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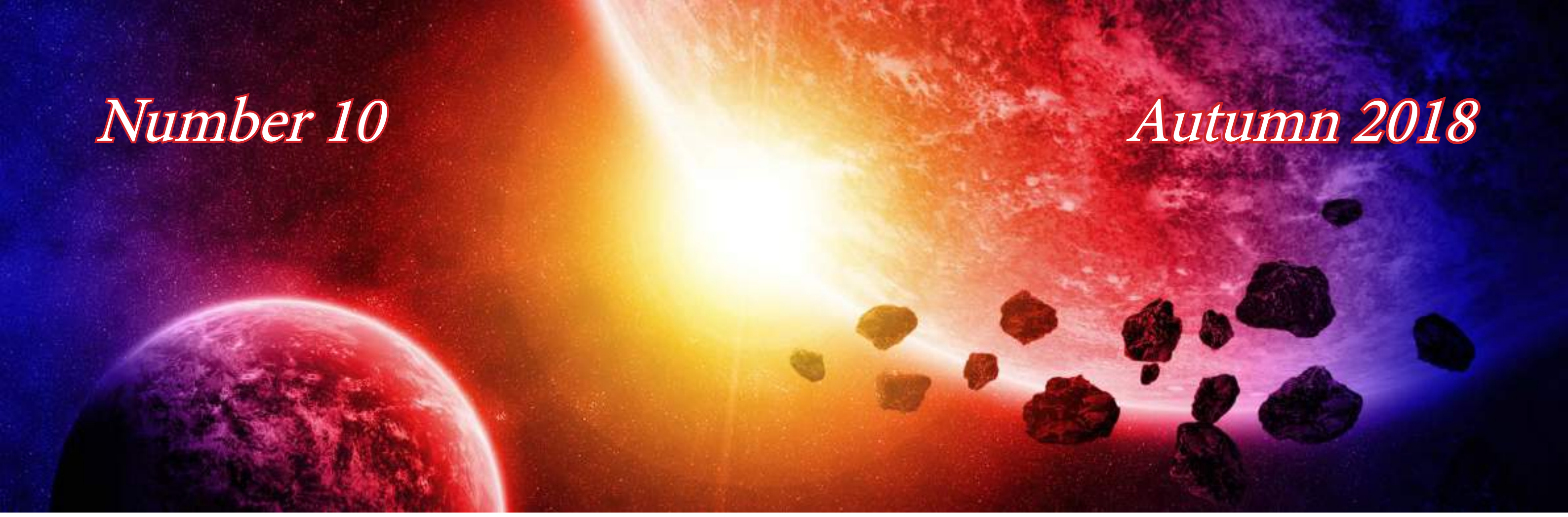

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SOLUTIONS

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JP.136. Let x, y, z be positive real numbers such that: $xyz = 1$. Find the maximum of the expression:

$$Q = \frac{1}{\sqrt[3]{2x^5 + y^4 - x^2 + 4}} + \frac{1}{\sqrt[3]{2y^5 + z^4 - y^2 + 4}} + \frac{1}{\sqrt[3]{2z^5 + x^4 - z^2 + 4}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Michael Stergiou-Greece

$$Q = \sum_{cyc} \frac{1}{\sqrt[3]{2x^5+y^4-x^3+4}} \quad (1)$$

The function $f(x) = \sqrt[3]{x}$ is concave so (1) becomes:

$$Q \le 3\sqrt[3]{\frac{1}{3}\sum \frac{1}{2x^5+y^4-x^2+4}} \to \frac{Q^3}{9} \le \sum_{cyc} \frac{1}{2x^5+y^4-x^2+4} \quad (2)$$

As $x^5 + x^5 + x^2 \ge 3x^4$ *(2)* $\to \frac{Q^3}{9} \le \sum_{cyc} \frac{1}{3x^4+y^4-2x^2+4}$ *(3)*

Let $x^2 = a, y^2 = b, z^2 = c, abc = 1$ *(3)* $\to \frac{Q^3}{9} \le \sum \frac{1}{3a^2+b^2-2a+4}$ *(4). But* $a^2 + b^2 \ge 2ab$, $a^2 + 1 \ge 2a$

(4) $\to \frac{Q^3}{9} \le \frac{1}{2}\sum_{cyc} \frac{1}{a+ab+1} \le \cdots \le 1$ *(after calculus the sum reduces to 1 using $abc = 1$)*

$$\textit{Therefore } Q \le \sqrt[3]{\frac{9}{2}}$$

JP.137. Let $x, y \ge 1$. Prove that:

$$\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \ge 2 + \frac{4(x-y)^2}{(2x+xy+1)(2y+xy+1)}$$

Proposed by Andrei Ștefan Mihalcea – Romania

Solution 1 by Ravi Prakash-New Delhi-India

Rewrite the inequality: $\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} - 2 \ge \frac{4(x-y)^2}{(2x+xy+1)(2y+xy+1)} \Leftrightarrow$

$$\Leftrightarrow \frac{(\sqrt{x}-\sqrt{y})^2}{\sqrt{xy}} \ge \frac{4(\sqrt{x}-\sqrt{y})^2(\sqrt{x}+\sqrt{y})^2}{(2x+xy+1)(2y+xy+1)} \quad (1)$$

If $x = y$, there is nothing to show. Suppose $x \ne y$, then (1) can be written as

$$\frac{1}{\sqrt{xy}} \geq \frac{4\left(\sqrt{x}+\sqrt{y}\right)^2}{(2x+xy+1)(2y+xy+1)} \Leftrightarrow (xy+1)^2+2(x+y)(xy+1)+4xy \geq$$

$$\geq 4\sqrt{xy}\left(x+y+2\sqrt{xy}\right) \Leftrightarrow x^2y^2+1+2(x+y)(xy+1)+6xy \geq 4\sqrt{xy}(x+y)+8xy$$

$$\Leftrightarrow x^2y^2-2xy+1+2(x+y)\left(xy-2\sqrt{xy}+1\right) \geq 0 \Leftrightarrow$$

$$\Leftrightarrow (xy-1)^2+2(x+y)\left(\sqrt{xy}-1\right)^2 \geq 0 \text{ which is true } \forall x,y>0.$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}} \overset{(1)}{\geq} 2+\frac{4(x-y)^2}{(2x+xy+1)(2y+xy+1)}$$

$$(1) \Leftrightarrow \frac{x}{y}+\frac{y}{x}+2 \geq 4+\frac{16(x-y)^4}{(2x+xy+1)^2(2y+xy+1)^2}+\frac{16(x-y)^2}{(2x+xy+1)(2y+xy+1)} \Leftrightarrow$$

$$\Leftrightarrow \frac{(x-y)^2}{xy} \geq \frac{16(x-y)^4}{(2x+y+1)^2(2y+xy+1)^2}+\frac{16(x-y)^2}{(2x+xy+1)(2y+xy+1)} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{xy} \geq \frac{16(x-y)^2+16(2x+xy+1)(2y+xy+1)}{(2x+xy+1)^2(2y+xy+1)^2} (\because (x-y)^2 \geq 0)$$

$$\Leftrightarrow (2x+xy+1)^2(2y+xy+1)^2 \geq 16xy(2x+xy+1)(2y+xy+1)+$$

$$+4xy\left((2x+xy+1)-(2y+xy+1)\right)^2 \Leftrightarrow$$

$$\Leftrightarrow a^2b^2 \geq 16xyab+4xy(a^2+b^2-2ab) \begin{pmatrix} \text{where } a=2x+xy+1 \\ b=2y+xy+1 \end{pmatrix}$$

$$\Leftrightarrow a^2b^2 \geq 4xy(a+b)^2 \Leftrightarrow ab \overset{(1)}{\geq} 2(a+b)\sqrt{xy}. \text{ Now,}$$

$$ab=(2x+xy+1)(2y+xy+1)=\left(x(1+y)+(1+x)\right)\left(y(1+x)+(1+y)\right)$$

$$\overset{A-G}{\geq} \left(2\sqrt{x(1+x)(1+y)}\right)\left(2\sqrt{y(1+x)(1+y)}\right)=4\sqrt{xy}(1+x)(1+y)=$$

$$=2\sqrt{xy}(2x+2x+2y+2xy)=2\sqrt{xy}(a+b) \Rightarrow \text{(1) is true (Proved)}$$

JP.138. Let $a,b,c>0$, with $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$. Prove that:

$$(4a-3)(4b-3)(4c-3) \geq 243\sqrt[3]{abc}$$

Proposed by Andrei Ștefan Mihalcea – Romania

Solution 1 by Boris Colakovic-Belgrade-Serbia

$$(4a-3)(4b-3)(4c-3) \geq 243\sqrt[3]{abc} \Leftrightarrow (4a-3)^3(4b-3)^3(4c-3)^3 \geq 243^3abc$$

$$(4a-3)^3 = 64a^3-144a^2+108a-27 \geq 243a \Leftrightarrow (a-3)(8a+3)^2 \geq 0 \Leftrightarrow a \geq 3 \quad (*)$$

$$(4b-3)^3 = 64b^3-144b^2+108b-27 \geq 234b \Leftrightarrow (b-3)(8b+3)^2 \geq 0 \Leftrightarrow b \geq 3 \quad (**)$$

$$(4c-3)^3 = 64c^3-144c^2+108c-27 \geq 243c \Leftrightarrow (c-3)(8c+3)^2 \geq 0 \Leftrightarrow c \geq 3 \quad (***)$$

(*) × (**) × (***) $\Rightarrow abc \geq 27$ ***true because of*** $1 = \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \overset{AM-GM}{\geq} \frac{3}{\sqrt[3]{abc}} \Rightarrow abc \geq 27$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a,b,c > 0$ ***and*** $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = 1$, ***give*** $a = x^3, b = y^3, c = z^3$. ***Hence*** $\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3} = 1 \Leftrightarrow$

$\Leftrightarrow x^3y^3+y^3z^3+z^3x^3 = (xyz)^3 \Leftrightarrow xyz \geq 3$. ***We want to show that***

$$64(xyz)^3+36(x^3+y^3+z^3) \geq 243xyz+48((xy)^3+(yz)^3+(zx)^3).$$

Iff $16(xyz)^3+36(x^3+y^3+z^3) \geq 243xyz+2y$

Iff $16(xyz)^3 \geq 135xyz+2y$

Iff $xyz(16((xyz)^2-135) \geq 24$

Iff $(16(xyz)^2-135) \geq 9$

Iff $16(xyz)^2 \geq 144: xyz \geq 3$

Hence $(4x^3-3)(4y^3-3)(4z^3-3) \geq 243xyz$

That is $(4a-3)(4b-3)(4c-3) \geq 243\sqrt[3]{abc}$

Therefore it is to be true.

Solution 3 by Soumava Chakraborty-Kolkata-India

$$LHS = 64abc-48\sum ab+36\sum a-27 = 64abc-48abc+36\sum a-27$$

$$\left(\because \sum ab = abc\right)$$

$$\overset{A-G}{\geq} 16abc+108\sqrt[3]{abc}-27 \overset{?}{\geq} 243\sqrt[3]{abc} \Leftrightarrow 16t^3-135t-27 \overset{?}{\geq} 0 \ \left(t = \sqrt[3]{abc}\right)$$

$\Leftrightarrow (t-3)(16t^2+48t+9) \overset{?}{\geq} 0 \rightarrow$ ***true*** $\because t = \sqrt[3]{abc} \geq 3$ ***(proved)***

Proof of $t \geq 3$:

$$\sum ab = abc \overset{A-G}{\geq} 3\sqrt[3]{a^2b^2c^2} \Rightarrow a^3b^3c^3 \geq 27a^2b^2c^2 \Rightarrow \sqrt[3]{abc} = t \geq 3$$

Solution 4 by proposer

From $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = 1$ ***we get*** $abc = ab+bc+ca \geq 3\sqrt[3]{(abc)^2}$. ***Now, we have:***

$(abc)^3 \geq 27(abc)^2 | : (abc)^2$ ***getting*** $abc \geq 27$ ***and from that*** $\sqrt[3]{abc} \geq 3$. ***Note that*** $a > 1$ ***because*** $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$ ***and we have*** $3a - 3 > 0, 3b - 3 > 0$ ***and*** $3c - 3 > 0$.

Using Hölder's inequality we get:

$$(4a-3)(4b-3)(4c-3) = [a + (3a-3)][b + (3b-3)][c + (3c-3)] \geq$$

$$\geq \left(\sqrt[3]{abc} + 3\sqrt[3]{(a-1)(b-1)(c-1)}\right)^3$$

$$(4a-3)(4b-3)(4c-3) \geq \left(\sqrt[3]{abc} + 3\sqrt[3]{abc - (ab+bc+ca) + (a+b+c) - 1}\right)^3$$

$$(4a-3)(4b-3)(4c-3) \geq \left(\sqrt[3]{abc} + 3\sqrt[3]{abc - abc + 3\sqrt[3]{abc} - 1}\right)^3$$

$$(4a-3)(4b-3)(4c-3) \geq \left(\sqrt[3]{abc} + 3\sqrt[3]{3 \cdot \sqrt[27]{27} - 1}\right)^3$$

$$(4a-3)(4b-3)(4c-3) \geq \left(\sqrt[3]{abc} + 3 + 3\right)^3$$

Using AM-GM we have: $(4a-3)(4b-3)(4c-3) \geq \left(3\sqrt[3]{\sqrt[3]{abc} \cdot 3 \cdot 3}\right)^3$

$$(4a-3)(4b-3)(4c-3) \geq 3^3 \cdot \sqrt[3]{abc} \cdot 9$$

$$(4a-3)(4b-3)(4c-3) \geq 243\sqrt[3]{abc}$$

JP.139. Let x, y, z be positive real numbers such that: $x^2 + y^2 + z^2 = 3$. Find the minimum of the expression:

$$P = \frac{x}{\sqrt[4]{\frac{y^8+z^8}{2}} + 3yz} + \frac{y}{\sqrt[4]{\frac{z^8+x^8}{2}} + 3zx} + \frac{z}{\sqrt[4]{\frac{x^8+y^8}{2}} + 3xy}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by proposer

- Using Cauchy and Bunhiacopxki inequality. We have:

$$\left(\sqrt{2(y^8+z^8)} + 2y^2z^2\right)^2 \leq 2(2(y^8+z^8) + 4y^4z^4) = 4(y^4+z^4)^2 \Leftrightarrow$$

$$\Leftrightarrow \sqrt{2(y^8+z^8)} + 2y^2z^2 \leq 2(y^4+z^4)$$

$$\Leftrightarrow \sqrt{2(y^8+z^8)} \le 2(y^4-y^2z^2+z^4) \Leftrightarrow \sqrt[4]{\frac{y^8+z^8}{2}} \le \sqrt{y^4-y^2z^2+z^4} \quad \textit{(1)}$$

- Other:

$$\sqrt{y^4-y^2z^2+z^4} = \sqrt{(y^2+z^2)^2-\left(yz\sqrt{3}\right)^2} = \sqrt{\left(y^2-yz\sqrt{3}+z^2\right)\left(y^2+yz\sqrt{3}+z^2\right)}$$

$$= \sqrt{\left(2+\sqrt{3}\right)\left(y^2-yz\sqrt{3}+z^2\right)\cdot\left(2-\sqrt{3}\right)\left(y^2+yz\sqrt{3}+z^2\right)} \le$$

$$\le \frac{\left(2+\sqrt{3}\right)\left(y^2-yz\sqrt{3}+z^2\right)+\left(2-\sqrt{3}\right)\left(y^2+yz\sqrt{3}+z^2\right)}{2}$$

$$= \frac{2(2y^2-3yz+2z^2)}{2} = 2y^2-3yz+2z^2$$

$$\Rightarrow \sqrt{y^4-y^2z^2+z^4} \le 2y^2-3yz+2z^2 \qquad \textit{(2)}$$

- Let (1), (2): $\Rightarrow \sqrt[4]{\frac{y^8+z^8}{2}} \le 2y^2-3yz+2z^2 \Leftrightarrow \sqrt[4]{\frac{y^8+z^8}{2}}+3yz \le 2(y^2+z^2)$

$$\Leftrightarrow \frac{1}{\sqrt[4]{\frac{y^8+z^8}{2}}+3yz} \ge \frac{1}{2(y^2+z^2)} \Leftrightarrow \frac{x}{\sqrt[4]{\frac{y^8+z^8}{2}}+3yz} \ge \frac{x}{2(y^2+z^2)} = \frac{x}{2(3-x^2)} \quad \textit{(Let } x^2+y^2+z^2=3\textit{)} \quad \textit{(3)}$$

- We have: $\frac{x}{3-x^2}-\frac{x^2}{2} = x\left(\frac{1}{3-x^2}-\frac{x}{2}\right) = \frac{x(x^3-3x+2)}{2(3-x^2)} = \frac{x(x-1)^2(x+2)}{2(3-x^2)} \ge 0$ ***(because*** $x>0$; $(x-1)^2 \ge 0$***)***

$$\Leftrightarrow \frac{x}{3-x^2}-\frac{x^2}{2} \ge 0 \Leftrightarrow \frac{x}{3-x^2} \ge \frac{x^2}{2}\textit{. Let (3):} \Rightarrow \frac{x}{\sqrt[4]{\frac{y^8+z^8}{2}}+3yz} \ge \frac{x^2}{4} \qquad \textit{(4)}$$

+ Similar: $\frac{y}{\sqrt[4]{\frac{z^8+x^8}{2}}+3zx} \ge \frac{y^2}{4}; \frac{z}{\sqrt[4]{\frac{x^8+y^8}{2}}+3xy} \ge \frac{z^2}{4}$ ***(5)***

- Let (4), (5) and $x^2+y^2+z^2=3$***:***

$$\Rightarrow P = \frac{x}{\sqrt[4]{\frac{y^8+z^8}{2}}+3yz}+\frac{y}{\sqrt[4]{\frac{z^8+x^8}{2}}+3zx}+\frac{z}{\sqrt[4]{\frac{x^8+y^8}{2}}+3xy} \ge \frac{x^2+y^2+z^2}{4} = \frac{3}{4} \Rightarrow$$

$$\Rightarrow Q_{min} = \frac{3}{4}$$

+ Equality occurs if:

$$\begin{cases} x,y,z>0; x^2+y^2+z^2=3 \\ \sqrt{2(x^8+y^8)} = 2x^2y^2; \sqrt{2(y^8+z^8)} = 2y^2z^2; \sqrt{2(z^8+x^8)} = 2z^2x^2 \\ x-1=y-1=z-1=0 \end{cases} \Leftrightarrow x=y=z=1$$

JP.140. Let $a,b,c>0$. Prove that:

$$\sum\frac{\sqrt{a+b}}{a}\le\left(\sum\frac{1}{a}\right)\sqrt{\sum a-\frac{\sum ab}{\sum a}}$$

