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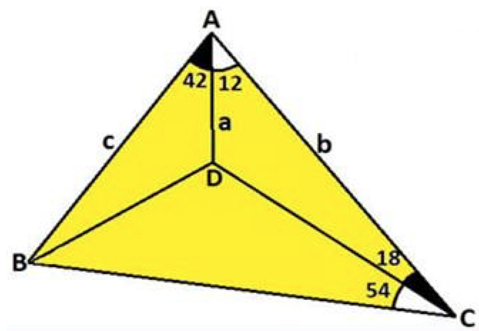
Prove that: $a^2 + b^2 = c^2$



Proposed by Mohamed Ozcelic-Turkey

Solution 1 by Serban George Florin-Romania, Solution 2 by Rovsen Pirguliyev-Sumgait-Azerbaijan, Solution 3 by Seyran Ibrahimov-Maasilli-Azerbaijan, Solution 4 by Soumava Chakraborty-Kolkata-India

Solution 1 by Serban George Florin-Romania

$$\Delta ADC,\ T\sin \frac{a}{\sin 18}=\frac{b}{\sin 150^\circ},\ \frac{a}{\sin 18}=\frac{b}{\frac{1}{2}}=2b \quad (1)$$

$$\Delta ABC,\ T\sin \frac{b}{\sin B}=\frac{c}{\sin C}=\frac{BC}{\sin A},\ \underset{\substack{\Downarrow\\ BC=b}}{\frac{BC}{\sin 54}}=\frac{c}{\sin 72}=\frac{b}{\sin 54} \quad (2)$$

$$a^2+b^2=c^2$$

$$x=18^\circ, 5x=90^\circ, 2x=90^\circ-3x, \sin 2x=\sin(90^\circ-3x), \sin 2x=\cos 3x,$$

$$2\sin x\cos x=4\cos^3 x-3\cos x, \cos x(2\sin x-4\cos^2 x+3)=0, \cos x\neq 0$$

$$2\sin x-4(1-\sin^2 x)+3=0, 4\sin^2 x+2\sin x-1=0$$

$$\sin 18^\circ=\frac{-1+\sqrt{5}}{3}, \cos^2 18^\circ=1-\sin^2 18^\circ=\frac{5+\sqrt{5}}{8}$$

$$\cos 18^\circ=\frac{\sqrt{10+2\sqrt{5}}}{4}, \sin 72^\circ=\sin(90^\circ-18^\circ)=\cos 18^\circ=\frac{\sqrt{10+2\sqrt{5}}}{4}$$

$$\cos 36^\circ=\cos(2\cdot 18^\circ)=1-2\sin^2 18^\circ=1-2\left(\frac{\sqrt{5}-1}{4}\right)^2=\frac{\sqrt{5}+1}{4}$$

$$\sin 54^\circ = \sin(90^\circ - 36^\circ) = \cos 36^\circ = \frac{\sqrt{5}+1}{4}$$

From (1) $\Rightarrow a = 2b\sin 18 \Rightarrow a = \frac{b(-1+\sqrt{5})}{2}, \frac{a}{b} = \frac{-1+\sqrt{5}}{2}$

From (2) $\Rightarrow \frac{c}{\sin 72} = \frac{b}{\sin 54} \Rightarrow \frac{c}{b} = \frac{\sqrt{10+2\sqrt{5}}}{4}\cdot\frac{4}{\sqrt{5}+1} = \frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}+1}$

$$a^2+b^2=c^2 |: b^2, \left(\frac{a}{b}\right)^2+1=\left(\frac{c}{b}\right)^2 \Rightarrow \left(\frac{-1+\sqrt{5}}{2}\right)^2+1=\left(\frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}+1}\right)^2,$$

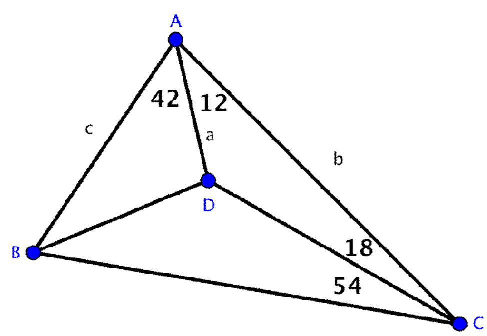
$$\frac{6-2\sqrt{5}}{4}+1=\frac{10+2\sqrt{5}}{(\sqrt{5}+1)^2}, \frac{10-2\sqrt{5}}{4}=\frac{10+2\sqrt{5}}{(\sqrt{5}+1)^2}, (10-2\sqrt{5})(\sqrt{5}+1)^2$$

$$=4(10+2\sqrt{5}) \Rightarrow (10-2\sqrt{5})(6+2\sqrt{5}) = 40+8\sqrt{5}$$

$$60+20\sqrt{5}-12\sqrt{5}-20=40+8\sqrt{5}$$

$$40+8\sqrt{5}=40+8\sqrt{5} \quad \textit{true}$$

Solution 2 by Rovsen Pirguliyev-Sumgait-Azerbaijan



By the sine theorem in $\Delta ADC \Rightarrow \frac{a}{\sin 18^\circ} = \frac{b}{\sin 30^\circ} \Rightarrow a = \frac{b\sin 18^\circ}{\sin 30^\circ} = 2b\sin 18^\circ$ ***(1)***

