bc^2-c^2d-cd^2+b^3\\ 1 & c^2+cd+d^2 & c^3+c^2d+d^2c+a^3\end{vmatrix}=$$

$$=\begin{vmatrix}a^2+ab-bc-c^2 & a^3+b^2a+a^2b-b^2c-bc^2-c^3\\ b^2+bc-cd-d^2 & b^3+b^2c+bc^2-c^2d-cd^2-d^3\end{vmatrix}\quad \textit{(2)}$$

$$a^3-c^3+b^2a+a^2b-b^2c=(a-c)(a^2+ac+c^2)+b(ab+a^2-bc-c^2)=$$

$$=(a-c)(a^2+ac+c^2)+b[(a-c)(a+c)+b(a-c)]=(a-c)(a^2+ac+c^2)+$$

$$+b(a+b+c)(a-c)=(a-c)(a^2+ac+c^2+ab+b^2+c^2)$$

$$=\begin{vmatrix}(a-c)(a+b+c) & (a-c)(a^2+b^2+c^2+ab+bc+ca)\\ (b-d)(a+c+a) & (b-d)(b^2+c^2+d^2+bc+cd+db)\end{vmatrix}=$$

$$=(a-c)(b-d)\begin{vmatrix}a+b+c & a^2+b^2+c^2+ab+bc+ca\\ b+c+d & b^2+c^2+d^2+bc+cd+db\end{vmatrix}=(a-b)(b-d)\cdot$$

$$\cdot\begin{vmatrix}a-d & a^2+ab+cd-d^2-cd-db\\ b+c+d & b^2+c^2+d^2+bc+cd+db\end{vmatrix}=$$

$$=(a-c)(b-d)\begin{vmatrix}a-d & (a-d)(a+b+c+d)\\ b+c+d & b^2+c^2+d^2+bc+cd+db\end{vmatrix}$$

$$=(a-c)(b-d)(a-d)\begin{vmatrix}1 & a+b+c+d\\ b+c+d & b^2+c^2+d^2+bc+cd+db\end{vmatrix}=$$

$$\left.\begin{matrix}=(a-c)(b-d)(a-d)\left[b^2+c^2+d^2+bc+cd+db-(a+b+c+d)+b+c+a)\right]\\ (1)\end{matrix}\right\}\Rightarrow$$

$$\Rightarrow\Delta=(a-b)(b-c)(a-d)(b-c)(b-d)(c-d)(ab+ac+ad+bc+cd+db)$$

So, we need to prove that $8(ab+ac+ad+bc+cd+db)\le 3\left(\sum a\right)^2$

$$\Leftrightarrow 2(ab+ac+ad+bc+cd+db)\le 3(a^2+b^2+c^2+d^2)\Leftrightarrow$$

$$\left.\begin{matrix}\Leftrightarrow 2(ab+bc+ca)+2(ad+cd+db)\le 2(a^2+b^2+c^2)+a^2+b^2+c^2+3d^2\\ \textit{But } ab+bc+ca\le a^2+b^2+c^2\Rightarrow 2(ab+bc+ca)\le(a^2+b^2+c^2)\end{matrix}\right\}\Rightarrow$$

$\Rightarrow$ *We need to prove that* $2ad+2cd+2db\le a^2+b^2+c^2+3d^2$

$$d^2+a^2\ge 2ad$$

$$b^2+d^2\ge 2bd$$

$$c^2+d^2\ge 2cd$$

$$-------$$

$$2(ad+bd+cd)\le 3d^2+a^2+b^2+c^2$$

(proved)

400. Solve for real positive numbers:

$$\begin{cases} 27\sqrt{\left(x^2+\frac{1}{y^2}\right)\left(y^2+\frac{1}{z^2}\right)\left(z^2+\frac{1}{x^2}\right)} = 8(x+y+z)^3 \\ x+y+z = \frac{1}{xyz} \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$27\sqrt{\left(x^2+\frac{1}{y^2}\right)\left(y^2+\frac{1}{z^2}\right)\left(z^2+\frac{1}{x^2}\right)} \overset{(1)}{=} 8(x+y+z)^3$$

$$x+y+z \overset{(2)}{=} \frac{1}{xyz}; \text{ LHS of (1)} = \frac{27}{xyz}\sqrt{(x^2y^2+1)(y^2z^2+1)(z^2x^2+1)}$$

$$\overset{(a)}{=} \frac{27}{xyz}\sqrt{\{x^2y^2+xyz(x+y+z)\}\{y^2z^2+xyz(x+y+z)\}\{z^2x^2+xyz(x+y+z)\}}$$

$$(\because 1 = xyz(x+y+z))$$

$$\text{Now, } x^2y^2+xyz(x+y+z) = xy(xy+zx+yz+z^2) \overset{(b)}{=} xy(y+z)(z+x)$$

$$\text{Similarly, } y^2z^2+xyz(x+y+z) \overset{(c)}{=} yz(x+y)(z+x) \ \&$$

$$z^2x^2+xyz(x+y+z) \overset{(d)}{=} zx(x+y)(y+z)$$

$$\text{(a), (b), (c), (d)} \Rightarrow \text{LHS} \overset{(i)}{=} 27(x+y)(y+z)(z+x)$$

$$\text{Now, } \sum x = \frac{1}{2}\{(x+y)+(y+z)+(z+x)\} \overset{A-G}{\geq} \frac{3}{2}\sqrt[3]{(x+y)(y+z)(z+x)}$$

$$\Rightarrow \left(2\sum x\right)^3 \geq 27(x+y)(y+z)(z+x) \Rightarrow 8\left(\sum x\right)^3 \overset{(ii)}{\geq} 27(x+y)(y+z)(z+x)$$

(i), (ii) ⇒ RHS of (1) ≥ LHS of (1), with equality occuring when $x=y=z$.

But LHS of (1) = RHS of (1)

$$\therefore x = y = z$$

$$\therefore \text{using (2), } 3x = \frac{1}{x^3} \Rightarrow x^4 = \frac{1}{3} \Rightarrow x = \frac{1}{\sqrt{\sqrt{3}}}$$

$\therefore$ ***only possible solution is:*** $(x, y, z) = \left(\frac{1}{\sqrt{\sqrt{3}}}, \frac{1}{\sqrt{\sqrt{3}}}, \frac{1}{\sqrt{\sqrt{3}}}\right)$ ***(answer)***

Its nice to be important but more important its to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

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