Proposed by Andrei Ștefan Mihalcea – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$LHS=\frac{bc\sqrt{a+b}+ca\sqrt{b+c}+ab\sqrt{c+a}}{abc}=\frac{\sqrt{bc(a+b)}\sqrt{bc}+\sqrt{ca(b+c)}\sqrt{ca}+\sqrt{ab(c+a)}\sqrt{ab}}{abc}\overset{CBS}{\le}$$

$$\le\frac{\sqrt{\sum ab}\sqrt{3abc+\sum a^2b}}{abc}\overset{?}{\le}\frac{\sum ab}{abc}\sqrt{\sum a-\frac{\sum ab}{\sum a}}\Leftrightarrow 3abc+\sum a^2b\overset{?}{\le}$$

$$\le\sum ab\left(\frac{\sum a^2+\sum ab}{\sum a}\right)\Leftrightarrow$$

$$\Leftrightarrow\left(\sum a^2+\sum ab\right)\left(\sum ab\right)\overset{?}{\ge}\left(\sum a\right)\left(3abc+\sum a^2b\right)\Leftrightarrow ab^2+bc^2+ca^3\underset{(1)}{\overset{?}{\ge}}$$

$$\ge abc\left(\sum a\right).\text{ But } ab^3+bc^3+ca^3=abc\left(\frac{b^2}{c}+\frac{c^2}{a}+\frac{a^2}{b}\right)\overset{Bergstrom}{\ge}abc\frac{(\sum a)^2}{\sum a}=$$

$$=abc\left(\sum a\right)\Rightarrow \textit{(1) is true (proved)}$$

JP.141. Let $a,b,c>0$. Prove that:

$$\left(\sum\sqrt{\frac{b+c}{a}}\right)^2\le\frac{2(\sum ab)^3}{3a^2b^2c^2}$$

Proposed by Andrei Ștefan Mihalcea – Romania

Solution 1 by proposer

$$\text{Let be } g:(0,\infty)\to\mathbb{R};\ g(x)=\sqrt{ab+bc+ca-\frac{abc}{x}}$$

$$g(x)=\left(ab+bc+ca-\frac{abx}{x}\right)^{\frac{1}{2}};\ g'(x)=\frac{abc}{2x^2}\left(ab+bc+ca-\frac{abc}{x}\right)^{-\frac{1}{2}}$$

$$g''(x)=\frac{abc}{2}\left[\left(-\frac{2}{x^3}\left(ab+bc+ca-\frac{abc}{x}\right)^{-\frac{1}{2}}-\frac{abc}{2x^4}\left(ab+bc+ca-\frac{abc}{x}\right)\right]$$

$g''(x) < 0; g$ – **concave. Denote** $q = ab + bc + ca; r = abc$

$$\sum_{cyc}\left(\frac{1}{a}f(a)\right) = \sum_{cyc}\left(\frac{1}{a}\sqrt{ab+bc}\right) = \sum_{cyc}\sqrt{\frac{b+c}{a}} \leq$$

$$\overset{JENSEN}{\leq} \frac{q}{r}f\left(\frac{3r}{q}\right) = \frac{q}{r}\sqrt{q - \frac{qr}{3r}} = \frac{q}{r}\sqrt{\frac{2q}{3}} = \frac{ab+bc+ca}{abc}\sqrt{\frac{2(ab+bc+ca)}{3}}$$

$$\left(\sum_{cyc}\sqrt{\frac{b+c}{a}}\right)^2 \leq \frac{2\left(\sum_{cyc}ab\right)^3}{3a^2b^2c^2}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\forall a,b,c > 0, \left(\sum\sqrt{\frac{b+c}{a}}\right)^2 \leq \frac{2(\sum ab)^3}{3a^2b^2c^2}$$

$$LHS \overset{CBS}{\leq} \left(\sqrt{\sum(b+c)}\sqrt{\sum\frac{1}{a}}\right)^2 = \frac{2(\sum a)(\sum ab)}{abc} \overset{?}{\leq} \frac{2(\sum ab)^3}{3a^2b^2c^2} \Leftrightarrow \left(\sum ab\right)^2 \geq 3abc\left(\sum a\right)$$

$$\Leftrightarrow \sum a^2b^2 \geq abc\left(\sum a\right) \rightarrow \textbf{true} \because \sum x^2 \geq \sum xy \quad (\text{where } x = ab, y = bc, z = ca)$$

(Proved)

JP.142. Let $a, b, c \geq 1$. Prove that:

$$\sum\sqrt{\frac{a-1}{bc}} \leq \left(\sum\frac{1}{ab}\right)\sqrt{abc - \frac{\sum a}{3}}$$

Proposed by Andrei Ștefan Mihalcea – Romania

Solution by proposer

Let be $f: (0,\infty) \rightarrow \mathbb{R}; f(x) = \sqrt{abc - \frac{abc}{x}}$

$$f(x) = \left(abc - \frac{abc}{x}\right)^{\frac{1}{2}}; \ f'(x) = \frac{1}{2}\cdot\frac{abc}{x^2}\left(abc - \frac{abc}{x}\right)^{-\frac{1}{2}}$$

$$f''(x) = \frac{abc}{2}\left[-\frac{2}{x^3}\left(abc - \frac{abc}{x}\right)^{-\frac{1}{2}} - \frac{abc}{2x^4}\left(abc - \frac{abc}{x}\right)^{-\frac{3}{2}}\right]$$

$f''(x) < 0; f$ **concave. Denote** $p = a + b + c; r = abc$

$$\sum_{cyc}\left(\frac{f(ab)}{ab}\right) = \sum_{cyc}\sqrt{\frac{a-1}{bc}} \overset{JENSEN}{\le} \frac{p}{r}f\left(\frac{3r}{p}\right) = \frac{p}{r}\sqrt{r-\frac{pr}{3r}} = \frac{p}{r}\sqrt{r-\frac{p}{3}}$$

$$abc\sum_{cyc}\sqrt{\frac{a-1}{bc}} \le \left(\sum_{cyc}a\right)\cdot\sqrt{abc-\frac{1}{3}\sum_{cyc}a}$$

$$\sum_{cyc}\sqrt{\frac{a-1}{bc}} \le \left(\sum_{cyc}\frac{1}{ab}\right)\sqrt{abc-\frac{a+b+c}{3}}$$

JP.143. In any ABC triangle the following relationship holds:

$$\frac{w_a^2}{h_b\cdot h_c} + \frac{w_b^2}{h_c\cdot h_a} + \frac{w_c^2}{h_a\cdot h_b} \le \left(\frac{R}{r}\right)^2 - 1$$

all notations are usual sense.

Proposed by Mehmet Şahin – Ankara – Turkey

Solution by Soumava Chakraborty-Kolkata-India

$$\because w_a^2 \le s(s-a), \text{etc} \therefore LHS \le \sum\frac{s(s-a)bc}{4S^2} = \frac{s}{4S^2}\left(s\sum ab - 12Rrs\right)$$

$$= \frac{s^2}{4r^2s^2}(s^2 - 8Rr + r^2)$$

$$\overset{?}{\le} \frac{R^2-r^2}{r^2} \Leftrightarrow s^2 \overset{?}{\le} 4R^2 + 8Rr - 5r^2. \text{ Now, } s^2 \overset{Gerretsen}{\le}$$

$$4R^2 + 4Rr + 3r^2 \overset{?}{\le} 4R^2 + 8Rr - 5r^2 \Leftrightarrow 4Rr \overset{?}{\ge} 8r^2 \Leftrightarrow R \overset{?}{\ge} 2r \to \text{true (Euler)}$$

JP.144. In any ABC triangle the following relationship holds:

$$\frac{a}{w_a} + \frac{b}{w_b} + \frac{c}{w_c} \ge \frac{4s}{3R}$$

Proposed by Mehmet Şahin – Ankara – Turkey

Solution by Soumava Chakraborty-Kolkata-India

$$LHS = \sum\frac{a^2}{aw_a} \underset{(1)}{\overset{Bergstrom}{\ge}} \frac{4s^2}{\sum aw_a}$$

WLOG, we may assume $a \geq b \geq c \therefore w_a \leq w_b \leq w_c$

$$(1) \Rightarrow LHS \underset{(2)}{\overset{Chebyshev}{\geq}} \frac{4s^2}{\frac{1}{3}(2s)\sum w_a} = \frac{2s\cdot 3}{\sum w_a}$$

$$\text{Now, } \sum w_a \underset{(3)}{\overset{w_a \leq \sqrt{s(s-a)},etc}{\leq}} \sqrt{3s}\sqrt{3s-2s} = \sqrt{3}s$$

$$(2),(3) \Rightarrow LHS \geq \frac{2s\cdot 3}{\sqrt{3}s} = \frac{2\cdot 3}{\sqrt{3}} = \frac{2\cdot 3\sqrt{3}}{3} = \frac{2\cdot 3\sqrt{3}R}{3R} \overset{Mitrinovic}{\geq} \frac{2\cdot 2s}{3R} = \frac{4s}{3R} \text{ (Done)}$$

JP.145. In any ABC triangle the following relationship holds:

$$\frac{m_a}{r+r_a} + \frac{m_b}{r+r_b} + \frac{m_c}{r+r_c} \leq \frac{s}{2r}$$

all notations are usual sense.

Proposed by Mehmet Şahin – Ankara – Turkey

Solution by Soumava Chakraborty-Kolkata-India

$$LHS = \sum \frac{m_a}{\frac{\Delta}{s} + \frac{\Delta}{s-a}} = \sum \frac{m_a s(s-a)}{rs(b+c)} \overset{m_a < \frac{b+c}{2},etc}{<} \sum \left(\frac{b+c}{2}\right)\cdot\frac{(s-a)}{r(b+c)} =$$

$$= \frac{\sum(s-a)}{2r} = \frac{3s-2s}{2r} = \frac{s}{2r} \therefore LHS < \frac{s}{2r} \Rightarrow LHS \leq \frac{s}{2r} \text{ (proved)}$$

JP.146. Let x, y, z be positive real numbers such that: $xyz = 1$. Find the maximum of the expression:

$$P = \frac{1}{\sqrt[3]{2(x^5-x^3+4)}} + \frac{1}{\sqrt[3]{2(y^5-y^3+4)}} + \frac{1}{\sqrt[3]{2(z^5-z^3+4)}}.$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by proposer

* *We have:* $x^5 - x^3 - 2x + 2 = x^4(x-1) + x^3(x-1) - 2(x-1) = (x-1)(x^4+x^3-2)$

$$= (x-1)\left(x^3(x-1) + 2x^2(x-1) + 2x(x-1) + 2(x-1)\right)$$

$$= (x-1)^2(x^3+2x^2+2x+2) \geq 0$$

$$\Rightarrow x^5 - x^3 - 2x + 2 \geq 0 \Leftrightarrow x^5 - x^3 + 4 \geq 2(x+1) \Leftrightarrow \frac{1}{\sqrt[3]{2(x^5-x^3+4)}} \leq \frac{1}{\sqrt[3]{2\cdot 2(x+1)}}$$

- Therefore, by AM-GM inequality for three positive real numbers:

$$\Rightarrow \frac{1}{\sqrt[3]{2(x^5-x^3+4)}} \le \frac{1}{\sqrt[3]{4(x+1)}} \le \frac{1}{3}\left(\frac{1}{\sqrt{2(x+1)}}+\frac{1}{\sqrt{2(x+1)}}+\frac{1}{2}\right) = \frac{1}{3}\left(\sqrt{\frac{2}{x+1}}+\frac{1}{2}\right)$$

+ Similar: $\frac{1}{\sqrt[3]{2(y^5-y^3+4)}} \le \frac{1}{3}\left(\sqrt{\frac{2}{y+1}}+\frac{1}{2}\right); \frac{1}{\sqrt[3]{2(z^5-z^3+4)}} \le \frac{1}{3}\left(\sqrt{\frac{2}{z+1}}+\frac{1}{2}\right)$

- Therefore:

$$\Rightarrow P = \frac{1}{\sqrt[3]{2(x^5-x^3+4)}}+\frac{1}{\sqrt[3]{2(y^5-y^3+4)}}+\frac{1}{\sqrt[3]{2(z^5-z^3+4)}} \le \frac{1}{3}\left(\sqrt{\frac{2}{x+1}}+\sqrt{\frac{2}{y+1}}+\sqrt{\frac{2}{z+1}}\right)+\frac{1}{2} \quad (1)$$

- Other, because $xyz = 1$ then exist positive real numbers a, b, c such that:

$$a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{x}$$

+ Therefore, by inequality Cauchy Schwarz:

$$\left(\sqrt{\frac{2}{x+1}}+\sqrt{\frac{2}{y+1}}+\sqrt{\frac{2}{z+1}}\right)^2 = \left(\sqrt{\frac{2}{\frac{a}{b}+1}}+\sqrt{\frac{2}{\frac{b}{c}+1}}+\sqrt{\frac{2}{\frac{c}{a}+1}}\right)^2 = 2\left(\sqrt{\frac{b}{a+b}}+\sqrt{\frac{c}{b+c}}+\sqrt{\frac{a}{c+a}}\right)^2$$

$$= 2\left(\sqrt{\frac{b}{(a+b)(b+c)}}\cdot\sqrt{b+c}+\sqrt{\frac{c}{(b+c)(c+a)}}\cdot\sqrt{c+a}+\sqrt{\frac{a}{(c+a)(a+b)}}\cdot\sqrt{a+b}\right)^2 \le$$

$$\le 2\big((b+c)+(c+a)+(a+b)\big)\cdot\left(\frac{b}{(a+b)(b+c)}+\frac{c}{(b+c)(c+a)}+\frac{a}{(c+a)(a+b)}\right)$$

$$\le 4(a+b+c)\cdot\frac{b(c+a)+c(a+b)+a(b+c)}{(a+b)(b+c)(c+a)} = \frac{8(a+b+c)(ab+bc+ca)}{(a+b)(b+c)(c+a)} \quad (2)$$

- We have: $9(a+b)(b+c)(c+a) - 8(a+b+c)(ab+bc+ca) = a(b-c)^2+b(c-a)^2+c(a-b)^2 \ge 0$

$$\Rightarrow 9(a+b)(b+c)(c+a) \ge 8(a+b+c)(ab+bc+ca) \Leftrightarrow \frac{8(a+b+c)(ab+bc+ca)}{(a+b)(b+c)(c+a)} \le 9$$

By (2): $\Rightarrow \left(\sqrt{\frac{2}{x+1}}+\sqrt{\frac{2}{y+1}}+\sqrt{\frac{2}{z+1}}\right)^2 \le 9 \Rightarrow \sqrt{\frac{2}{x+1}}+\sqrt{\frac{2}{y+1}}+\sqrt{\frac{2}{z+1}} \le 3 \quad$ ***(3)***

- Let (1), (3): $\Rightarrow P \le \frac{1}{3}\cdot 3+\frac{1}{2} = \frac{3}{2} \Rightarrow P_{\max} = \frac{3}{2}$. ***Equality occurs if:*** $\begin{cases} xyz = 1 \\ x = y = z > 0 \end{cases} \Leftrightarrow$

$$x = y = z = 1$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a > 0$, we have this fact:

$a^5-a^3+4\geq a^2+3$ *because* $a^5-a^3-a^2+1=a^3(a^2-1)-(a^2-1)=$

$=(a^2-1)(a^3-1)=(a^2-1)(a-1)(a^2+a+1)=(a-1)^3(a+1)(a^2+a+1)\geq 0$

Hence $\frac{1}{\sqrt[3]{2(x^5-x^3+4)}}+\frac{1}{\sqrt[3]{2(y^5-y^3+4)}}+\frac{1}{\sqrt[3]{2(z^5-z^3+4)}}\leq\frac{1}{\sqrt[3]{2(x^2+3)}}+\frac{1}{\sqrt[3]{2(y^3+3)}}+\frac{1}{\sqrt[3]{2(z^3+3)}}\leq$

$$\leq\sqrt{\frac{3^2}{2}\left(\frac{1}{x^2+3}+\frac{1}{y^2+3}+\frac{1}{z^2+3}\right)}\leq\frac{3}{2}$$

Iff $\frac{9}{2}\left(\frac{1}{x^2+3}+\frac{1}{y^2+3}+\frac{1}{z^2+3}\right)\leq\frac{27}{8}$

Iff $\frac{1}{x^2+3}+\frac{1}{y^2+3}+\frac{1}{z^2+3}\leq\frac{3}{4}$

Iff $\frac{b^2}{a^2+3b^2}+\frac{c^2}{b^2+3c^2}+\frac{a^2}{c^2+3a^2}\leq\frac{3}{4}: xyz=1; x=\frac{a}{b}, y=\frac{b}{c}, z=\frac{c}{a}$ ***and it is be true.***

Because $5(a^4b^2+b^4c^2+c^4a^2)+3(a^4c^2+c^4b^2+b^4a^2)\geq 24(abc)^2\Rightarrow$

$\Rightarrow 9(a^4b^2+b^4c^2+c^4a^2)+27(a^4e^2+c^4b^2+b^4a^2)+84(abc)^2\geq$

$\geq 4(a^4b^2+b^4c^2+c^4a^2)+24(a^4c^2+c^4b^2+b^4a^2)+108(abc)^2\Rightarrow$

$$\Rightarrow\frac{a^2}{c^2+3a^2}+\frac{b^2}{a^2+3b^2}+\frac{c^2}{b^2+3c^2}\leq\frac{3}{4}$$

Therefore it is to be true.

JP.148. Let a,b,c be positive real numbers such that: $ab+bc+ca=12$. Prove that:

$$\frac{a^3+b^3}{2b^2-bc+2c^2}+\frac{b^3+c^3}{2c^2-ca+2a^2}+\frac{c^3+a^3}{2a^2-ab+2b^2}\geq 4$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Marian Ursărescu – Romania

Because $ab+bc+ca=12\Rightarrow\exists x,y,z>0$ ***such that:***

$$a=\frac{2\sqrt{3}x}{\sqrt{xy+yz+zx}}, b=\frac{2\sqrt{3}y}{\sqrt{xy+yz+zx}}, c=\frac{2\sqrt{3}z}{\sqrt{xy+yz+zx}}$$

Inequality becomes:

$$\sum\left(\frac{\frac{24\sqrt{3}(x^3+y^3)}{\left(\sqrt{xy+yz+zx}\right)^3}}{\frac{24y^2-12yz+24z^2}{xy+yz+zx}}\right)\ge 4\Leftrightarrow 2\sqrt{3}\sum\frac{x^3+y^3}{\sqrt{xy+yz+zx}(2y^2-yz+z^2)}\ge 4\Leftrightarrow$$

$$\Leftrightarrow\sum\frac{x^3+y^3}{2y^3-yz+2z^2}\ge\frac{2}{\sqrt{3}}\sqrt{xy+yz+zx}\quad (1)$$

But $(x+y+z)^2\ge 3(xy+xz+yz)\Rightarrow\sqrt{xy+xz+yz}\le\frac{x+y+z}{\sqrt{3}}$ ***(2)***

From (1) + (2) we must show: $\sum\frac{x^3+y^3}{2y^2-yz+2z^2}\ge\frac{2}{3}(x+y+z)$ ***(3)***

But $2y^2-yz+2z^2\le 3(y^2-yz+z^2)$ ***(4) (because*** $\Leftrightarrow y^2-2yz+z^2\ge 0$***)***

$$\sum\frac{x^3+y^3}{y^2-yz+z^2}\ge 2(x+y+z)\quad (5)$$

But this inequality its proposed and solved by Vasile Cîrtoaje in 2009, solved by S.O.S method. (Or its solved used $4(x^3+y^3)\ge(x+y)^3$ ***and Hölder's inequality)***

Completion: We must (5): $\sum\frac{x^3+y^3}{y^2-yz+z^2}\ge 2(x+y+z)$

We show: (6) $\sum\frac{x^3}{y^2-yz+z^2}\ge x+y+z$ ***and*** $\sum\frac{y^3}{y^2-yz+z^2}\ge x+y+z$ ***(7)***

From (6) + (7) $\Rightarrow$ ***(5)***

For (6): $\sum\frac{x^3}{y^2-yz+z^2}=\sum\frac{x^4}{x(y^2-yz+z^2)}\ge\frac{(x^2+y^2+z^2)^2}{\sum x(y^2-yz+z^2)}$

(from Cauchy's or Bergström's inequality) $\Rightarrow$

We must show: $\frac{(x^2+y^2+z^2)^2}{\sum x(y^2-yz+z^2)}\ge x+y+z\Leftrightarrow$

$$\Leftrightarrow(x^2+y^2+z^2)^2\ge(x+y+z)\cdot\sum x(y^2-yz+z^2)\Leftrightarrow$$

$$\Leftrightarrow(x^2+y^2+z^2)^2-(x+y+z)\sum x(y^2-yz+z^2)\ge 0\quad (8)$$

Now we use Cîrtoaje's theorem: If $f_n(x,y,z)$ ***is a symmetric and homogeneous polynom of degree*** 4 ***then*** $f_4(x,y,z)\ge 0\ \forall x,y,z\in\mathbb{R}\Leftrightarrow f_4(x_1,1,1)\ge 0\ \forall x\in\mathbb{R}$ ***in our case:*** $f_4(x,y,z)=(x^2+y^2+z^2)^2-(x+y+z)\sum x(y^2-yz+z^2)$

