R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

Let $a, b, c > 0$ and $a + b + c = 3$. Prove that:

$$a \cdot arcsin\left(\frac{b}{b+1}\right) + b \cdot arcsin\left(\frac{c}{c+1}\right) + c \cdot arcsin\left(\frac{a}{a+1}\right) \leq \frac{\pi}{2}$$

Proposed by Dimitris Kastriotis-Athens-Greece

Solution 1 by Soumitra Mandal-Chandar Nagore-India, Solution 2 by Chris Kyriazis-Athens-Greece, Solution 3 by Soumava Chakraborty-Kolkata-India

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$a + b + c = 3$ ***then*** $3(ab + bc + ca) \leq (a + b + c)^2 \Rightarrow (a + b + c)(ab + bc + ca) \leq$

$\leq (a + b + c)^2 \Rightarrow ab + bc + ca \leq a + b + c$. ***Let*** $f(x) = \sin^{-1}\left(\frac{x}{x+1}\right)$ ***for all*** $x \in (0, 1)$

then

$$f'(x) = \frac{1}{(1+x)\sqrt{2x+1}} \Rightarrow f''(x) = -\frac{1}{(1+x)^2\sqrt{2x+1}} - \frac{1}{(1+x)(2x+1)^{\frac{3}{2}}} < 0 \text{ for all } x \in (0, 1)$$

Hence f is concave function, then:

$$\sum_{cyc} \frac{a}{a+b+c} \sin^{-1}\left(\frac{b}{b+1}\right) \leq \sin^{-1} \frac{\frac{ab+bc+ca}{a+b+c}}{\frac{ab+bc+ca}{a+b+c} + 1} \overset{AM\geq GM}{\leq} \sin^{-1}\left(\frac{\sqrt{\frac{ab+bc+ca}{a+b+c}}}{2}\right)$$

$$\leq \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} \Rightarrow \sum_{cyc} a \sin^{-1}\left(\frac{b}{b+1}\right) \leq \frac{\pi}{2} \quad \textit{(proved)}$$

Solution 2 by Chris Kyriazis-Athens-Greece

I will use that:

1) Function $f(x)=\arcsin\left(\frac{x}{x+1}\right), x>0$ *is concave (because* $f''(x)=-\frac{3x+2}{(x+2)^2(2x+1)^{\frac{3}{2}}}<0$

2) Function $\arcsin x$ *is strictly increasing when* $0<x<1$, $\left((\arcsin x)'=\frac{1}{\sqrt{1-x^2}}>0\right)$

3) $\frac{a\frac{b}{b+1}+b\frac{c}{c+1}+c\frac{a}{a+1}}{3}\le\frac{1}{2}$, *when* $a,b,c>0, a+b+c=3$

Proof:

$$a\frac{b}{b+1}+b\frac{c}{c+1}+c\frac{a}{a+1}\overset{GM-AM}{\le}a\frac{(b+1)^2}{4(b+1)}+b\frac{(c+1)^2}{4(c+1)}+c\frac{(a+1)^2}{4(a+1)}=$$

$$=\frac{ab+a+bc+b+ca+c}{4}\le\frac{\left(\frac{a+b+c}{3}\right)^2+3}{4}=\frac{6}{4}=\frac{3}{2}$$

So, $\frac{a\frac{b}{b+1}+b\frac{c}{c+1}+c\frac{a}{a+1}}{3}\le\frac{1}{2}$

Now, (using (1)) applying Jensen's inequality with weights a,b,c, *gives then:*

$$LHS\le(a+b+c)\arcsin\left(\frac{a\frac{b}{b+1}+b\frac{c}{c+1}+c\frac{a}{a+1}}{a+b+c}\right)=$$

$$=3\arcsin\frac{\left(a\frac{b}{b+1}+b\frac{c}{c+1}+c\frac{a}{a+1}\right)}{3}\underset{(2)}{\overset{(3)}{\le}}3\arcsin\left(\frac{1}{2}\right)=3\cdot\frac{\pi}{6}=\frac{\pi}{2}$$

because $\arcsin\left(\frac{1}{2}\right)=\frac{\pi}{6}$

Solution 3 by Soumava Chakraborty-Kolkata-India

Given inequality can be written as:

$$\left(\frac{a}{\sum a}\right)\sin^{-1}\left(\frac{b}{b+1}\right)+\left(\frac{b}{\sum a}\right)\sin^{-1}\left(\frac{c}{c+1}\right)+\left(\frac{c}{\sum a}\right)\sin^{-1}\left(\frac{a}{a+1}\right)\overset{(1)}{\le}\frac{\pi}{6}$$

Let $\frac{a}{\sum a}=p_1, \frac{b}{\sum a}=p_2, \frac{c}{\sum a}=p_3$. *Then* $p_1+p_2+p_3=1$. *Now,*

$\because f''(x)=-\frac{(3x+2)}{(x+1)^5\left(\frac{2x+1}{(x+1)^2}\right)^{\frac{3}{2}}}<0, \forall x>0 \because f(x)=\sin^{-1}\left(\frac{x}{x+1}\right), \forall x>0$ *is concave,*

$\therefore$ ***by Jensen, LHS of (1)*** $= p_1 f(b) + p_2 f(c) + p_3 f(a) \overset{(2)}{\leq} f(p_1 b + p_2 c + p_3 a) =$

$$= \sin^{-1}\left(\frac{\frac{ab+bc+bc}{\sum a}}{\frac{\sum ab}{\sum a}+1}\right) = \sin^{-1}\left(\frac{\sum ab}{\sum ab+3}\right) \because 3\left(\sum ab\right) \leq \left(\sum a\right)^2 = 9 \therefore \sum ab \leq 3$$

$$\therefore 1 - \frac{3}{\sum ab + 3} \leq 1 - \frac{3}{3+3} = \frac{1}{2} \Rightarrow \frac{\sum ab}{\sum ab + 3} \overset{(3)}{\leq} \frac{1}{2}$$

(2),(3)⇒ LHS of (1) $\leq \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} =$ ***RHS of (1)***

(proved)