In $\Delta\, ABC \Rightarrow \frac{c}{\sin 72^\circ} = \frac{b}{\sin 54^\circ} \Rightarrow c = \frac{b\sin 72^\circ}{\sin 54^\circ}$

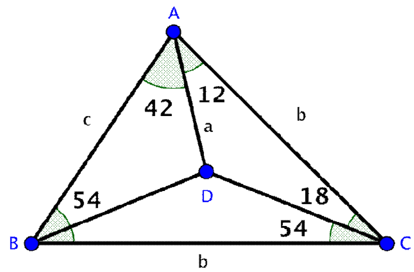
We verify the equality $a^2+b^2=c^2$.

$$4b^2\sin^2 18^\circ + b^2 = \frac{b^2\sin^2 72^\circ}{\cos^2 36^\circ}, 4\sin^2 18^\circ + 1 = 4\sin^2 36^\circ$$

It is known that $\sin 18^\circ = \frac{\sqrt{5}-1}{4}, \sin 36^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4}$,

we have $(\sqrt{5}-1)^2+4 = 10-2\sqrt{5}$. ***Q.E.D.***

Solution 3 by Seyran Ibrahimov-Maasilli-Azerbaijan



$$BC = AC = b$$

$$\frac{B}{\sin 150} = \frac{a}{\sin 18} \Rightarrow a = 2b \cdot \sin 18 = b \cdot \frac{\sqrt{5}-1}{2} \quad \textbf{(1)}$$

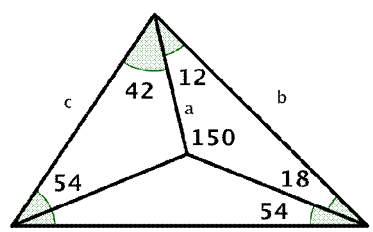
$$\frac{b}{\sin 54} = \frac{c}{\sin 72} \Rightarrow c = \frac{b \sin 72}{\sin 54} = \frac{\sqrt{10+2\sqrt{5}}}{\sqrt{6+2\sqrt{5}}} b \quad \textbf{(2)}$$

$$a^2 + b^2 \overset{(1)}{=} \frac{6 - 2\sqrt{5}}{4} b^2 + b^2 = b^2 \cdot \frac{10 - 2\sqrt{5}}{4} \overset{?}{=} c^2$$

$$c^2 \overset{(2)}{=} b^2 \cdot \frac{10 + 2\sqrt{5}}{6 + 2\sqrt{5}} = b^2 \cdot \frac{10 - 2\sqrt{5}}{4} \quad (=)$$

(proved)

Solution 4 by Soumava Chakraborty-Kolkata-India



$$\frac{a}{\sin 18} = \frac{b}{\sin 150} \Rightarrow b \overset{(1)}{=} \frac{a}{2 \sin 18}$$

$$\frac{c}{\sin 72} = \frac{b}{\sin 54} = \frac{a}{2 \sin 18 \sin 54} \Rightarrow c \overset{(2)}{=} \frac{a \sin 72}{2 \sin 18 \sin 54}$$

$$\textbf{(1), (2)} \Rightarrow c^2 - b^2 = \frac{a^2}{4 \sin^2 18}\left(\frac{\sin^2 72}{\sin^2 54} - 1\right) = \frac{a^2}{4 \sin^2 18}\left(\frac{2 \sin^2 72 - 2 \sin^2 54}{2 \sin^2 54}\right)$$

$$= \frac{a^2}{4 \sin^2 18}\left(\frac{1 - \cos 144 - 1 + \cos 108}{1 - \cos 108}\right) = \frac{a^2}{4 \sin^2 18}\left(\frac{\cos 36 - \cos 72}{1 + \cos 72}\right) =$$

$$= \frac{a^2}{4\sin^2 18}\left(\frac{1-2\sin^2 18-\sin 18}{1+\sin 18}\right) = \frac{a^2(1-2\sin 18)(1+\sin 18)}{4\sin^2 18\,(1+\sin 18)}$$

$$= \frac{a^2(1-2\sin 18)}{4\sin^2 18} = \frac{a^2\left(1-\frac{\sqrt{5}-1}{2}\right)}{4\frac{\left(\sqrt{5}-1\right)^2}{16}} = \frac{a^2\left(3-\sqrt{5}\right)}{2} \times \frac{4}{6-2\sqrt{5}} = a^2$$

$$\Rightarrow a^2 + b^2 = c^2$$ ***(Proved)***