$f_4(x_1,1,1)=x^4-2x^3+x^2=x^2(x-1)^2\ge 0$ ***true*** $\Rightarrow$ ***(6) its true.***

For (7): $\sum\frac{y^3}{y^2-yz+z^2}\ge x+y+z\Leftrightarrow\frac{y^3}{y^2-yz+z^2}+\frac{z^3}{z^2-zx+x^2}+\frac{x^3}{x^2-xy+y^2}\ge x+y+z\Leftrightarrow$

$$\sum x^3(y^2-yz+z^2)(z^2-xz+x^2) \geq (x+y+z)\prod(x^2-xy+y^2) \Leftrightarrow$$

$$\Leftrightarrow \sum x^3(y^2-yz+z^2)(z^2-xz+x^2)-$$

$$-(x+y+z)(x^2-xy+y^2)(y^2-yz+z^2)(z^2-xz+x^2) \geq 0 \quad (9)$$

Now again use Cîrtoaje's theorem: If $f_5(x,y,z)$ it's a symmetric polynomial function of degree 5 then: $f_5(x,y,z) \geq 0\ \forall x,y,z \geq 0 \Leftrightarrow f_5(0,4,4) \geq 0$. In our case:

$$f_5(x,y,z) = \sum x^3(y^2-yz+z^2)(z^2-xz+x^2)-$$

$$-(x+y+z)(x^2-xy+y^2)(y^2-yz+z^2)(z^2-xz+x^2)$$

$f_5(0,4,4) = 2y^7 - 2y^7 \geq 0$ ***true*** $\Rightarrow$ ***(9) its true*** $\Rightarrow$ ***(7) its true.***

Vasile Cartoaje proof:

Let a,b,c be non-negative real numbers. Prove that:

$$\frac{a^3}{b^2-bc+c^2}+\frac{b^3}{c^2-ca+a^2}+\frac{c^3}{a^2-ab+b^2} \geq a+b+c$$

Solution. Applying Cauchy – Schwarz inequality, we have:

$$\sum_{cyc}\frac{a^3}{b^2-bc+c^2} = \sum_{cyc}\frac{a^4}{a(b^2-bc+c^2)} \geq \frac{(a^2+b^2+c^2)^2}{\sum_{cyc}a(b^2-bc+c^2)}$$

It remains to prove that: $\left(\sum_{cyc}a^2\right)^2 \geq \left(\sum_{cyc}a(b^2-bc+c^2)\right)\left(\sum_{cyc}a\right)$ ***or***

$\sum_{cyc}a^4 + 2\sum_{cyc}a^2b^2 \geq (a+b+c)\sum_{cyc}a^2(b+c) - 3abc\sum_{cyc}a$ ***or***

$$\sum_{cyc}a^4 + abc\sum_{cyc}a \geq \sum_{cyc}a^3(b+c)$$

This is exactly the fourth degree-Schur's inequality, so we are done.

Equality holds for $a=b=c$ or $a=b, c=0$ up to permutation.

Solution 2 by Michael Sterghiou-Greece

$$\sum_{cyc}\frac{a^3+b^3}{2b^2-bc+2c^2} \geq 4 \quad (1)$$

(1) $\rightarrow \sum_{cyc}\frac{a^3}{2b^2-bc+2c^2}+\sum_{cyc}\frac{b^3}{2b^2-bc+2c^2} \geq 4$. ***But***

$$\sum_{cyc}\frac{a^3}{2b^2-bc+2c^2} \overset{BCS}{\geq} \frac{(a^2+b^2+c^2)^2}{\sum_{cyc}2ab^2-abc+2ac^2} \quad (2)$$

Let $\left(\sum_{cyc} a, \sum_{cyc} ab, abc\right) = (p, q, r)$ *with* $q = 12$

(2) $\equiv \frac{(p^2-2q)^2}{24p-9r} \geq \frac{(p^2-2q)^2}{24p-9\cdot\frac{48p-p^3}{9}} = \frac{p^2-24}{p}$ *(3) because from the 3rd degree Schur*

$$q = 12 \leq \frac{4p^3 + 9r}{4p} \to r \geq \frac{48p - p^3}{9}$$

Now $\sum_{cyc} \frac{b^3}{2b^2-bc+2c^2} \overset{Holder}{\geq} \frac{p^3}{3\cdot\sum_{cyc}(2b^2-bc+2c^2)} = \frac{p^3}{12(p^2-27)}$ *(4)*

It suffices to prove that $\frac{p^2-24}{p} + \frac{p^3}{12(p^2-27)} - 4 \geq 0$ *(5)*

(5) $\to 13p^4 - 48p^3 - 612p^2 + 1296p + 7776 \geq 0$ *or*

$(p-6)^2(13p^2 + 108p + 216) \geq 0$ *which holds as* $p^2 \geq 3q \to p \geq 6$ *and the trinomial is clearly* ≥ 0 *for* $p \geq 6$*. Done!*

JP.149. Find all functions: $f: (0, +\infty) \to \mathbb{R}$ **which verify the relationship:**

$$ln(xy) \leq xf(x) + yf(y) \leq xyf(xy), \forall x, y > 0$$

Proposed by Marian Ursărescu – Romania

Solution by Ravi Prakash-New Delhi-India

$$\ln(xy) \leq xf(x) + yf(y) \leq xyf(xy), \forall x, y > 0$$

Put $x = y = 1$*, we get:* $0 \leq f(1) + f(1) \leq f(1) \Rightarrow f(1) = 0.$

Put $y = \frac{1}{x}$ *to obtain*

$0 \leq xf(x) + \frac{1}{x} f\left(\frac{1}{x}\right) \leq 0 \Rightarrow f\left(\frac{1}{x}\right) = -x^2 f(x)$*. Taking* $y = 1$*, we get:*

$$\ln(x) \leq xf(x), \forall x > 0 \quad (1)$$

$$\Rightarrow \ln\left(\frac{1}{x}\right) \leq \frac{1}{x} f\left(\frac{1}{x}\right) = -xf(x), \forall x > 0 \Rightarrow -\ln(x) \leq -xf(x), \forall x > 0$$

$$\Rightarrow xf(x) \leq \ln x, \forall x > 0 \quad (2)$$

From (1), (2): $xf(x) = \ln(x), \forall x > 0 \Rightarrow f(x) = \frac{1}{x}\ln(x), \forall x > 0$

JP.150. Let be $z_1, z_2, z_3 \in \mathbb{C}^*$ different in pairs such that: $|z_1| = |z_2| = |z_3|$. If $(z_1 + z_2)(z_2 + z_3)(z_3 + z_1) + z_1z_2z_3 = 0$, then z_1, z_2, z_3 are the affixes of an equilateral triangle.

Proposed by Marian Ursărescu – Romania

Solution 1 by Ravi Prakash-New Delhi-India

Let $z = z_1 + z_2 + z_3$

$$(z_1 + z_2)(z_2 + z_3)(z_3 + z_1) + z_1z_2z_3 = 0 \Rightarrow (z - z_3)(z - z_1)(z - z_2) + z_1z_2z_3 = 0 \Rightarrow$$

$$\Rightarrow z^3 - (z_1 + z_2 + z_3)z^2 + (z_2z_3 + z_3z_1 + z_1z_2)z - z_1z_2z_3 + z_1z_2z_3 = 0 \Rightarrow$$

$$\Rightarrow z^3 - z(z^2) + z(z_2z_3 + z_3z_1 + z_1z_2) = 0 \Rightarrow z\left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right)z_1z_2z_3 = 0.$$

As $|z_1| = |z_2| = |z_3| = k > 0, z_1z_2z_3 \neq 0$. *Thus*

$$z\left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right) = 0 \quad (1)$$

Also, $k^2 = z_1\overline{z_1} = z_2\overline{z_2} = z_3\overline{z_3} \quad (2)$

From (1), (2): $k^2z(\overline{z_1} + \overline{z_2} + \overline{z_3}) = 0 \Rightarrow k^2z\overline{z} = 0$. *As* $k^2 \neq 0, |z|^2 = 0 \Rightarrow z = 0 \Rightarrow$

$\Rightarrow z_1 + z_2 + z_3 = 0$. *Now,* $|z_2 - z_3|^2 + |z_1|^2 = |z_2 - z_3|^2 + |-z_2 - z_3|^2 =$

$= |z_2 - z_3|^2 + |z_2 + z_3|^2 = 2|z_2|^2 + 2|z_3|^2 = 4k^2 \Rightarrow |z_2 - z_3|^2 + k^2 = 4k^2 \Rightarrow$

$\Rightarrow |z_2 - z_3| = \sqrt{3}k$. *Similarly,* $|z_3 - z_1| = |z_1 - z_2| = \sqrt{3}k$. *Thus,*

$|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1| \Rightarrow$ *triangle with vertices* z_1, z_2, z_3 *is an equilateral triangle.*

Solution 2 by Rovsen Pirguliyev-Sumgait-Azerbaijan

Lemma: Let $|z_1| = |z_2| = |z_3|$. *Points* z_1, z_2, z_3 *are vertices of an equilateral triangle if and only if* $z_1 + z_2 + z_3 = 0$. *Since* $(z_1 + z_2)(z_2 + z_3)(z_3 + z_1) + z_1z_2z_3 =$

$= (z_1 + z_2 + z_3)(z_1z_2 + z_1z_3 + z_2z_3)$,

considering condition, we have: $z_1 + z_2 + z_3 = 0$ *or*

$z_1z_2 + z_1z_3 + z_2z_3 = 0$. *Using Lemma* $\Rightarrow$ *Q.E.D.*

SP.136. Let x, y, z be positive real numbers such that: $x^4 + y^4 + z^4 = xy + yz + zx$.

Find the maximum of the expression:

$$P = \sqrt[3]{\frac{x^6 + y^6}{2}} + \sqrt[3]{\frac{y^6 + z^6}{2}} + \sqrt[3]{\frac{z^6 + x^6}{2}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by proposer

* Let $x, y, z > 0$, we will prove that inequality:

$$x^4 + y^4 + z^4 + xyz(x + y + z) \geq xy(x^2 + y^2) + yz(y^2 + z^2) + zx(z^2 + x^2) \quad (1)$$

$$(1) \Leftrightarrow x^4 + y^4 + z^4 + xyz(x + y + z) - xy(x^2 + y^2) - yz(y^2 + z^2) - zx(z^2 + x^2) \geq 0$$

$$\Leftrightarrow x^2(x^2 - xy - xz + yz) + y^2(y^2 - yz - yx + zx) + z^2(z^2 - zx - zy + xy) \geq 0$$

$$\Leftrightarrow x^2(x - y)(x - z) + y^2(y - z)(y - x) + z^2(z - x)(z - y) \geq 0 \quad (2)$$

- Supposed $x \geq y \geq z > 0$

$$+ \text{ We have: } \begin{cases} z \leq x \\ z \leq y \end{cases} \Leftrightarrow \begin{cases} z - x \leq 0 \\ z - y \leq 0 \end{cases} \Rightarrow (z - x)(z - y) \geq 0 \Rightarrow z^2(z - x)(z - y) \geq 0 \quad (3)$$

$$+ \text{ Other: } x^2(x - y)(x - z) + y^2(y - z)(y - x)$$

$$= (x - y)[x^2(x - z) - y^2(y - z)] = (x - y)[(x^3 - y^3) - z(x^2 - y^2)]$$

$$= (x - y)[(x - y)(x^2 + xy + y^2) - z(x - y)(x + y)] = (x - y)^2(x^2 + xy + y^2 - zx - zy) \geq 0 \quad (4)$$

(because $x \geq y \geq z > 0, x^2 + xy + y^2 - zx - zy = x(x - z) + y(x - z) + y^2 \geq y^2 > 0$

and $(x - y)^2 \geq 0$)

- Let (3), (4): $\Rightarrow x^2(x - y)(x - z) + y^2(y - z)(y - x) + z^2(z - x)(z - y) \geq 0$

$\Rightarrow$ Inequality (2) true $\Rightarrow$ (1) true.

* We have: $x^6 + y^4 = (x^2 + y^2)(x^4 - x^2y^2 + y^4) = (x^2 + y^2)(x^2 - xy\sqrt{3} + y^2)(x^2 + xy\sqrt{3} + y^2)$

- Therefore, by AM-GM inequality: $\sqrt[3]{\frac{x^6+y^6}{2}} = \sqrt[3]{\frac{(x^2+y^2)(x^2-xy\sqrt{3}+y^2)(x^2+xy\sqrt{3}+y^2)}{2}}$

$$= \sqrt[3]{\frac{(x^2 + y^2)}{2} \cdot (2 + \sqrt{3})(x^2 - xy\sqrt{3} + y^2)(2 - \sqrt{3})(x^2 + xy\sqrt{3} + y^2)}$$

$$\leq \frac{\frac{x^2 + y^2}{2} + (2 + \sqrt{3})(x^2 - xy\sqrt{3} + y^2) + (2 - \sqrt{3})(x^2 + xy\sqrt{3} + y^2)}{3} = \frac{3x^2 - 3xy + 3y^2}{2}$$

$$\Rightarrow \sqrt[3]{\frac{x^6+y^6}{2}} \le \frac{3x^2-4xy+3y^2}{2}.\ \textit{Similar: } \sqrt[3]{\frac{y^6+z^6}{2}} \le \frac{3y^2-4yz+3z^2}{2};\ \sqrt[3]{\frac{z^6+x^6}{2}} \le \frac{3z^2-4zx+3x^2}{2}$$

$$\Rightarrow P = \sqrt[3]{\frac{x^6+y^6}{2}} + \sqrt[3]{\frac{y^6+z^6}{2}} + \sqrt[3]{\frac{z^6+x^6}{2}} \le \frac{3x^2-4xy+3y^2}{2} + \frac{3y^2-4yz+3z^2}{2} + \frac{3z^2-4zx+3x^2}{2}$$

$$\Leftrightarrow P \le 3(x^2+y^2+z^2) - 2(xy+yz+zx) \quad \textit{(5)}$$

We will prove:** $3(x^2+y^2+z^2) - 2(xy+yz+zx) \le 3$ ***(6)

$$\Leftrightarrow 3(x^2+y^2+z^2) - 2(xy+yz+zx) \le \frac{3(x^4+y^4+z^4)}{xy+yz+zx} \quad \left(x^4+y^4+z^4 = xy+yz+zx \textit{ then } \frac{x^4+y^4+z^4}{xy+yz+zx} = 1\right)$$

$$\Leftrightarrow \big(3(x^2+y^2+z^2) - 2(xy+yz+zx)\big)(xy+yz+zx) \le 3(x^4+y^4+z^4)$$

$$\Leftrightarrow 3(x^2+y^2+z^2)(xy+yz+zx) \le 3(x^4+y^4+z^4) + 2(xy+yz+zx)^2$$

$$\Leftrightarrow 3xy(x^2+y^2) + 3yz(y^2+z^2) + 3zx(z^2+x^2) + 3xyz(x+y+z) \le$$

$$\le 3(x^4+y^4+z^4) + 2(x^2y^2+y^2z^2+z^2z^2) + 4xyz(x+y+z)$$

$$\Leftrightarrow 3(x^4+y^4+z^4) + xyz(x+y+z) + 2(x^2y^2+y^2z^2+z^2z^2) \ge 3xy(x^2+y^2) + 3yz(y^2+z^2) + 3zx(z^2+x^2) \quad \textit{(7)}$$

- By AM-GM inequality for 2 real numbers:

$$x^2y^2+y^2z^2+z^2z^2 = \frac{x^2(y^2+z^2)}{2} + \frac{y^2(z^2+x^2)}{2} + \frac{z^2(x^2+y^2)}{2} \ge \frac{x^2\cdot 2yz}{2} + \frac{y^2\cdot 2zx}{2} + \frac{z^2\cdot 2xy}{2}$$

$$\Rightarrow x^2y^2+y^2z^2+z^2z^2 \ge xyz(x+y+z) \quad \textit{(8)}$$

$$\Rightarrow 3(x^4+y^4+z^4) + xyz(x+y+z) + 2(x^2y^2+y^2z^2+z^2z^2) \ge 3\big(x^4+y^4+z^4+xyz(x+y+z)\big) \quad \textit{(9)}$$

- Let (1), (9):

$$\Rightarrow 3(x^4+y^4+z^4) + xyz(x+y+z) + 2(x^2y^2+y^2z^2+z^2z^2) \ge 3xy(x^2+y^2) + 3yz(y^2+z^2) + 3zx(z^2+x^2)$$

⇒ (7) True ⇒ Inequality (6) true.

- Let (5), (6): $\Rightarrow P \le 3 \Rightarrow P_{max} = 3$

$$+\ \textit{Equality occurs if: } \Leftrightarrow \begin{cases} x, y, z > 0 \\ x^4+y^4+z^4 = xy+yz+zx \\ x = y = z \\ x^2 = y^2+z^2 \end{cases} \Leftrightarrow x = y = z = 1$$

SP.137. Let $a, b, c > 0$ such that $a+b+c = 3$. Prove that:

$$\frac{a}{\sqrt[3]{4(b^6+c^6)+7bc}} + \frac{b}{\sqrt[3]{4(c^6+a^6)+7ca}} + \frac{c}{\sqrt[3]{4(a^6+b^6)+7ab}} + \frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{12} \ge \frac{7}{12} \quad \textbf{(1)}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by proposer

*** We have:**

$$b^6+c^6=(b^2+c^2)(b^4-b^2c^2+c^4)=(b^2+c^2)\left[(b^2+c^2)^2-\left(bc\sqrt{3}\right)^2\right]$$
$$=(b^2+c^2)\left(b^2-bc\sqrt{3}+c^2\right)\left(b^2+bc\sqrt{3}+c^2\right)$$

- By inequality AM-GM for three positive real numbers:

$$\sqrt[3]{4(b^6+c^6)}=\sqrt[3]{(b^2+c^2)\cdot 2(2+\sqrt{3})\left(b^2-bc\sqrt{3}+c^2\right)\cdot 2(2-\sqrt{3})\left(b^2+bc\sqrt{3}+c^2\right)}\le$$
$$\le\frac{(b^2+c^2)+2(2+\sqrt{3})\left(b^2-bc\sqrt{3}+c^2\right)+2(2-\sqrt{3})\left(b^2+bc\sqrt{3}+c^2\right)}{3}=\frac{9b^2-12bc+9c^2}{3}$$
$$\Leftrightarrow\sqrt[3]{4(b^6+c^6)}\le 3b^2-4bc+3c^2\Leftrightarrow\sqrt[3]{4(b^6+c^6)}+7bc\le 3b^2+3bc+3c^2$$
$$\Leftrightarrow\frac{1}{\sqrt[3]{4(b^6+c^6)}+7bc}\ge\frac{1}{3(b^2+bc+c^2)}\Leftrightarrow\frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc}\ge\frac{a}{3(b^2+bc+c^2)}\quad\textbf{(2)}$$

+ Similar: $$\frac{b}{\sqrt[3]{4(c^6+a^6)}+7ca}\ge\frac{b}{3(c^2+ca+a^2)}\quad\textbf{(3)}$$

$$\frac{c}{\sqrt[3]{4(a^6+b^6)}+7ab}\ge\frac{c}{3(a^2+ab+b^2)}\quad\textbf{(4)}$$

- Then (2), (3), (4): $$\Rightarrow\frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc}+\frac{b}{\sqrt[3]{4(c^6+a^6)}+7ca}+\frac{c}{\sqrt[3]{4(a^6+b^6)}+7ab}\ge$$
$$\ge\frac{a}{3(b^2+bc+c^2)}+\frac{b}{3(c^2+ca+a^2)}+\frac{c}{3(a^2+ab+b^2)}\quad\textbf{(5)}$$

– Other, by Cauchy-Schwarz we have:

$$\frac{a}{b^2+bc+c^2}+\frac{b}{c^2+ca+a^2}+\frac{c}{a^2+ab+b^2}=\frac{a^2}{ab^2+abc+ac^2}+\frac{b^2}{bc^2+bca+ba^2}+\frac{c^2}{ca^2+cab+cb^2}\ge$$
$$\ge\frac{(a+b+c+)^2}{(ab^2+abc+ac^2)+(bc^2+bca+ba^2)+(ca^2+cab+cb^2)}\quad\textbf{(6)}$$

- That $$\frac{(a+b+c)^2}{(ab^2+abc+ac^2)+(bc^2+bca+ba^2)+(ca^2+cab+cb^2)}$$
$$=\frac{(a+b+c)^2}{ab(a+b)+bc(b+c)+ca(c+a)+3abc}=\frac{(a+b+c)^2}{(a+b+c)(ab+bc+ca)}=\frac{a+b+c}{ab+bc+ca}\quad\textbf{(7)}$$

- Then (6), (7): $$\Rightarrow\frac{a}{b^2+bc+c^2}+\frac{b}{c^2+ca+a^2}+\frac{c}{a^2+ab+b^2}\ge\frac{a+b+c}{ab+bc+ca}\quad\textbf{(8)}$$

+ And $a+b+c=3$. ***Then (8):***

$$\Rightarrow\frac{a}{b^2+bc+c^2}+\frac{b}{c^2+ca+a^2}+\frac{c}{a^2+ab+b^2}\ge\frac{3}{ab+bc+ca}\quad\textbf{(9)}$$

- Then (5), (9): $\Rightarrow \frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc}+\frac{b}{\sqrt[3]{4(c^6+a^6)}+7ca}+\frac{c}{\sqrt[3]{4(a^6+b^6)}+7ab}\ge\frac{1}{ab+bc+ca}$ **(10)**

- By AM-GM for five positive real numbers:

$$\sqrt[3]{a}+\sqrt[3]{a}+\sqrt[3]{a}+a^2+a^2\ge 5\sqrt[5]{\sqrt[3]{a}\cdot\sqrt[3]{a}\cdot\sqrt[3]{a}\cdot a^2\cdot a^2}=5\sqrt[5]{a^5}=5a$$

$$\Leftrightarrow 3\cdot\sqrt[3]{a}+2a^2\ge 5a\Leftrightarrow 3\sqrt[3]{a}\ge 5a-2a^2 \quad \textbf{(11)}$$

+ Similar: $\sqrt[3]{b}\ge 5b-2b^2; 3\sqrt[3]{c}\ge 5c-2c^2$ **(12)**

– Then (11), (12): $\Rightarrow 3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)\ge 5(a+b+c)-2(a^2+b^2+c^2)$

$$\Leftrightarrow 3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)\ge 15-2(a^2+b^2+c^2)\ \ (a+b+c=3)$$

$$\Leftrightarrow 3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+1\right)\ge 18-2(a^2+b^2+c^2)=2(a+b+c)^2-2(a^2+b^2+c^2)$$

(Because $a+b+c=3\Rightarrow 2(a+b+c)^2=18$**)**

$$\Leftrightarrow 3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+1\right)\ge 2(a^2+b^2+c^2+2ab+2bc+2ca)-2(a^2+b^2+c^2)$$

$$\Leftrightarrow 3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+1\right)\ge 4(ab+bc+ca)\Leftrightarrow 3\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)\ge 4(ab+bc+ca)-3$$

$$\Leftrightarrow \frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{12}\ge\frac{4(ab+bc+ca)-3}{36}\Leftrightarrow\frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{12}\ge\frac{ab+bc+ca}{9}-\frac{1}{2} \quad \textbf{(13)}$$

- Then (10), (13):

$$\Rightarrow \frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc}+\frac{b}{\sqrt[3]{4(c^6+a^6)}+7ca}+\frac{c}{\sqrt[3]{4(a^6+b^6)}+7ab}+\frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{12}\ge$$

$$\ge\frac{1}{ab+bc+ca}+\frac{ab+bc+ca}{9}-\frac{1}{12} \quad \textbf{(14)}$$

- By AM-GM we have:

$$\frac{1}{ab+bc+ca}+\frac{ab+bc+ca}{9}\ge 2\cdot\sqrt{\frac{1}{ab+bc+ca}\cdot\frac{ab+bc+ca}{9}}=2\sqrt{\frac{1}{9}}=\frac{2}{3}$$

$$\Rightarrow \frac{1}{ab+bc+ca}+\frac{ab+bc+ca}{9}-\frac{1}{12}\ge\frac{2}{3}-\frac{1}{12}=\frac{7}{12}\Leftrightarrow\frac{1}{ab+bc+ca}+\frac{ab+bc+ca}{9}-\frac{1}{12}\ge\frac{7}{12} \quad \textbf{(15)}$$

- Then (14), (15):

$$\Rightarrow \frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc}+\frac{b}{\sqrt[3]{4(c^6+a^6)}+7ca}+\frac{c}{\sqrt[3]{4(a^6+b^6)}+7ab}+\frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{12}\ge\frac{7}{12}$$

$\Rightarrow$ ***Inequality (1) True and we get the result***

$$+\ \textit{Equality occurs if:}\begin{cases} a,b,c>0; a+b+c=3 \\ a=b=c \\ \frac{1}{b^2+bc+c^2}=\frac{1}{c^2+ca+a^2}=\frac{1}{a^2+ab+b^2} \\ \sqrt[3]{a}=a^2; \sqrt[3]{b}=b^2; \sqrt[3]{c}=c^2 \\ \frac{1}{ab+bc+ca}=\frac{ab+bc+ca}{9} \end{cases} \Leftrightarrow a=b=c=1.$$

SP.138. Let a,b,c be positive real numbers such that: $a+b+c=3$.

Prove that:

$$\frac{a^2}{\sqrt{5(b^4+4)}}+\frac{b^2}{\sqrt{5(c^4+4)}}+\frac{c^2}{\sqrt{5(a^4+4)}}\geq\frac{3}{5}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Heikichi Ezakiya-Jakarta-Indonesia

$$\textit{Let}: \varphi=\frac{a^2}{\sqrt{5(b^4+4)}}+\frac{b^2}{\sqrt{5(a^4+4)}}+\frac{c^2}{\sqrt{5(a^4+4)}}=\frac{1}{\sqrt{5}}\left(\frac{a^2}{\sqrt{b^4+4}}+\frac{b^2}{\sqrt{c^4+4}}+\frac{c^2}{\sqrt{a^4+4}}\right)$$

$$\textit{Using CBS: } \varphi\geq\frac{1}{\sqrt{5}}\cdot\frac{(a+b+c)^2}{\left(\sqrt{a^4+4}+\sqrt{b^4+4}+\sqrt{c^4+4}\right)}=\varphi^{(1)}$$

$$\textit{Using QM-AM for } \left(\sqrt{a^4+4}+\sqrt{b^4+4}+\sqrt{c^4+4}\right):$$

$$\sqrt{\frac{a^4+b^4+c^4+12}{3}}\geq\frac{\sqrt{a^4+4}+\sqrt{b^4+4}+\sqrt{c^4+4}}{3}\Leftrightarrow\frac{1}{\left(\sqrt{a^4+4}+\sqrt{b^4+4}+\sqrt{c^4+4}\right)}\geq\frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{a^4+b^4+c^4+12}}\ \textit{so,}$$

$$\varphi^{(1)}\geq\frac{1}{\sqrt{15}}\cdot\frac{1}{\sqrt{a^4+b^4+c^4+12}}=\varphi^{(2)}\quad (\#)$$

$$\textit{Using QM-AM for } a^2+b^2+c^2$$

$$\sqrt{\frac{a^2+b^2+c^2}{3}}\geq\frac{a+b+c}{3}\Leftrightarrow a^2+b^2+c^2\geq\frac{(a+b+c)^2}{3}$$

$$\textit{because } a+b+c, \textit{ then: } a^2+b^2+c^2\geq 3\quad (1)$$

$$\textit{Using QM-AM for } a^4+b^4+c^4$$

$$\sqrt{\frac{a^4+b^4+c^4}{3}}\geq\frac{a^2+b^2+c^2}{3}\Leftrightarrow a^4+b^4+c^4\geq\frac{(a^2+b^2+c^2)}{3}\quad (2)$$

$$\textit{From (1) \& (2): } a^4+b^4+c^4\geq\frac{(3)^2}{3}=3$$

From (#), if we choose $a^4+b^4+c^4=3$, # becomes equal, then

$$\varphi^{(1)} \geq \varphi^{(2)} = \frac{1}{\sqrt{15}} \cdot \frac{(a+b+c)^2}{\sqrt{3+12}} = \frac{(a+b+c)^2}{15}$$

Because $a+b+c=3$, then: $\varphi \geq \varphi^{(1)} \geq \varphi^{(2)} = \frac{(3)^2}{15} = \frac{3}{5}$

SP.139. In ABC triangle the lengths of sides BC, CA, AB are a, b, c. Let h_a, h_b, h_c be the distances from A, B, C to BC, CA, AB; l_a, l_b, l_c are the lengths of the bisectors A, B, C. Prove that:

$$\frac{l_a l_b}{l_c} + \frac{l_b l_c}{l_a} + \frac{l_c l_a}{l_b} \geq \frac{h_a h_b}{h_c} + \frac{h_b h_c}{h_a} + \frac{h_c h_a}{h_b}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Soumava Chakraborty-Kolkata-India

In any $\Delta ABC, \sum \frac{w_b w_c}{w_a} \geq \sum \frac{h_b h_c}{h_a}$. *Firstly,*

$$\prod(a+b) = 2abc + \sum ab\,(2s-c) = 2s(s^2+4Rr+r^2) - 4Rrs \overset{(1)}{=} 2s(s^2+2Rr+r^2)$$

Also, $\sum(s-b)(s-c) = \sum(s^2 - s(b+c) + bc) = 3s^2 - 4s^2 + \sum ab \overset{(2)}{=} 4Rr + r^2$

Also, $\prod w_a = \prod\left(\frac{2bc}{b+c}\cos\frac{A}{2}\right) = \frac{8(16R^2r^2s^2)}{\prod(b+c)}\left(\frac{s}{4R}\right) \overset{by\,(1)}{=} \frac{128R^2r^2s^2}{2s(s^2+2Rr+r^2)}\left(\frac{s}{4R}\right) \overset{(3)}{=} \frac{16Rr^2s^2}{s^2+2Rr+r^2}$

Now, $\sum \frac{h_b h_c}{h_a} = \sum \frac{(h_b h_c)^2}{h_a h_b h_c} = \left(\frac{8R^3}{a^2b^2c^2}\right)\sum\left(\frac{ca}{2R}\cdot\frac{ab}{2R}\right)^2 = \left(\frac{8R^3}{a^2b^2c^2}\right)\left(\frac{a^2b^2c^2}{16R^4}\right)\sum a^2 \overset{(a)}{=} \frac{\sum a^2}{2R}$

Now, $\sum \frac{w_b w_c}{w_a} = \left(\frac{1}{\prod w_a}\right)\sum w_b^2 w_a^2 \overset{by\,(3)}{=} \left(\frac{s^2+2Rr+r^2}{16Rr^2s^2}\right)\sum\left[\frac{4c^2a^2}{(c+a)^2}\cdot\frac{s(s-b)}{ca}\cdot\frac{4a^2b^2}{(a+b)^2}\cdot\frac{s(s-c)}{ab}\right]$

$$= \left(\frac{s^2+2Rr+r^2}{16Rr^2s^2}\right)\cdot\frac{16\cdot 4Rrs}{(\prod(a+b))^2}\left[\sum a\,(s-b)(s-c)(b+c)^2\right]$$

$$= \left(\frac{s^2+2Rr+r^2}{16Rr^2s^2}\right)\cdot\frac{64Rrs\cdot r^2s}{4s^2(s^2+2Rr+r^2)^2}\left[\sum\frac{a(b+c)^2}{s-a}\right] \overset{(4)}{=} \left(\frac{r}{s^2+2Rr+r^2}\right)\left[\sum\frac{a(b+c)^2}{s-a}\right]$$

Now, $\sum\frac{a(b+c)^2}{s-a} = \sum\frac{a(s+s-a)^2}{s-a} = \sum\frac{as^2+a(s-a)^2+2as(s-a)}{s-a} = s^2\sum\frac{a-s+s}{s-a} + \sum a\,(s-a) + 2s(2s)$

$$= s^2\left(-3+4+\frac{s}{r^2s}\sum(s-b)(s-c)\right) + s(2s) - 2(s^2-4Rr-r^2)$$

$$\overset{by\,(2)}{=} s^2\left(1+\frac{4R+r}{r}\right)+2(4Rr+r^2)\overset{(5)}{=}\frac{s^2(4R+2r)+2r^2(4R+r)}{r}$$

$$\text{(4),(5)} \Rightarrow \sum\frac{w_bw_c}{w_a}\overset{(b)}{=}\frac{s^2(4R+2r)+2r^2(4R+r)}{s^2+2Rr+r^2}$$

$$\text{(a), (b)} \Rightarrow \text{given inequality} \Leftrightarrow \frac{s^2(4R+2r)+2r^2(4R+r)}{s^2+2Rr+r^2}\ge\frac{\sum a^2}{2R}=\frac{s^2-4Rr-r^2}{R}$$

$$\Leftrightarrow s^2(4R^2+2Rr)+2Rr^2(4R+r)\ge(s^2-4Rr-r^2)(s^2+2Rr+r^2)$$

$$\Leftrightarrow s^2(4R^2+2Rr)+2Rr^2(4R+r)\ge s^4-2Rrs^2-r^2(4R+r)(2R+r)$$

$$\Leftrightarrow s^2(4R^2+4Rr)+r^2(4R+r)^2\overset{(c)}{\ge} s^4$$

$$\text{Now, RHS of (4)}\overset{Gerretsen}{\le} s^2(4R^2+4Rr+3r^2)\overset{?}{\le}s^2(4R^2+4Rr)+r^2(4R+r)^2\Leftrightarrow$$

$$\Leftrightarrow (4R+r)^2\ge 3s^2\to \text{true (Trucht) (Proved)}$$

SP.140. Let a,b,c be positive real numbers. Prove that:

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\ge\frac{4(a^2+b^2+c^2)}{ab+bc+ca}+\frac{2(ab+bc+ca)}{a^2+b^2+c^2}$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution 1 by Rade Krenkov-Sturmica-Macedonia

From Cauchy – Schwarz inequality we have:

$$\left(\frac{b}{a}+\frac{c}{a}+\frac{c}{b}+\frac{a}{b}+\frac{a}{c}+\frac{b}{c}\right)(b^3a+c^3a+c^3b+a^3b+a^3c+b^3c)\ge 4(a^2+b^2+c^2)^2 \quad (1)$$

$$\left(\frac{b}{a}+\frac{c}{a}+\frac{c}{b}+\frac{a}{b}+\frac{a}{c}+\frac{b}{c}\right)(abc^2+acb^2+bca^2+bac^2+cab^2+cba^2)\ge 4(ab+bc+ca)^2$$

$$\left(\frac{b}{a}+\frac{c}{a}+\frac{c}{b}+\frac{a}{b}+\frac{a}{c}+\frac{b}{c}\right)(abc^2+bca^2+cab^2)\ge 2(ab+bc+ca)^2 \quad (2)$$

From (1) and (2) we get:

$$\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)(ab+bc+ca)(a^2+b^2+c^2)\ge 4(a^2+b^2+c^2)^2+2(ab+bc+ca)^2$$

Now, we have that:

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\ge\frac{4(a^2+b^2+c^2)}{ab+bc+ca}+\frac{2(ab+bc+ca)}{a^2+b^2+c^2}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$\sum\frac{b+c}{a}\geq\frac{4\sum a^2}{\sum ab}+\frac{2\sum ab}{\sum a^2}$. *Given inequality transforms into:*

$$\left(\sum ab\right)\left(\sum a^2\right)\left\{\sum ab\,(a+b)\right\}-abc\left\{4\left(\sum a^2\right)^2+2\left(\sum ab\right)^2\right\}\geq 0\Leftrightarrow$$

$$\Leftrightarrow\sum a^5b^2+\sum a^2b^5+\sum a^4b^3+\sum a^3b^4+2abc\left(\sum a^3b+\sum ab^3\right)\geq$$

$\overset{(a)}{\geq}6abc(\sum a^2b^2)+2abc(\sum a^4)$, **we have** $2abc(\sum a^3b+\sum ab^3)\underset{(1)}{\overset{A-G}{\geq}}2abc(\sum ab(2ab))=$

$=4abc(\sum a^2b^2)$. **Also,** $\sum a^4b^3+\sum a^3b^4=\sum c^4\,(a^3+b^3)\geq\sum c^4\,ab(a+b)=$

$=abc(\sum a^3b+\sum ab^3)=abc(\sum ab(a^2+b^2))\underset{(2)}{\overset{A-G}{\geq}}2abc(\sum a^2b^2)$. **Moreover,**

$$\sum a^5b^2+\sum a^2b^5=\sum c^5(a^2+b^2)\underset{(3)}{\overset{A-G}{\geq}}\sum c^5\,(2ab)=2abc\left(\sum a^4\right)$$

(1)+(2)+(3)⇒ (a) is true (proved)

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a,b,c>0$ ***we have:***

$$a^3b^2+a^2b^3+b^3c^2+b^2c^3+c^3a^2+c^2a^3\geq 2(a^2bc+ab^3c+abc^3)$$

Hence $(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2)(ab+bc+ca)$

$$\geq 4abc(a^2+b^2+c^2)+2(abc)(ab+bc+ca)\Rightarrow\frac{(a^2b+ab^2)}{abc}+\frac{(b^2c+bc^2)}{abc}+\frac{(c^2a+ca^2)}{abc}\geq$$

$$\geq\frac{4(a^2+b^2+c^2)}{(ab+bc+ca)}+\frac{2(ab+bc+ca)}{(ab+bc+ca)}\Rightarrow\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}\geq\frac{4(a^2+b^2+c^2)}{(ab+bc+ca)}+\frac{2(ab+bc+ca)}{a^2+b^2+c^2}.$$

Therefore it is to be true.

SP.141. Let $a,b,c>0$ such that: $a+b+c=3$. Prove that:

$$\frac{a^4}{b^4(2ab-\sqrt{c}+2)}+\frac{b^4}{c^4(2bc-\sqrt{a}+2)}+\frac{c^4}{a^4(2ca-\sqrt{b}+2)}\geq\frac{a^2+b^2+c^2}{3}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Michael Sterghiou-Greece

$$\sum_{cyc}\frac{a^4}{b^4(2ab-\sqrt{c}+2)}\geq\frac{\sum_{cyc}a^2}{3}\quad\textit{(1)}$$

Let $\left(\sum_{cyc} a, \sum_{cyc} ab, abc\right) = (p, q, r).\ p = 3.\ \sum_{cyc} a^3 = 9 - 2q$

LHS of (1) $\geq \frac{\left(\sum_{cyc}\frac{a^2}{b^2}\right)^2}{\sum_{cyc}(2ab-\sqrt{c}+2)}\ [BCS] \geq \frac{\left(\sum_{cyc}\frac{a}{b}\right)^4}{9[2q+6-\sum_{cyc}\sqrt{a}]}$ *[again] BCS (2)*

It suffices that (2) $\geq \frac{9-2q}{3}$. *But it holds that*

$\sum_{cyc}\frac{a}{b} \geq \frac{p}{r^{\frac{1}{3}}}$ *(AM-GM) and* $\sum_{cyc}\sqrt{a} \geq q$ *(as* $p = 3$*)*

The last one: as $\sum_{cyc} a^2 + 2\sum_{cyc} ab = \left(\sum_{cyc} a\right)^2 = 9$ *it suffices that*

$\sum_{cyc} a^2 + 2\sum_{cyc}\sqrt{a} \geq 9$. *But*

$$\sum_{cyc}\left(a^2 + \sqrt{a} + \sqrt{a}\right) \overset{AM-GM}{\geq} 3 \cdot \sum_{cyc} a = 9$$

Therefore we have to show that $\frac{81}{9r^{\frac{4}{3}}(q+6)} \geq \frac{9-2q}{3}$ *or*

$f(q) = \left(\frac{q}{3}\right)^2(q+6)(9-2q) - 27 \leq 0$ *because this stronger inequality arises from the fact that* $r \leq \left(\frac{q}{3}\right)^{\frac{3}{2}}$. *But* $f(q) = \frac{1}{9}(3-q)(2q^3+9q^2-27q-81)$ *and* $q \leq 3$,

$9(q) = q(2q^2+9q-27) \leq 81$ *as* $q \leq 3$ *and* $2q^2+9q-27 \leq 27$

The proof is complete.

Solution 2 by proposer

** By Inequality Cauchy- Schwarz. We have:*

$$\sum\frac{a^4}{b^4(2ab-\sqrt{c}+2)} = \sum\frac{\left(\frac{a^2}{b^2}\right)^2}{(2ab-\sqrt{c}+2)} \geq \frac{\left(\sum\frac{a^2}{b^2}\right)^2}{\sum(2ab-\sqrt{c}+2)} = \frac{\left(\sum\frac{a^2}{b^2}\right)^2}{2\sum ab-\sum\sqrt{a}+6} \quad (2)$$

- Other, by AM-GM:

$$\sum\frac{a^2}{b^2} = \sum\frac{\frac{a^2}{b^2}+\frac{a^2}{b^2}+\frac{b^2}{c^2}}{3} \geq \sum\frac{3\cdot\sqrt[3]{\frac{a^2}{b^2}\cdot\frac{a^2}{b^2}\cdot\frac{b^2}{c^2}}}{3} = \sum\sqrt[3]{\frac{a^4}{b^2c^2}} = \frac{\sum a^2}{\sqrt[3]{a^2b^2c^2}} \quad (3)$$

$3 = a+b+c \geq 3\sqrt[3]{abc} \Rightarrow \sqrt[3]{abc} \leq 1 \Rightarrow \sqrt[3]{a^2b^2c^2} \leq 1$. *Let (3):* $\Rightarrow \sum\frac{a^2}{b^2} \geq \sum a^2$

- Let (2): $\Rightarrow \sum\frac{a^4}{b^4(2ab-\sqrt{c}+2)} \geq \frac{\left(\sum a^2\right)^2}{2\sum ab-\sum\sqrt{a}+6} \quad (4)$

- By AM-GM and $a+b+c=3$. *We have:*

$$2\sum\sqrt{a}+\sum a^2=\sum(\sqrt{a}+\sqrt{a}+a^2)>\sum 3\cdot\sqrt[3]{\sqrt{a}\cdot\sqrt{a}\cdot a^2}=3\sum a=9=\left(\sum a\right)^2\Rightarrow\sum\sqrt{a}\geq\sum ab$$

- Let (4):

$$\Rightarrow\sum\frac{a^4}{b^4(2ab-\sqrt{c}+2)}\geq\frac{(\sum a^2)^2}{2\sum ab-\sum ab+6}=\frac{(\sum a^2)^2}{\sum ab+6}\geq\frac{(\sum a^2)^2}{\sum a^2+6}\ \textit{(because } \sum ab\leq\sum a^2\textit{)}\ \ (5)$$

**** We will prove that:*** $\frac{(\sum a^2)^2}{\sum a^2+6}\geq\frac{\sum a^2}{3}\Leftrightarrow\frac{\sum a^2}{\sum a^2+6}\geq\frac{1}{3}\Leftrightarrow 3\sum a^2\geq\sum a^2+6\Leftrightarrow\sum a^2\geq 3$

(True because by AM-GM: $\sum a^2\geq\frac{(\sum a)^2}{3}=\frac{3^2}{3}=3$***)***

- Therefore, let (5): $\Rightarrow\sum\frac{a^4}{b^4(2ab-\sqrt{c}+2)}\geq\frac{\sum a^2}{3}$

$$\Leftrightarrow\frac{a^4}{b^4(2ab-\sqrt{c}+2)}+\frac{b^4}{c^4(2bc-\sqrt{a}+2)}+\frac{c^4}{a^4(2ca-\sqrt{b}+2)}\geq\frac{a^2+b^2+c^2}{3}\Rightarrow \textit{Q.E.D.}$$

SP.142. Let a,b,c be positive real numbers such that: $abc=1$. Prove that:

$$\frac{a^2b^2}{a^2-2a+b^2+2}+\frac{b^2c^2}{b^4-2b+c^2+2}+\frac{c^2a^2}{c^4-2c+a^2+2}\leq\frac{a^2+b^2+c^2+3}{4}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Marian Ursarescu-Romania

$a^4-2a+b^2+2=a^4-2a^2+1+2a^2+b^2+2^{-2a}=\left(a^2-1\right)^2+a^2-2a+1+a^2+$
$+b^2=(a^2-1)^2+(a-1)^2+a^2+b^2\geq a^2+b^2\geq 2ab$, **with equality for $a=b=1$.**

Inequality becomes: $\frac{a^2b^2}{2ab}+\frac{b^2c^2}{2bc}+\frac{a^2c^2}{2ac}\leq\frac{a^2+b^2+c^2+3}{4}\Leftrightarrow$

$a^2+b^2+c^2+3\geq 2(ab+bc+ac),\forall a,b,c>0$ ***with*** $abc=1$ ***(1)***

$a^2+b^2\geq 2ab$ ***(2)**; $c^2+1\geq 2c$ **(3)**;* $2+2c\geq 2ac+2bc$ ***(4)*** $\Leftrightarrow 1+\frac{1}{ab}\geq\frac{1}{b}+\frac{1}{a}\Leftrightarrow$

$\Leftrightarrow ab+1\geq a+b\Leftrightarrow(a-1)(b-1)\geq 0$***, true because we can choose two numbers so that*** $a,b\geq 1$ ***or*** $a,b\leq 1$***. From (2)+(3)+(4)***$\Rightarrow a^2+b^2+c^2+1+2+2c\geq$
$\geq 2ab+2c+2ac+2bc\Rightarrow a^2+b^2+c^2+3\geq 2(ab+bc+ac)\Rightarrow$ ***then (1) its true.***

Solution 2 by Michael Sterghiou-Greece

$$a,b,c>0\wedge abc=1\rightarrow\sum_{cyc}\frac{a^2b^2}{a^4-2a+b^2+2}\leq\frac{(\sum_{cyc}a^2)+3}{4}\quad (1)$$

$$\frac{a^2b^2}{a^4+2a+b^2+2}=\frac{a^2b^2}{a^4-a^2-2a+b^2+a^2+2}\le\frac{a^2b^2}{\underbrace{(a-1)^2(a^2+2a+2)}_{>0}+2ab}\le\frac{ab}{2}$$

Applying this in a cyclic manner it suffices to prove

$a^2+b^2+c^2+3-2(ab+bc+ca)\ge 0$ ***(2). From the numbers*** $a-1, b-1, c-1$ ***two will have the same sign (pigeonhole principle). Let WLOG these be*** $a-1, b-1$

(2)$\to a^2+b^2+c^2+1+2abc-2ab-2bc-2ca=(a-1)^2+(b-c)^2+$ $+2a(1-b)(1-c)$ ***which is clearly*** ≥ 0***. Done.***

Solution 3 by Rade Krenkov-Sturmica-Macedonia

Using AM-GM we get: $\frac{a^2b^2}{a^4-2a+b^2+2}+\frac{b^2c^2}{b^2-2bc+c^2+2}+\frac{c^2a^2}{c^4-2c+a^2+2}\le$

$$\le\frac{a^2b^2}{a^4-a^2+b^2+4}+\frac{b^2c^2}{b^4-b^2+c^2+4}+\frac{c^2a^2}{c^4-c^2+a^2+4}$$

$$\frac{a^2b^2}{a^4-2a+b^2+2}+\frac{b^2c^2}{b^4-2b+c^2+2}+\frac{c^2a^2}{c^4-2c+a^2+2}\le\frac{a^2b^2}{a^2+b^2}+\frac{b^2c^2}{b^2+c^2}+\frac{c^2a^2}{c^2+a^2}$$

$$\frac{a^2b^2}{a^4-2a+b^2+2}+\frac{b^2c^2}{b^4+2b+c^2+2}+\frac{c^2a^2}{c^4-2c+a^2+2}\le$$

$$\le\frac{1}{2}\left[\frac{ab\cdot 2ab}{a^2+b^2}+\frac{bc\cdot 2bc}{b^2+c^2}+\frac{ca(2ca)}{c^2+a^2}\right]$$

$$\frac{a^2b^2}{a^4-2a+b^2+2}+\frac{b^2c^2}{b^4-2b+c^2+2}+\frac{c^2a^2}{c^4-2c+a^2+2}\le\frac{1}{2}(ab+bc+ca)$$

We have to prove that: $a^2+b^2+c^2+3\ge 2(ab+bc+ca)$

We will prove that: $(a+b+c)(a^2+b^2+c^2+3)\ge 2(ab+bc+ca)(a+b+c)$

Note that: $a+b+c\ge 3\sqrt[3]{abc}=3\sqrt[3]{(abc)^3}=3abc$***. Now,***

$$(a+b+c)(a^2+b^2+c^2+3)=a^3+b^3+c^3+\sum a^2b+\sum ab^2+3(a+b+c)$$

$$(a+b+c)(a^2+b^2+c^2+3)\ge a^3+b^3+c^3+\sum a^2b+\sum ab^2+6abc+3abc$$

Using Schur's inequality we get:

$$(a+b+c)(a^2+b^2+c^2+3)\ge 2\left(\sum a^2b+\sum ab^2+3abc\right)$$

$(a+b+c)(a^2+b^2+c^2+3)\ge 2(a+b+c)(ab+bc+ca)$ ***(Done)***

Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand

For $a,b,c>0$ and $abc=1$, give $a=\frac{x}{y},b=\frac{y}{z},c=\frac{z}{x}$ and since $\frac{(ab)^2}{a^4-2a+b^2+z}\le\frac{ab}{2}$

$$\frac{(bc)^2}{b^4-2b+c^2+2}\le\frac{bc}{2};\frac{(ca)^2}{c^4-2c+a^2+2}\le\frac{ca}{2}\text{ and }\frac{ab}{2}+\frac{bc}{2}+\frac{ca}{2}\le\frac{a^2+b^2+c^2+3}{4}$$

Because $ab+bc+ca=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{y}{x}+\frac{z}{y}+\frac{x}{z}$

$a^2+b^2+c^2=\left(\frac{x}{y}\right)^2+\left(\frac{y}{z}\right)^2+\left(\frac{z}{x}\right)^2$ and since

$$x^4z^2+y^4x^2+z^4y^2+3(xyz)^2\ge 2(xy^3z^2+x^3y^2z+x^2yz^3)\Rightarrow$$

$$\Rightarrow\frac{x^2}{y^2}+\left(\frac{y}{z}\right)^2+\left(\frac{z}{x}\right)^2+3\ge 2\left(\frac{y}{x}+\frac{x}{z}+\frac{z}{y}\right)\Rightarrow a^2+b^2+c^2+3\ge 2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\Rightarrow$$

$\Rightarrow\frac{a^2+b^2+c^2+3}{4}\ge\frac{ab+bc+ca}{2}$. Therefore it is to be true.

SP.143. Let x,y,z be non-negative real numbers. Prove that:

$$x\sqrt{3x^2+yz}+y\sqrt{3y^2+zx}+z\sqrt{3z^2+xy}\ge x^2+y^2+z^2+xy+yz+zx$$

Proposed by Do Quoc Chinh – Ho Chi Minh – Vietnam

Solution by Rade Krenkov-Strumica-Macedonia

Using Cauchy – Schwarz inequality we have: $(3x^2+yz)(3x^2+x^2)\ge\left(3x^2+x\sqrt{yz}\right)^2$.

$$2x\sqrt{3x^2+yz}\ge 3x^2+\sqrt{yz}\quad(1)$$

Now, we get: $2y\sqrt{2y^2+zx}\ge 3y^2+\sqrt{zx}\quad(2)$

$$2z\sqrt{3z^2+xy}\ge 3z^2+\sqrt{xy}\quad(3)$$

From (1), (2) and (3) we get: $2\left(x\sqrt{3x^2+yz}+y\sqrt{3y^2+zx}+z\sqrt{3z^2+xy}\right)\ge$

$$\ge 2(x^2+y^2+z^2)+\left(x^2+y^2+z^2+\sqrt{xy}+\sqrt{yz}+\sqrt{zx}\right).$$

It is enough to prove that:

$x^2+y^2+z^2+x\sqrt{yz}+y\sqrt{zx}+z\sqrt{xy}\ge 2(xy+yz+zx)$. Introducing substitution

$$x=a^2,y=b^2,z=c^2$$

we get: $a^4+b^4+c^4+abc(a+b+c)\ge 2(a^2b^2+b^2c^2+c^2a^2)$.

Using Schur's inequality we have:

$$\sum_{cyc} a^4 + abc\sum_{cyc} a = \left(\sum_{cyc} a^3 + 3abc\right)\cdot\sum_{cyc} a - \left(\sum_{cyc} a^3b + \sum_{cyc} ab^3\right)$$

$$\sum_{cyc} a^4 + abc\sum_{cyc} a \ge \left(\sum_{cyc} a^2b + \sum_{cyc} ab^2\right)\sum_{cyc} a - 2abc\sum_{cyc} a - \left(\sum_{cyc} a^3b + \sum_{cyc} ab^3\right)$$

$$\sum_{cyc} a^4 + abc\sum_{cyc} a \ge 2\sum_{cyc} a^2b^2$$

SP.144. Let A, B, C be the corners in a triangle ABC. Prove that:

$$\left(\frac{\sin\frac{A}{2}}{\tan\frac{B}{2}}\right)^2 + \left(\frac{\sin\frac{B}{2}}{\tan\frac{C}{2}}\right)^2 + \left(\frac{\sin\frac{C}{2}}{\tan\frac{A}{2}}\right)^2 \ge \frac{9}{4}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Marian Ursarescu-Romania

$$s = \frac{a+b+c}{2}$$

$$\left(\frac{\sin\frac{A}{2}}{\tan\frac{B}{2}}\right)^2 = \frac{\frac{(s-b)(s-c)}{bc}}{\frac{(s-a)(s-c)}{s(s-b)}} = \frac{s(s-b)^2}{bc(s-a)} = 1 \text{ we must show: } \sum \frac{s(s-b)^2}{bc(s-a)} \ge \frac{9}{4} \Leftrightarrow \sum \frac{as(s-b)^2}{abc(s-a)} \ge \frac{9}{4} \Leftrightarrow$$

$$\Leftrightarrow \frac{a(s-b)^2}{s-a} \ge \frac{9}{4}\cdot\frac{abc}{s} \Leftrightarrow \sum \frac{a^2(s-b)^2}{a(s-a)} \ge \frac{9abc}{4s} \quad (1)$$

From Bergstrom inequality we have: $\sum \frac{a(s-b)^2}{a(s-a)} \ge \frac{(\sum a(s-b))^2}{\sum a(s-a)}$ ***(2)***

From (1)+(2) we must show this: $\frac{\left(2s^2-(ab+bc+ac)\right)^2}{2s^2-(a^2+b^2+c^2)} \ge \frac{9abc}{4s}$ ***(3)***

But we have $abc = 4sRr$ ***(4)***

$$ab + bc + ac = s^2 + r^2 + 4Rr \quad (5)$$

$$a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr) \quad (6)$$

From (3)+(4)+(5)+(6) we must show: $\frac{(2s^2-s^2-r^2-4Rr)^2}{2s^2-2s^2+2r^2+8Rr} \ge \frac{94sRr}{4s} \Leftrightarrow \frac{(s^2-r^2-4Rr)^2}{2r(4R+r)} \ge 9Rr \Leftrightarrow$

$$(s^2 - r^2 - 4Rr)^2 \ge 18Rr^2(4R + r) \quad (7)$$

From Gerretsen inequality we have: $s^2 \ge 16Rr - 5r^2$ ***(8)***

From (7)+(8) we must show: $(12Rr-6r^2)^2\ge 18Rr^2(4R+r)\Leftrightarrow$

$$\Leftrightarrow 36r^2(2R-r)^2\ge 18Rr^2(4Rr+r)\Leftrightarrow 2(2R-r)^2\ge R(4R+r)\Leftrightarrow$$

$$\Leftrightarrow 8R^2-8Rr+2r^2\ge 4R^2+Rr\Leftrightarrow 4R^2-9Rr+2r^2\ge 0\Leftrightarrow (R-2r)(4R-r)\ge 0$$

true, because from Euler $R\ge 2r$.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$LHS=\frac{(s-b)(s-c)s(s-b)}{bc(s-c)(s-a)}+\frac{(s-c)(s-a)s(s-c)}{ca(s-a)(s-b)}+\frac{(s-a)(s-b)s(s-a)}{ab(s-b)(s-c)}=$$

$$=\frac{s}{4Rrs}\left[\frac{a(s-b)^2}{(s-a)}+\frac{b(s-c)^2}{(s-b)}+\frac{c(s-a)^2}{(s-c)}\right]=$$

$$=\frac{1}{4Rr}\left[\frac{\{s-(s-a)\}(s-b)^2}{s-a}+\frac{\{s-(s-b)\}(s-c)^2}{s-b}+\frac{\{s-(s-c)\}(s-a)^2}{s-c}\right]=$$

$$=\frac{s}{4Rr}\left[\frac{(s-b)^3}{(s-a)(s-b)}+\frac{(s-c)^3}{(s-b)(s-c)}+\frac{(s-a)^3}{(s-c)(s-a)}\right]-\frac{1}{4Rr}\left[\sum(s-b)^2\right]\ge$$

$$\overset{Hölder}{\ge}\frac{s}{4Rr}\cdot\frac{(3s-2s)^3}{3\sum(s-a)(s-b)}-\frac{1}{4Rr}\sum(s^2-2bs+b^2)=$$

$$=\frac{s^4}{12Rr\sum\{s^2-s(a+b)+ab\}}-\frac{1}{4Rr}\{3s^2-4s^2+2(s^2-4Rr-r^2)\}$$

$$=\frac{s^4}{1Rr(3s^2-4s^2+s^2+4Rr+r^2)}-\frac{s^2-8Rr-2r^2}{4Rr}=$$

$$=\frac{s^4-3(4Rr+r^2)(s^2-8Rr-2r^2)}{12Rr(4Rr+r^2)}\overset{Gerretsen}{\ge}\frac{s^2(16Rr-5r^2-12Rr-3r^2)+6r^2(4R+r)^2}{12R(4R+r)r^2}$$

$$\overset{Gerretsen}{\ge}\frac{(16Rr-5r^2)(4Rr-8r^2)+6r^2(4R+r)^2}{12R(4R+r)r^2}\overset{?}{\ge}\frac{9}{4}\Leftrightarrow 52R^2-127Rr+46r^2\overset{?}{\ge}0$$

$$\Leftrightarrow (R-2r)(52R-23r)\overset{?}{\ge}0\rightarrow \text{true}\because R\overset{Euler}{\ge}2r\quad \textit{(proved)}$$

Solution 3 by Rade Krenkov-Strumica-Macedonia

From $\sin\left(\frac{A}{2}+\frac{B}{2}\right)=\cos\frac{C}{2}\Leftrightarrow\frac{\sin\frac{A}{2}}{\tan\frac{B}{2}}+\cos\frac{A}{2}=\frac{\cos\frac{C}{2}}{\sin\frac{B}{2}}|^2$

$$2\left(\frac{\sin^2\frac{A}{2}}{\tan^2\frac{B}{2}}+\cos^2\frac{A}{2}\right)\ge\left(\frac{\sin\frac{A}{2}}{\tan\frac{B}{2}}+\cos\frac{A}{2}\right)^2=\frac{\cos^2\frac{C}{2}}{\sin^2\frac{B}{2}}.$$ *Now,*

$$2LHS+2\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)\ge\frac{\cos^2\frac{A}{2}}{\sin^2\frac{C}{2}}+\frac{\cos^2\frac{B}{2}}{\sin^2\frac{A}{2}}+\frac{\cos^2\frac{C}{2}}{\sin^2\frac{B}{2}}\quad (1)$$

Using equality $\tan\frac{A}{2}\cdot\tan\frac{B}{2}+\tan\frac{B}{2}\cdot\tan\frac{C}{2}+\tan\frac{C}{2}\cdot\tan\frac{A}{2}=1$ *and Cauchy-Schwarz inequality we get:* $\left(\frac{\cos^2\frac{A}{2}}{\sin^2\frac{C}{2}}+\frac{\cos^2\frac{B}{2}}{\sin^2\frac{A}{2}}+\frac{\cos^2\frac{C}{2}}{\sin^2\frac{B}{2}}\right)\left(\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}\cdot\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}+\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}\cdot\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}\cdot\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)\ge$

$$\ge\left(\sqrt{\frac{\sin A}{\sin C}}+\sqrt{\frac{\sin B}{\sin A}}+\sqrt{\frac{\sin C}{\sin B}}\right)^2$$

We have that $\frac{\cos^2\frac{A}{2}}{\sin^2\frac{C}{2}}+\frac{\cos^2\frac{B}{2}}{\sin^2\frac{A}{2}}+\frac{\cos^2\frac{C}{2}}{\sin^2\frac{B}{2}}\ge 9$ ***(2)***

From Jensen's inequality on a function $f(x)=\cos^2\frac{x}{2}$ *which is concave* $(\cap)$ *we have:*

$$3\cos^2\left(\frac{A+B+C}{6}\right)\ge\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}$$

$$\frac{9}{4}\ge\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\quad (3)$$

Using (1), (2) and (3) we get: $2LHS+2\cdot\frac{9}{4}\ge 9;\ LHS\ge\frac{9}{4}$

Solution 4 by Lahiru Samarakoon-Sri Lanka

For $\Delta ABC,\ \frac{\sin^2\frac{A}{2}}{\tan^2\frac{B}{2}}+\frac{\sin^2\frac{B}{2}}{\tan^2\frac{C}{2}}+\frac{\sin^2\frac{C}{2}}{\tan^2\frac{A}{2}}\ge\frac{9}{4}$

$$LHS=\sum\frac{\sin^2\frac{A}{2}\cdot\cos^2\frac{B}{2}}{\sin^2\frac{B}{2}}=\sum\frac{\sin^2\frac{A}{2}}{\sin^2\frac{B}{2}}\left(1-\sin^2\frac{B}{2}\right)=\sum\left(\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}}\right)^2-\sum\sin^2\frac{A}{2}$$

Let's consider, $\left(\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}}\right)^2+\left(\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}}\right)^2+\left(\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}\right)^2$*. Using AM-GM*

$$2\left(\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}}\right)^2+\left(\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}\right)^2\ge 3\sqrt[3]{\frac{\sin^4\frac{A}{2}\cdot\sin^2\frac{B}{2}}{\sin^4\frac{B}{2}\cdot\sin^2\frac{C}{2}}}=\frac{3\sin^2\frac{A}{2}}{\sqrt[3]{\sin^2\frac{A}{2}\cdot\sin^2\frac{B}{2}\cdot\sin^2\frac{C}{2}}}\ge 12\sin^2\frac{A}{2}\quad (1)\ \left(\because\prod\sin\frac{A}{2}\le\frac{1}{8}\right)$$

Similalry, $2\left(\frac{\cos^2\frac{B}{2}}{\sin^2\frac{C}{2}}\right)+\left(\frac{\sin^2\frac{C}{2}}{\cos^2\frac{A}{2}}\right)\ge 12\sin^2\frac{B}{2}$ ***(2)***

$$2\left(\frac{\sin^2\frac{C}{2}}{\cos^2\frac{A}{2}}\right)+\left(\frac{\sin^2\frac{A}{2}}{\sin^2\frac{B}{2}}\right)\ge 12\sin^2\frac{C}{2}\quad (3)$$

$$(1)+(2)+(3), \sum\left(\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}}\right)^2 \geq 4\sum\sin^2\frac{A}{2}$$

$$\therefore LHS \geq 4\sum\sin^2\frac{A}{2} - \sum\cos^2\frac{A}{2} = 3\sum\sin^2\frac{A}{2}$$

But, $\sum\sin^2\frac{A}{2} \geq \frac{3}{4}$. So, $LHS \geq 3 \times \frac{3}{4} = \frac{9}{4}$ (proved)

SP.145. If $1 < a \leq b$ then:

$$\int_a^b\int_a^b\int_a^b \frac{dx\,dy\,dz}{1+\sqrt[3]{xyz}} \leq \log\left(\sqrt[3]{\frac{b+1}{a+1}}\right)^{(b-a)^2}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

Let $x = e^m, y = e^n$ and $e^p = z$ where $m, n, p > 0$

Let $f(m) = \frac{1}{1+e^m}$ for all $m > 0, f'(m) = -\frac{e^m}{(1+e^m)^2}, f''(m) = \frac{e^m(e^m-1)}{(1+e^m)^3} > 0$

hence f is convexe function, $\therefore \sum_{cyc}\frac{1}{1+e^m} \geq \frac{3}{1+e^{\frac{m+n+p}{3}}}$

$$\Rightarrow \sum_{cyc}\frac{1}{1+x} \geq \frac{3}{1+\sqrt[3]{xyz}} \Rightarrow \frac{1}{3}\sum_{cyc}\int_a^b\int_a^b\int_a^b \frac{1}{1+x}dx \geq \int_a^b\int_a^b\int_a^b \frac{dx\,dy\,dz}{1+\sqrt[3]{xyz}}$$

$$\Rightarrow \frac{(a-b)^2}{3}\sum_{cyc}[\log(x+1)]_{x=a}^{x=b} \geq \int_a^b\int_a^b\int_a^b \frac{dx\,dy\,dz}{1+\sqrt[3]{xyz}}$$

$$\Rightarrow \log\left(\frac{b+1}{a+1}\right)^{(a-b)^2} \geq \int_a^b\int_a^b\int_a^b \frac{dx\,dy\,dz}{1+\sqrt[3]{xyz}}$$

SP.146. Let be $A, B \in M_3(\mathbb{R})$ such that:

$$AB = \begin{pmatrix} 2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

Find: $det((BA)^2 - 3I_3)$

Proposed by Marian Ursărescu – Romania

Solution by Ravi Prakash-New Delhi-India

$$AB = \begin{pmatrix} 2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\det(AB) = 2 \Rightarrow \det(A)\det(B) = 2 \neq 0 \Rightarrow \det(A) \neq 0, \det(B) \neq 0$$

$\therefore A^{-1}, B^{-1}$ ***both exist. Now,*** $(BA)^2 - 3I_3 = A^{-1}A((BA)^2 - 3I_3)BB^{-1} =$

$= A^{-1}[A(BA)^2B - 3AB]B^{-1} = A^{-1}[(AB)^3 - 3(AB)]B^{-1}$ ***(1)***

Characteristic equation of AB

$$\begin{vmatrix} 2-t & 1 & 1 \\ 0 & -1-t & 1 \\ 0 & 0 & -1-t \end{vmatrix} = 0 \Rightarrow (1+t)^2(2-t) = 0 \Rightarrow (1+2t+t^2)(2-t) = 0 \Rightarrow$$

$\Rightarrow 2 + 4t + 2t^2 - t - 2t^2 - t^3 = 0$ ***or*** $2 + 3t - t^3 = 0$***. As*** AB ***satisfies this equation***

$2I_3 = (AB)^3 - (3AB)$ ***(2)***

From (1), (2): $(BA)^2 - 3I_3 = A^{-1}(2I_3)B^{-1} = 2A^{-1}B^{-1}$

$$\det((BA)^2 - 3I_3) = 8\det(A^{-1})\det(B^{-1}) = \frac{8}{\det(A)\det(B)} = \frac{8}{\det(BA)} = \frac{8}{2} = 4$$

SP.147. Find all continuous functions $f: \mathbb{R} \to \mathbb{R}$ having the property:

$$f(x) + 2f(2x) + f(4x) = 25x^2 + 9x + 4, \forall x \in \mathbb{R}$$

Proposed by Marian Ursărescu – Romania

Solution 1 by Rovsen Pirguliyev-Sumgait-Azerbaijan

$f(x) + 2f(2x) + f(4x) = 25x^2 + 9x + 4$ ***(1)*** $\Rightarrow [f(x) - (x^2 + x + 1)] +$

$+2[f(2x) - (4x^2 + 2x + 1)] + [f(4x) - (16x^2 + 4x + 1)] = 0$***. Let*** $g(x) = f(x) -$

$-(x^2 + x + 1)$***. Then we have:*** $g(x) + 2g(2x) + g(4x) = 0$ ***(2)***

(2) $\overset{x=0}{\Rightarrow} g(0) = 0$ ***(3)***

(2) $\Rightarrow g(4x) + g(2x) = -\big(g(2x) + g(x)\big) \Rightarrow$

$$g(2x) + g(x) = -\left(g(2^{1-n}x) + g(2^{-n}x)\right)$$

$\overset{n\to+\infty}{\Rightarrow}$ ***using continuity*** $\Rightarrow g(2x) + g(2x) = 0 \Rightarrow g(2x) = -g(x)$ ***and so***

$g(x) = -g(2^{-n}x); g(x) = -g(2^{-n}x) \overset{n\to\infty}{\Rightarrow}$ ***and using continuity, we have***

$g(x) = -g(0) \stackrel{(3)}{=} 0$. ***Hence*** $g(x) = 0 \Rightarrow f(x) - (x^2 + x + 1) = 0 \Rightarrow f(x) = x^2 + x + 1.$

Solution 2 by Sagar Kumar-Kolkata-India

$$f(x) + 2f(2x) + f(4x) = (x^2 + 2x + 1) + 2((2x)^2 + (2x) + 1) + ((4x)^2 + (4x) + 1)$$

$$\Rightarrow \left(f(x) - (x^2 + x + 1)\right) + 2\left(f(2x) - ((2x)^2 + 2x + 1)\right) + f(4x) -$$

$$-((4x)^2 + 4x + 1) = 0 \text{ on comparing } f(x) = x^2 + x + 1$$

Solution 3 by Tran Hong-Vietnam

Let $g(x) = f(x) - [x^2 + x + 1]; \forall x \in \mathbb{R}$ ***since*** f ***continuous*** $\Rightarrow g$ ***continuos on*** $\mathbb{R} \Rightarrow$

$$\Rightarrow g(x) + 2g(2x) + g(ax) = 0, \forall x \in \mathbb{R} \Rightarrow g\left(\frac{x}{2^{n+2}}\right) + 2g\left(\frac{x}{2^{n+1}}\right) + g\left(\frac{x}{2^n}\right) = 0, \forall x \in \mathbb{R}$$

$$\Rightarrow \lim_{n\to\infty}\left[g\left(\frac{x}{2^{n+2}}\right) + 2g\left(\frac{x}{2^{n+1}}\right) + g\left(\frac{x}{2^n}\right)\right] = 0; (\forall)x \in \mathbb{R} \Rightarrow g(0) + 2g(0) + g(0) = 0 \Rightarrow$$

$$\Rightarrow 3g(0) = 0 \Rightarrow g(0) = 0; [g(x) + g(2x)] + [g(2x) + g(x)] = 0;$$

Let $h(x) = g(x) = g(x); h(0) = 0 \Rightarrow h(x) + h(2x) = 0 \Rightarrow$

$$h(2x) = (-1)^n h\left(\frac{x}{2^n}\right); \forall n \in \mathbb{N}$$

$$\Rightarrow h(x) = \lim_{n\to\infty}(-1)6n\, h\left(\frac{x^2}{2^n}\right) = 0 \Rightarrow g(2x) = \lim_{n\to\infty}(-1)^n g\left(\frac{x}{2^n}\right) = 0$$

$$f(x) = x^2 + x + 1$$

Solution 4 by Chris Kyriazis-Greece

Set $g(x) = f(x) - x^2 - x - 1, \forall x \in \mathbb{R}$. ***Then, the given functional equation becomes:***

$$g(x) + 2g(2x) + g(4x) = 0 \quad (1)$$

If we put $x = 0$, ***then*** $g(0) = 0$. ***Setting*** $h(x) = g(x) + g(2x)$ ***(1)***

Becomes $h(x) + h(2x) = 0, \forall x \in \mathbb{R}$ ***or (putting*** $x \to x/2$ ***):*** $h(x) = -h\left(\frac{x}{2}\right)$

Once more and $h(x) = h\left(\frac{x}{4}\right) = h\left(\frac{x}{2^2}\right)$. ***By induction, we can show that:***

$h(x) = (-1)^n h\left(\frac{x}{2^n}\right)$. ***Using continuity of*** h ***and taking limits, we have that:***

$$\lim_{n\to+\infty} h(x) = \lim_{n\to\infty}(-1)^n h\left(\frac{x}{2^n}\right) = 0 \Rightarrow h(x) = 0, \forall x \in \mathbb{R}.$$

This means $g(x) + g(2x) = 0$

$\forall x \in \mathbb{R}$. ***Working the same way,*** $g(x) = 0, \forall x \in \mathbb{R}$. ***So,*** $f(x) = x^2 + x + 1, \forall x \in \mathbb{R}$ ***which satisfies the given equation.***

SP.148. Let be $x_0 > 0$ **and** $x_{n+1} = \arctan\frac{x_n}{1+x_n}, \forall n \in \mathbb{N}$.

Find:

$$\Omega = \lim_{n\to\infty}(n \cdot x_n)$$

Proposed by Marian Ursărescu – Romania

Solution by Remus Florin Stanca-Romania

We prove by using Mathematical induction that $x_n > 0, \forall n \in \mathbb{N}$.

1) We prove that $P(0): x_0 > 0$ ***is true (true)***

2) We suppose that $P(n): x_n > 0$ ***is true.***

3) We prove by using $P(n)$ ***that*** $P(n+1): x_{n+1} > 0$ ***is true.***

$$x_n > 0 \Rightarrow x_{n+1} \in \left(0; \frac{\pi}{2}\right] \Rightarrow x_{n+1} > 0 \Rightarrow x_n > 0; \forall n \in \mathbb{N}$$

$x_{n+1} = \arctan\frac{x_n}{x_n+1}$ ***and because*** $x_n > 0; \forall n \in \mathbb{N} \Rightarrow x_n \in \left(0; \frac{\pi}{2}\right]$

We study the sign of $x_1 - x_0 = \arctan\frac{x_0}{x_0+1} - x_0$

Let $f: \left(0; \frac{\pi}{2}\right] \to \mathbb{R}$ ***such that*** $f(x) = \arctan\frac{x}{x+1} - x$

$$\Rightarrow f'(x) = \frac{1}{1 + \frac{x^2}{(x+1)^2}} \cdot \frac{1}{(x+1)^2} - 1 = \frac{1}{2x^2 + 2x + 1} - 1 \Rightarrow f'(x) < 0$$

$\Rightarrow f(x)$ ***is a decreasing function*** $f(0) = 0 \Rightarrow f(x) < 0$ ***for*** $x \in \left(0, \frac{\pi}{2}\right] \Rightarrow x_1 < x_0$

We prove by using the Mathematical induction that $x_n > x_{n+1}$

1) We proved that $P(0): x_0 > x_1$ ***is true***

2) We suppose that $P(n): x_n > x_{n+1}$ ***is true***

3) We prove that $P(n+1): x_{n+1} > x_{n+2}$ ***is true by using*** $P(n)$

$$x_{n+1} - x_{n+2} = \arctan\frac{x_n}{x_n + 1} - \arctan\frac{x_{n+1}}{x_{n+1} + 1}$$

We prove that the function $f: \left(0; \frac{\pi}{2}\right] \to \mathbb{R}, f(x) = \arctan\frac{x}{x+1}$ ***is an increasing function.***

$$f(x)=\frac{1}{1+\frac{x^2}{(x+1)^2}}\cdot\frac{1}{(x+1)^2}>0\Rightarrow$$ ***true* so** $x_{n+1}-x_{n+2}>0$

$\succ x_{n+1}>x_{n+2}\succ x_n>x_{n+1}$ ***for*** $n\in\mathbb{N}$.

$x_n\in\left(0;\frac{\pi}{2}\right]$ ***and*** x_n ***is a decreasing sequence*** $\Rightarrow -l=\lim_{n\to\infty}x_n$ ***such that*** $l\in\mathbb{R}$

$\Rightarrow l=\arctan\frac{l}{l+1}, l\in\left(0,\frac{\pi}{2}\right]$ ***the funciton*** $f(l)=\arctan\frac{l}{l+1}-l$ ***is a decreasing function***

so $l=0$ ***is the unique solution.***

$$\Rightarrow\lim_{n\to\infty}x_n=0\Rightarrow\lim_{n\to\infty}nx_n=\lim_{n\to\infty}\frac{n}{\frac{1}{x_n}}\overset{Stolz-Cesaro}{=}\lim_{n\to\infty}\frac{1}{\frac{1}{x_{n+1}}-\frac{1}{x_n}}=$$

$$=\lim_{n\to\infty}\frac{x_{n+1}x_n}{x_n-x_{n+1}}=\lim_{n\to\infty}\frac{\arctan\frac{x_n}{x_n+1}x_n}{x_n-\arctan\frac{x_n}{x_n+1}}$$

$$\lim_{n\to 0}\frac{\arctan\frac{x}{x+1}\cdot x}{x-\arctan\frac{x}{x+1}}\overset{L'H}{\underset{\frac{0}{0}}{=}}\lim_{x\to 0}\frac{\arctan\frac{x}{x+1}+x\cdot\frac{1}{1+\frac{x^2}{(x+1)^2}}\cdot\frac{1}{(x+1)^2}}{1-\frac{1}{1+\frac{x^2}{(x+1)^2}\cdot\frac{1}{(x+1)^2}}}=$$

$$=\lim_{x\to 0}\frac{\left(\arctan\frac{x}{x+1}\right)(2x^2+2x+1)-x}{2x^2+2x}=\lim_{x\to 0}\frac{\arctan\frac{x}{x+1}}{\frac{x}{x+1}\cdot(x+1)(2x+2)}(2x^2+2x+1)=$$

$$=\frac{1}{2x+2}=\frac{1}{2}-\frac{1}{2}=0$$

SP.149. Let be the sequence $(x_n)_{n\in\mathbb{N}}: x_0>1$ and $x_{n+1}=1+\ln\left(\frac{2x_n}{1+x_n}\right)$,

$\forall n\in\mathbb{N}$. Find:

$$\lim_{n\to\infty}(n\ln x_n)$$

Proposed by Marian Ursărescu – Romania

Solution by Remus Florin Stanca-Romania

We prove that $x_n>1,\forall n\in\mathbb{N}$ ***by using the Mathematical induction:***

1) we prove $P(0): x_0>1$ ***(true)***

2) we suppose that $P(n): x_n > 1$ ***is true***

3) we prove $P(n+1): x_{n+1} > 1$ ***by using*** $P(n)$

$$x_n > 1 \Rightarrow \frac{2x_n}{x_n+1} > 1 \Rightarrow \ln\left(\frac{2x_n}{x_n+1}\right) + 1 > 1 \Rightarrow x_{n+1} > 1 \Rightarrow x_n > 1\ \forall n \in \mathbb{N}$$

We study the sign of $x_1 - x_0 = 1 + \ln\left(\frac{2x_0}{1+x_0}\right) - x_0$

Let $f: (1; +\infty) \to \mathbb{R}; f(x) = 1 + \ln\left(\frac{2x}{1+x}\right) - x$

$f'(x) = \frac{1+x}{2x} \cdot \frac{2}{(x+1)^2} - 1 = \frac{1}{x(x+1)} - 1 < 0 \Rightarrow f$ ***is a decreasing function***

$f(1) = 0 \succ f(x) < 0$ ***for*** $x > 1 \succ x_1 < x_0$

$g(x) = \frac{2x}{2x+1}$ ***is an increasing function so*** $x_{n+1} < x_n$

$x_{n+1} < x_n$ ***and*** $x_n > 1 \succ l = \lim_{n\to\infty} x_n \in \mathbb{R}$

$l = 1 + \ln\left(\frac{2l}{l+1}\right) \Rightarrow f(l) = 1 + \ln\left(\frac{2l}{l+1}\right) - f$ ***is a decreasing function*** $\Rightarrow$

$l = 1$ ***is an unique solution*** $\Rightarrow \lim_{n\to\infty} x_n = 1$

$$\lim_{n\to\infty} n \ln x_n = \lim_{n\to\infty} \frac{n}{\frac{1}{\ln x_n}} = \lim_{n\to\infty} \frac{1}{\frac{1}{\ln x_{n+1}} - \frac{1}{\ln x_n}} =$$

$$= \lim_{n\to\infty} \frac{1}{\frac{1}{\ln\left(1 + \ln\frac{2x_n}{x_n+1}\right)} - \frac{1}{\ln x_n}} = \lim_{n\to\infty} \frac{\ln x_n \ln\left(1 + \ln\frac{2x_n}{x_n+1}\right)}{\ln x_n - \ln\left(1 + \ln\frac{2x_n}{x_n+1}\right)}$$

$$\lim_{x\to1} \frac{\ln x \ln\left(1 + \ln\frac{2x}{x+1}\right)}{\ln x - \ln\left(1 + \ln\frac{2x}{x+1}\right)} = \lim_{x\to1} \frac{\frac{\ln x}{x-1} \cdot (x-1) \cdot \frac{\ln\left(1+\ln\frac{2x}{x+1}\right)}{\ln\frac{2x}{x+1}} \cdot \ln\frac{2x}{x+1}}{\ln\left(\frac{x}{1+\ln\frac{2x}{x+1}} - 1 + 1\right)} \cdot$$

$$\cdot \frac{x - 1 - \ln\frac{2x}{x+1}}{1 + \ln\frac{2x}{x+1}} \cdot \frac{1 + \ln\frac{2x}{x+1}}{x - 1 - \ln\frac{2x}{x+1}} = \lim_{x\to1} \frac{(x-1)\ln\frac{2x}{x+1}}{x - 1 - \ln\frac{2x}{x+1}} = \lim_{x\to1} \frac{(x-1)\ln\left(\frac{x-1}{x+1} + 1\right)}{x - 1 - \ln\frac{2x}{x+1}}$$

$$= \lim_{x\to 1}\frac{(x-1)\cdot\frac{\ln\left(\frac{x-1}{x+1}+1\right)}{\frac{x-1}{x+1}}}{x+1-\frac{\ln\left(\frac{x-1}{x+1}+1\right)}{\frac{x-1}{x+1}}} = \lim_{x\to 1}\frac{x-1}{x} = 0 \Rightarrow \lim_{n\to\infty} n\ln x_n = 0$$

SP.150. Let be $f \in \mathbb{Z}, f = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$, such that $a_1, a_2, \dots, a_n \in \{\pm 1, \pm 2, \dots, \pm n\}$. If a_0 is a prime number, $a_0 > n^2$ then f is irreducible over $\mathbb{Z}$.

Proposed by Marian Ursărescu – Romania

Solution by proposer

Suppose $\exists g, h \in \mathbb{Z}$ *such that* $f = g\cdot h$, *grade* $f, h \ge 1$. $f(0) = g(0) = h(0) \Rightarrow$

$\Rightarrow a_0 = g(0)\cdot h(0)$. *But* a_0 *being prime* $\Rightarrow g(0) = 1$ *or* $h(0) = 1$.

Suppose $g(0) = 1 \Rightarrow g(x) = b_k x^k + \dots + b_1 x + 1$

Let be $x_1, x_2, \dots, x_k$ *the roots of* f. *From the last Viete relationship* $\Rightarrow$

$$\Rightarrow |x_1 x_2 \dots x_n| = \left|\frac{(-1)^k}{b_k}\right| = \frac{1}{|b_k|} \le 1, \text{ because } b_k \in \mathbb{Z}$$

$|x_1 x_2 \dots x_k| \le 1 \Rightarrow \exists p \in \{1, 2, \dots, k\}$ *such that* $|x_p| \le 1$.

But x_p *is root and for* $f \Rightarrow a_n x_p^n + a_{n-1}x_p^{n-1} + \dots + a_1 x_p + a_0 = 0 \Rightarrow$

$$\Rightarrow |a_0| = |a_n x_p^n + a_{n-1}x_p^{n-1} + \dots + a_1 x_p| \le |a_n||x_p|^n + \dots + |a_1||x_p| \le$$

$$\le |a_1| + |a_2| + \dots + |a_n| \Rightarrow |a_0| \le |a_1| + |a_2| + \dots + |a_n| \le n^2 \Rightarrow a_0 \le n^2 \textit{ false} \Rightarrow$$

$\Rightarrow f$ *is irreducible over* $\mathbb{Z}$

UP.136. Prove that:

$$\sum_{k=0}^{n} T_{4k}(x) = \frac{1}{4}\left[\frac{2 + U_{4n+2}(x)}{x\sqrt{1-x^2}}\right]$$

where, $T_n(x)$ and $U_n(x)$ denotes the Chebyshev Polynomials of First and Second Kind.

Proposed by Shivam Sharma-New Delhi-India

Solution by proposer

$$S=\sum_{k=0}^{n}T_{4k}(x)$$

Let, $x=\cos(\theta)$***. Then, by definition, we have,***

$$\Rightarrow\sum_{k=0}^{n}\cos(4k\cos^{-1}(x))\Rightarrow\sum_{k=0}^{n}\cos(4k\cos^{-1}(\cos\theta))\Rightarrow\sum_{k=0}^{n}\cos(4k\theta)\Rightarrow$$

$\Rightarrow$ ***Real Part*** $\sum_{k=0}^{n}e^{4ki\theta}\Rightarrow 1+e^{4i\theta}+e^{8i\theta}+\cdots+(n+1)$ ***terms*** $\Rightarrow\frac{1-\left(e^{4i\theta}\right)^{n+1}}{1-e^{4i\theta}}\Rightarrow$

$$\Rightarrow\frac{\left(1-e^{i(4n+4)\theta}\right)\left(1-e^{-4i\theta}\right)}{\left(1-e^{4i\theta}\right)\left(1-e^{-4i\theta}\right)}\Rightarrow\frac{\left(1-e^{i(4n+4)\theta}\right)\left(1-e^{-4i\theta}\right)}{2\left(e^{4i\theta}+e^{-4i\theta}\right)}\Rightarrow$$

$$\Rightarrow\frac{(1-\cos(4n+4)\theta)-i\sin\big((4n+4)\theta\big)\,(1-(\cos(4\theta))+i\sin(4\theta))}{2-2\cos(4\theta)}\Rightarrow$$

$$\Rightarrow\frac{1-\cos\big((4n+4)\theta\big)-\cos(4\theta)+\cos\big((4n+4)\theta\big)\cos(4\theta)+\sin\big((4n+4)\theta\big)\sin(4\theta)}{2(1-\cos(4\theta))}$$

$$\Rightarrow\frac{(1-\cos(4\theta))-\cos\big((4n+4)\theta\big)\,(1-\cos(4\theta))+\sin\big((4n+4)\theta\big)\sin(4\theta)}{2(1-\cos(4\theta))}$$

$$\Rightarrow\frac{1}{2}\left[1-\cos\big((4n+4)\theta\big)+\frac{\sin\big((4n+4)\theta\big)\sin(4\theta)}{1-\cos(4\theta)}\right]$$

$$\Rightarrow\frac{1}{2}\left[1-\cos\big((4n+4)\theta\big)+\frac{\sin\big((4n+4)\theta\big)\,2\sin(2\theta)\cos(2\theta)}{2\sin^{2}(2\theta)}\right]$$

$$\Rightarrow\frac{1}{2}\left[\frac{1+\sin\big((4n+4)\theta\big)\cos(2\theta)-\cos\big((4n+4)\theta\big)\sin(2\theta)}{\sin(2\theta)}\right]$$

$$\Rightarrow\frac{1}{2}\left[1+\frac{\sin\Big(\big((4n+4)\theta\big)-2\theta\Big)}{\sin(2\theta)}\right]\Rightarrow\frac{1}{2}\left[1+\frac{\sin\big((4n+2)\theta\big)}{\sin(2\theta)}\right]\Rightarrow$$

$$\Rightarrow\frac{1}{2}\left[1+\frac{\sin\big((4n+2)\theta\big)}{2\left[\cos(\theta)\sqrt{1-\cos^{2}\theta}\right]}\right]$$

As, $x=\cos\theta\Rightarrow U_{4n+2}(x)=\sin\Big((4n+2)\cos^{-1}(\cos\theta)\Big)$***, so,***

$U_{4n+2}(x)=\sin\big((4n+2)\theta\big)$***. Using above, we get,***

$$S=\frac{1}{4}\left[\frac{2+U_{4n+2}(x)}{x\sqrt{1-x^2}}\right] \text{ (Answer)}$$

UP.137. Let $f,g:\mathbb{R}_+^*\to\mathbb{R}_+^*$ be functions such that:

$$\lim_{x\to\infty}\left(f(x+1)-f(x)\right)=a\in\mathbb{R}_+^*, \lim_{x\to\infty}\frac{g(x+1)}{xg(x)}=b\in\mathbb{R}_+^*$$

and exists $\lim_{x\to\infty}\frac{f(x)}{x}$ and $\lim_{x\to\infty}\frac{(g(x))^{\frac{1}{x}}}{x}$. For $t\in\mathbb{R}$ calculate the limit:

$$\lim_{x\to\infty}\left(f(x)\right)^{\cos^2 t}\left(\left(g(x)\right)^{\frac{\sin^2 t}{x+1}}-g((x))^{\frac{\sin^2 t}{x}}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution by Shafiqur Rahman-Bangladesh

$$\lim_{n\to\infty}\frac{f(x)}{x}=\lim_{n\to\infty}\left(f(x+1)-f(x)\right)=a;\ \lim_{n\to\infty}\frac{\left(g(x)\right)^{\frac{1}{x}}}{x}=\lim_{n\to\infty}\frac{g(x+1)x^x}{g(x)\cdot(x+1)^{x+1}}=\frac{b}{e}$$

$$\textit{Now, } \lim_{n\to\infty}\left(f(x)\right)^{\cos^2 t}\left(\left(g(x)\right)^{\frac{\sin^2 t}{x+1}}-\left(g(x)\right)^{\frac{\sin^2 t}{x}}\right)=$$

$$=\lim_{n\to\infty}\left(\frac{f(x)}{x}\right)^{\cos^2 t}\cdot x^{\cos^2 t}\left(\frac{\left(g(x)^{\frac{1}{x}}\right)}{x}\right)^{\sin^2 t}\cdot x^{\sin^2 t}\left(\left(g(x)\right)^{-\frac{\sin^2 t}{x(x+1)}}-1\right)=$$

$$=a^{\cos^2 t}\cdot\left(\frac{b}{e}\right)^{\sin^2 t}\cdot\lim_{n\to\infty}\left(\frac{e^{-\frac{\sin^2 t}{x(x+1)}\ln(g(x))}-1}{-\frac{\sin^2 t}{x(x+1)}\ln(g(x))}\cdot\ln\left(g(x)\right)^{-\frac{\sin^2 t}{x+1}}\right)=a^{\cos^2 t}\cdot\left(\frac{b}{e}\right)^{\sin^2 t}\cdot 1\cdot\ln 0$$

$$\therefore \lim_{n\to\infty}\left(f(x)\right)^{\cos^2 t}\left(\left(g(x)\right)^{\frac{\sin^2 t}{x+1}}-\left(g(x)\right)^{\frac{\sin^2 t}{x}}\right)=-\infty$$

Note: $\lim_{n\to\infty}\frac{\sin^2 t}{x(x+1)}\ln\left(g(x)\right)=\lim_{n\to\infty}\frac{\sin^2 t}{x+1}\ln\left(\frac{bx}{e}\right)=0$ *and* $\lim_{n\to\infty}\left(g(x)\right)^{-\frac{\sin^2 t}{x+1}}=$

$$\lim_{n\to\infty}\left(\frac{bx}{e}\right)^{-\frac{x\sin^2 t}{x+1}}=0$$

$$\lim_{n\to\infty}\frac{f(x)}{x}=\lim_{n\to\infty}\left(f(x+1)-f(x)\right)=a;\ \lim_{n\to\infty}\frac{\left(g(x)\right)^{\frac{1}{x}}}{x}=\lim_{n\to\infty}\frac{g(x+1)x^x}{g(x)\cdot(x+1)^{x+1}}=\frac{b}{e}$$

Now,

$$\lim_{n\to\infty}(f(x))^{\cos^2 t}\left((g(x))^{\frac{\sin^2 t}{x+1}}-(g(x))^{\frac{\sin^2 t}{x}}\right)=\lim_{n\to\infty}\left(\frac{f(x)}{x}\right)^{\cos^2 t}\cdot x^{\cos^2 t}\cdot\left(\frac{(g(x))^{\frac{1}{x}}}{x}\right)^{\sin^2 t}\cdot$$

$$\cdot x^{\sin^2 t}\left((g(x))^{-\frac{\sin^2 t}{x(x+1)}}-1\right)=a^{\cos^2 t}\left(\frac{b}{e}\right)^{\sin^2 t}\cdot x\cdot\lim_{n\to\infty}\left(\left(\frac{bx}{e}\right)^{-\frac{\sin^2 t}{x+1}}-1\right)=$$

$$=a^{\cos^2 t}\cdot\left(\frac{b}{e}\right)^{\sin^2 t}\lim_{n\to\infty}\left(-\frac{x\sin^2 t}{x+1}\ln\left(\frac{bx}{e}\right)+0\left(\frac{\ln^2\left(\frac{bx}{e}\right)}{x+1}\right)\right)=-a^{\cos^2 t}\cdot\left(\frac{b}{e}\right)^{\sin^2 t}\lim_{n\to\infty}\frac{x\sin^2 t}{x+1}\ln\left(\frac{bx}{e}\right)$$

$$\therefore\lim_{n\to\infty}(f(x))^{\cos^2 t}\left((g(x))^{\frac{\sin^2 t}{x+1}}-(g(x))^{\frac{\sin^2 t}{x}}\right)=-\infty$$

$$\lim_{n\to\infty}\frac{f(x)}{x}=\lim_{n\to\infty}(f(x+1)-f(x))=a;\ \lim_{n\to\infty}\frac{(g(x))^{\frac{1}{x}}}{x}=\lim_{n\to\infty}\frac{g(x+1)\cdot x^x}{g(x)(x+1)^{x+1}}=\frac{b}{e}$$

Now, $\lim_{n\to\infty}(g(x))^{\cos^2 t}\left(g(x+1)^{\frac{\sin^2 t}{x+1}}-(g(x))^{\frac{\sin^2 t}{x}}\right)=$

$$=\lim_{n\to\infty}\left(\frac{f(x)}{x}\right)^{\cos^2 t}\cdot\left((x+1)^{\sin^2 t}\left(\frac{g(x+1)}{(x+1)^{x+1}}\right)^{\frac{\sin^2 t}{x+1}}-x^{\sin^2 t}\left(\frac{g(x)}{x^x}\right)^{\frac{\sin^2 t}{x}}\right)=$$

$$=a^{\cos^2 t}\cdot\sin^2 t\cdot\lim_{n\to\infty}\left(\frac{\frac{g(x+1)}{(x+1)^{x+1}}}{\frac{g(x)}{x^x}}\right)^{\sin^2 t}$$

$$\therefore\lim_{n\to\infty}(f(x))^{\cos^2 t}\left((g(x))^{\frac{\sin^2 t}{x+1}}-(g(x))^{\frac{\sin^2 t}{x}}\right)=a^{\cos^2 t}\cdot\sin^2 t\left(\frac{b}{e}\right)^{\sin^2 t}$$

UP.138. Let $f,g:\mathbb{R}_+^*\to\mathbb{R}_+^*$ such that: $\lim_{x\to\infty}(f(x+1)-f(x))\ a\in\mathbb{R}_+^*$, $\lim_{x\to\infty}\frac{g(x+1)}{xg(x)}=b\in\mathbb{R}_+^*$ and there is $\lim_{x\to\infty}\frac{f(x)}{x}$, $\lim_{x\to\infty}\frac{(g(x))^{\frac{1}{x}}}{x}$. For $t\in\mathbb{R}$, calculate:

$$\lim_{x\to\infty}(f(x))^{\sin^2 t}\left((g(x))^{\frac{\cos^2 t}{x+1}}-(g(x))^{\frac{\cos^2 t}{x}}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution by proposers

By Cesaro – Stolz theorem we have:

$$\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{\substack{x\to\infty\\ n\to\mathbb{N}^*}}\frac{f(n)}{n}\overset{C-S}{=}\lim_{x\to\infty}\frac{f(n+1)-f(n)}{(n+1)-n}=\lim_{x\to\infty}\big(f(n+1)-f(n)\big)=a,$$

and by Cauchy-D'Alembert theorem we deduce that:

$$\lim_{x\to\infty}\frac{(g(x))^{\frac{1}{x}}}{x}=\lim_{\substack{x\to\infty\\ n\in\mathbb{N}^*}}\frac{(g(n))^{\frac{1}{n}}}{n}=\lim_{x\to\infty}\sqrt[n]{\frac{g(n)}{n^2}}\overset{C-D'A}{=}\lim_{n\to\infty}\left(\frac{g(n+1)}{(n+1)^{n+1}}\cdot\frac{n^n}{g(n)}\right)=$$

$=\lim_{n\to\infty}\left(\frac{g(n+1)}{ng(n)}\left(\frac{n}{n+1}\right)^{n+1}\right)=\frac{b}{e}$. **So,** $\lim_{x\to\infty}(f(x))^{\sin^2 t}\left((g(x))^{\frac{\cos^2 t}{x+1}}-(g(x))^{\frac{\cos^2 t}{x}}\right)=$

$$=\lim_{x\to\infty}\left(\frac{f(x)}{x}\right)^{\sin^2 t}\lim_{x\to\infty}\left(\frac{(g(x))^{\frac{1}{x}}}{x}\right)^{\cos^2 t}(u(x)-1)x^{\sin^2 t+\cos^2 t}=$$

$$=a^{\sin^2 t}\cdot\frac{b^{\cos^2 t}}{e^{\cos^2 t}}\cdot\lim_{x\to\infty}\left(\frac{u(x)-1}{\ln u(x)}\cdot\ln(u(x))^x\right),$$

where $u(x)=\left(\frac{(g(x+1))^{\frac{1}{x+1}}}{(g(x))^{\frac{1}{x}}}\right)^{\cos^2 t}$ **with** $\lim_{x\to\infty}u(x)=1$**, then** $\lim_{x\to\infty}\frac{u(x)-1}{\ln u(x)}=1$**. We have:**

$$\lim_{x\to\infty}(u(x))^x=\lim_{x\to\infty}\left(\frac{(g(x+1))^{\frac{x}{x+1}}}{g(x)}\right)^{\cos^2 t}=\lim_{x\to\infty}\left(\frac{g(x+1)}{g(x)}\cdot\frac{1}{(g(x+1))^{\frac{1}{x+1}}}\right)^{\cos^2 t}=$$

$$=\lim_{x\to\infty}\left(\frac{g(x+1)}{xg(x)}\cdot\frac{x+1}{(g(x+1))^{\frac{1}{x+1}}}\cdot\frac{x}{x+1}\right)^{\cos^2 t}=\left(b\cdot\frac{e}{b}\cdot 1\right)^{\cos^2 t}=e^{\cos^2 t}$$

Therefore: $\lim_{x\to\infty}(f(x))^{\sin^2 t}\left((g(x))^{\frac{\cos^2 t}{x+1}}-(g(x))^{\frac{\cos^2 t}{x}}\right)=\frac{a^{\sin^2 t}\cdot b^{\cos^2 t}}{e^{\cos^2 t}}\cdot\ln e^{\cos^2 t}=$

$=\frac{a^{\sin^2 t}\cdot b^{\cos^2 t}}{e^{\cos^2 t}}\cdot\cos^2 t$**, and we are done.**

UP.139. Calculate: $\lim_{x\to\infty}\left(x^{\cosh^2 t}\left(\left(\Gamma(x+1)\right)^{-\frac{\sinh^2 t}{x}}-\left(\left(\Gamma(x+2)\right)\right)^{-\frac{\sinh^2 t}{x+1}}\right)\right)$,

Where $t\in\mathbb{R}$ and Γ is the Gamma function (Euler integral of the second kind).

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution by proposers

We have: $\lim_{x\to\infty}\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x}=\lim_{\substack{n\to\infty\\ n\in\mathbb{N}^*}}\frac{(\Gamma(n+1))^{\frac{1}{n}}}{n}=\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}}\overset{C-D'A}{=}$

$=\lim_{n\to\infty}\left(\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}\right)=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\frac{1}{e}$. *Let*

$$f(x)=x^{\cosh^2 t}\left(\left(\Gamma(x+1)\right)^{-\frac{\sinh^2 t}{x}}-\left(\Gamma(x+2)\right)^{-\frac{\sinh^2 t}{x+1}}\right)=-x^{\cosh^2 t}\left(\Gamma(x+1)\right)^{-\frac{\sinh^2 t}{x}}(u(x)-1),$$

where $u:\mathbb{R}_+^*\to\mathbb{R}$, $u(x)=\left(\frac{(\Gamma(x+2))^{\frac{1}{x+1}}}{(\Gamma(x+1))^{\frac{1}{x}}}\right)^{-\sinh^2 t}$. *We deduce that:*

$$\lim_{n\to\infty}u(x)=\lim_{n\to\infty}\left(\frac{(\Gamma(x+2))^{\frac{1}{x+1}}}{x+1}\cdot\frac{x}{(\Gamma(x+1))^{\frac{1}{x}}}\cdot\frac{x+1}{x}\right)^{-\sinh^2 t}=\left(\frac{1}{e}\cdot e\cdot 1\right)^{-\sinh^2 t}=1$$

We have, $\lim_{n\to\infty}\frac{u(x)-1}{\ln u(x)}=1$. *Also, we have:*

$$\lim_{x\to\infty}\left(u(x)\right)^x=\lim_{x\to\infty}\left(\frac{(\Gamma(x+2))^{\frac{1}{x+1}}}{(\Gamma(x+1))^{\frac{1}{x}}}\right)^{-x\sin^2 t}=\lim_{x\to\infty}\left(\frac{\Gamma(x+2)}{\Gamma(x+1)}\cdot\frac{1}{(\Gamma(x+2))^{\frac{1}{x+1}}}\right)^{-\sin^2 t}=$$

$=\lim_{x\to\infty}\left(\frac{x+1}{(\Gamma(x+2))^{\frac{1}{x+1}}}\right)^{-\sin^2 t}=e^{-\sin^2 t}$. *Therefore:*

$$\lim_{x\to\infty}f(x)=-\lim_{x\to\infty}\left(x^{ch^2 t}\left(\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x}\cdot x\right)^{-\sin^2 t}\cdot\frac{u(x)-1}{\ln u(x)}\cdot\ln u(x)\right)=$$

$$=-\lim_{x\to\infty}\left(\left(\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x}\right)^{-\sin^2 t}\cdot x^{\cosh^2 t-\sinh^2 t}\cdot\frac{u(x)-1}{\ln u(x)}\cdot\ln u(x)\right)=$$

$$= -\left(\frac{1}{e}\right)^{-\sinh^2 t} \lim_{x\to\infty} \frac{u(x)-1}{\ln u(x)} \cdot \ln\left(\lim_{x\to\infty}(u(x))^x\right) = -e^{\sinh^2 t} \cdot 1 \cdot \ln e^{-\sinh^2 t} = e^{\sinh^2 t} \cdot \sinh^2 t$$

UP.140. Calculate: $\lim_{x\to\infty}\left(x^{\sin^2 t}\left((\Gamma(x+2))^{\frac{\cos^2 t}{x+1}} - (\Gamma(x+1))^{\frac{\cos^2 t}{x}}\right)\right)$, where $t \in \mathbb{R}$ and Γ is the Gamma function (Euler integral of the second kind).

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution by proposers

We have: $\lim_{x\to\infty}\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x} = \lim_{\substack{n\to\infty \\ n\in\mathbb{N}^*}}\frac{(\Gamma(n+1))^{\frac{1}{n}}}{n} = \lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = \lim_{x\to\infty}\sqrt[n]{\frac{n!}{n^n}} \overset{C-D'A}{=}$

$$= \lim_{x\to\infty}\left(\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n!}{n!}\right) = \lim_{x\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{1}{e}.$$ *We denote:*

$$f(x) = x^{\sin^2 t}\left((\Gamma(x+2))^{\frac{\cos^2 t}{x+1}} - (\Gamma(x+1))^{\frac{\cos^2 t}{x}}\right) = x^{\sin^2 t}(\Gamma(x+1))^{\frac{\cos^2 t}{x}}(u(x)-1),$$

where $u: \mathbb{R}_+^* \to \mathbb{R}$, $u(x) = \left(\frac{(\Gamma(x+2))^{\frac{1}{x+1}}}{(\Gamma(x+1))^{\frac{1}{x}}}\right)^{\cos^2 t}$. *We deduce that:*

$$\lim_{x\to\infty} u(x) = \lim_{x\to\infty}\left(\frac{(\Gamma(x+2))^{\frac{1}{x+1}}}{x+1} \cdot \frac{x}{(\Gamma(x+1))^{\frac{1}{x}}} \cdot \frac{x+1}{x}\right)^{\cos^2 t} = \left(\frac{1}{e} \cdot e \cdot 1\right)^{\cos^2 t} = 1.$$

We have, $\lim_{x\to\infty}\frac{u(x)-1}{\ln u(x)} = 1$ *and also we have:*

$$\lim_{x\to\infty}(u(x))^x = \lim_{x\to\infty}\left(\frac{(\Gamma(x+2))^{\frac{1}{x+1}}}{(\Gamma(x+1))^{\frac{1}{x}}}\right)^{x\cos^2 t} = \lim_{x\to\infty}\left(\frac{\Gamma(x+2)}{\Gamma(x+1)} \cdot \frac{1}{(\Gamma(x+2))^{\frac{1}{x+1}}}\right)^{\cos^2 t} =$$

$$= \lim_{x\to\infty}\left(\frac{x+1}{(\Gamma(x+2))^{\frac{1}{x+1}}}\right)^{\cos^2 t} = e^{\cos^2 t}.$$ *Therefore:*

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(x^{\sin^2 t}\left(\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x} \cdot x\right) \cdot \frac{u(x)-1}{\ln u(x)} \cdot \ln u(x)\right) =$$

$$= -\lim_{x\to\infty}\left(\left(\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x}\right)^{\cos^2 t}\cdot x^{\sin^2 t+\cos^2 t}\cdot\frac{u(x)-1}{\ln u(x)}\cdot\ln u(x)\right)=$$

$$= e^{\cos^2 t}\lim_{x\to\infty}\frac{u(x)-1}{\ln u(x)}\cdot\ln\left(\lim_{x\to\infty}(u(x))^x\right)=e^{\cos^2 t}\cdot 1\cdot\ln e^{\cos^2 t}=e^{\cos^2 t}\cdot\cos^2 t$$

UP.141. For $\{a_n\}_{n\geq 0}, a_n=\frac{(n+2)^{n+1}}{(n+1)^n}, x\in(-\infty,\infty), \{b_n(x)\}_{n\geq 1}$,

$b_n(x)=n^{\sin^2 x}\left(a_{n+1}^{\cos^2 x}-a_n^{\cos^2 x}\right)$, find $\lim_{n\to\infty} b_n(x)$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution by Marian Ursărescu – Romania

$$L=\lim_{n\to\infty} n^{\sin^2 x}\left[\left(\frac{(n+3)^{n+2}}{(n+2)^{n+1}}\right)^{\cos^2 x}-\left(\frac{(n+2)^{n+1}}{(n+1)^n}\right)^{\cos^2 x}\right]$$

Let $f:[n,n+1]\to\mathbb{R}, f(t)=\left(\frac{(t+2)^{t+1}}{(t+1)^t}\right)^{\cos^2 x}$

From Lagrange's theorm we have: $\exists c\in(n,n+1)$ *such that* $f(n+1)-f(n)=f'(c)\Rightarrow$

$$L=\lim_{n\to\infty} n^{\sin^2 x}\left(f(n+1)-f(n)\right)=\lim_{n\to\infty} n^{\sin^2 x} f'(c)\quad (1)$$

$$f(t)=\left((t+2)\left(\frac{t+2}{t+1}\right)^t\right)^{\cos^2 x}=(t+2)^{\cos^2 x}\left(1+\frac{1}{t+1}\right)^{t\cos^2 x}$$

$$f'(t)=\cos^2 x\,(t+2)^{\cos^2 x-1}\left(1+\frac{1}{t+1}\right)^{t\cos^2 x}+(t+2)^{\cos^2 x}\cdot$$

$$\cdot\left(\left(1+\left(\frac{1}{t+1}\right)\right)^{t\cos^2 x}\left(\cos^2 x\cdot\ln\left(1+\frac{1}{t+1}\right)\right)+\tan^2 x\cdot\frac{-\frac{1}{t+1}}{1+\frac{1}{t+1}}\right)\Rightarrow$$

$$f'(t)=\cos^2 x\left(1+\frac{1}{t+1}\right)^{t\cos^2 x}\left[(t+2)^{-\sin^2 x}+(t+2)^{\cos^2 x}\left(\ln\left(1+\frac{1}{t+1}\right)-\frac{t}{(t+1)(t+2)}\right)\right]\quad (2)$$

From (1)+(2) and because $c\in(n,n+1)$ we must calculate:

$$L=\lim_{n\to\infty} n^{\sin^2 x}\cdot\cos^2 x\left(1+\frac{1}{n+1}\right)^{\cos^2 x}\left[(n+2)^{-\sin^2 x}+(n+2)^{\cos^2 x}\left(\ln\left(1+\frac{1}{n+1}\right)-\frac{n}{(n+1)(n+2)}\right)\right]\quad (3)$$

$$\lim_{n\to\infty}\cos^2 x\left(1+\frac{1}{n+1}\right)^{n\cos^2 x}=\cos^2 x\cdot e^{\cos^2 x}\quad (4)$$

$$\lim_{n\to\infty}\left(\frac{n+2}{n}\right)^{\sin^2 x}+(n+2)^{\cos^2 x-1}\cdot n\cdot n^{\sin^2 x}\left(\ln\left(1+\frac{1}{n+1}\right)-\frac{n}{(n+1)(n+2)}\right)$$

$$=1+\lim_{n\to\infty}\left(\frac{n}{n+2}\right)^{\sin^2 x}\cdot\left(\frac{\ln\left(1+\frac{1}{n+1}\right)}{\frac{1}{n}}-\frac{n}{(n+1)(n+2)}\right)=$$

$$1+1(1-1)=1\quad (5)$$

$$\text{From } (3)+(4)+(5)\Rightarrow L=\cos^2 x\, e^{\cos^2 x}$$

UP.142. Let $(x_n)_{n\geq 1}$ be a sequence which satisfy:

$$-\ln(mn+x_n)+\sum_{k=1}^{mn}\frac{1}{k}=\gamma$$

where m is positive integer and γ is Euler – Mascheroni's constant. Compute:

$$\lim_{n\to\infty} x_n$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution 1 by Shafiqur Rahman-Bangladesh

$$-\ln(mn+x_n)+\sum_{k=1}^{mn}\frac{1}{k}=\gamma\Rightarrow\lim_{n\to\infty}x_n=\lim_{x\to\infty}\left(e^{\left(\sum_{k=1}^{mn}\frac{1}{k}-\gamma\right)}-mn\right)=$$

$$=\lim_{n\to\infty}\left(e^{\left(\sum_{k=1}^{mn}\frac{1}{k}-\sum_{k=1}^{n}\frac{1}{k}+\ln n\right)}-mn\right)=\lim_{n\to\infty}n\left(e^{\left(\sum_{k=n+1}^{mn}\frac{1}{k}\right)}-m\right)=$$

$$=\lim_{n\to\infty}\left(e^{\int_1^m\frac{dx}{d}}-m\right)=\lim_{n\to\infty}n\left(e^{\ln m}-m\right)\therefore\lim_{n\to\infty}x_n=0$$

Solution 2 by Khalef Ruhemi-Jarash-Jordan

$$-\ln(mn+x_n)+\sum_{k=1}^{k=mn}\frac{1}{k}=\gamma.\ \textit{Find}\ \lim_{n\to\infty}(x_n)\Rightarrow x_n+mn=$$

$$=e^{-\gamma+\sum_{k=1}^{k=mn}\frac{1}{k}}\left(\sum_{e^{k=1}}^{k=mn}\frac{1}{k}\right)-\ln(mn)+\ln(mn)-\gamma$$

$$\therefore x_n+mn=(mn)e^{\left(\sum_{k=1}^{k=mn}\frac{1}{k}\right)-\ln(mn)-\gamma}\ \therefore x_n=(mn)\left(e^{\left(\sum_{k=1}^{k=mn}\frac{1}{k}\right)-\ln(mn)-\gamma}-1\right)$$

$$\therefore \lim_{n\to\infty}(x_n) = \lim_{n\to\infty}\left(n\left(e^{\left(\sum_{k=1}^{k=n}\frac{1}{k}\right)-\ln(n)-\gamma} - 1\right)\right) \therefore \lim_{n\to\infty}(x_n) = \lim_{n\to\infty}\left(n\left(e^{-\gamma+\int_0^1\left(\frac{1-x^n}{1-x}+\frac{1-x^{n-1}}{\ln(x)}\right)dx} - 1\right)\right)$$

$$= \lim_{p\to 0}\frac{-1+e^{-\gamma+\int_0^1\left(\frac{1-x^{\frac{1}{p}}}{1-x}+\frac{1-x^{\frac{1}{p}-1}}{\ln(x)}\right)dx}}{p}, \textit{ using Lop-rulle}$$

$$\lim_{n\to\infty}(x_n) = \lim_{p\to 0}\left(e^{-\gamma}\cdot e^{\int_0^1\left(\frac{1-x^{\frac{1}{p}}}{1-x}+\frac{1-x^{\frac{1}{p}-1}}{\ln(x)}\right)dx}\cdot\int_0^1\left(\frac{x^{\frac{1}{p}}\ln(x)}{p^2(1-x)}+\frac{x^{\frac{1}{p}-1}}{p^2}\right)dx\right)$$

$$= \lim_{p\to 0}\int_0^1\frac{x^{\frac{1}{p}}}{p^2}\left(\frac{\ln(x)}{1-x}+\frac{1}{x}\right)dx = \lim_{n\to\infty}\int_0^1 n^2x^n\cdot\frac{\ln(x)}{1-x}dx + \lim_{n\to\infty}\int_0^1 n^2\cdot x^{n-1}\,dx$$

Since $\lim_{n\to\infty}(n^2x^n) = 0$***, and*** $\lim_{n\to\infty}(n^2x^{n-1}) = 0, 1 > x > 0$

$$\therefore \lim_{n\to\infty}(x_n) = 0 + 0 = 0 \Rightarrow \lim_{n\to\infty}(x_n) = 0$$

UP.143. Let $a, b \in \mathbb{R}_+, \gamma_n(a,b) = -\ln(n+a) + \sum_{k=1}^{n}\frac{1}{k+b}$ **with**

$$\lim_{n\to\infty}\gamma_n(a,b) = \gamma(a,b) \in \mathbb{R}$$

Calculate:

$$\lim_{n\to\infty}\left(\ln\frac{e}{n+a}+\sum_{k=1}^{n}\frac{1}{k+b}-\gamma(a,b)\right)^n$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$\lim_{n\to\infty} n\big(\gamma_n(a,b)-\gamma(a,b)\big) = \lim_{n\to\infty}\frac{\gamma_n(a,b)-\gamma(a,b)}{\frac{1}{n}} = \lim_{n\to\infty}\frac{\gamma_{n+1}(a+b)-\gamma_n(a,b)}{\frac{1}{n+1}-\frac{1}{n}}$$

$$= \lim_{n\to\infty}\frac{\sum_{k=1}^{n+1}\frac{1}{k+b}-\ln(n+1+a)-\sum_{k=1}^{n}\frac{1}{k+b}+\ln(n+a)}{\frac{1}{n+1}-\frac{1}{n}}$$

$$=\lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n+a}\right)-\frac{1}{n+1+b}}{\frac{1}{(n+1)}}=\lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)-\frac{1}{n+1}}{\frac{1}{n(n+1)}}=$$

$$=\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)\ln\left(1+\frac{1}{n}\right)-\frac{1}{n}}{\frac{1}{n^2}}=$$

$$=\lim_{\substack{x\to 0\\ x>0}}\frac{(1+x)\ln(1+x)-x}{x^2}\overset{L'HOSPITAL}{=}\lim_{\substack{x\to 0\\ x>0}}\frac{\ln(1+x)+1-1}{2x}=\frac{1}{2}\lim_{\substack{x\to 0\\ x>0}}\ln\sqrt[x]{1+x}=$$

$$=\frac{\ln e}{2}=\frac{1}{2}$$

$$\lim_{n\to\infty}\left(\ln\frac{e}{n+a}+\sum_{k=1}^{n}\frac{1}{k+b}-\gamma(a,b)\right)^n=e^{b-a+\frac{1}{2}}$$

Solution 2 by Shafiqur Rahman-Bangladesh

$$\textit{Let } \Omega=\lim_{n\to\infty}\left(\ln\left(\frac{e}{n+a}\right)+\sum_{k=1}^{n}\frac{1}{k+b}-\gamma(a,b)\right)^n\Rightarrow$$

$$\Rightarrow\ln\Omega=\lim_{n\to\infty}n\ln\left(1+\sum_{k=1}^{n}\frac{1}{k+b}-\ln(n+a)-\gamma(a,b)\right)$$

$$=\lim_{n\to\infty}\frac{\left(\sum_{k=1}^{n}\frac{1}{k+b}-\ln(n+a)-\gamma(a,b)\right)}{\frac{1}{n}}=\lim_{n\to\infty}\left(\frac{\frac{1}{n+b+1}-\ln\left(1+\frac{1}{n+a}\right)}{\frac{1}{n+1}-\frac{1}{n}}\right)=$$

$$=\lim_{n\to\infty}n(n+1)\left(\frac{1}{n+a}-\frac{1}{2(n+a)^2}+0\left(\frac{1}{(n+a)^3}\right)-\frac{1}{n+b+1}\right)$$

$$=\lim_{n\to\infty}\left(\frac{(2b-2a+1)n^2(n+1)+(2ab-2a^2+2a-b-1)n(n+1)}{2(n+a)^2(n+b+1)}\right)=b-a+\frac{1}{2}$$

$$\therefore\ \Omega=\lim_{n\to\infty}\left(\ln\left(\frac{e}{n+a}\right)+\sum_{k=1}^{n}\frac{1}{k+b}-\gamma(a,b)\right)^n=e^{b-a+\frac{1}{2}}$$

Solution 3 by Remus Florin Stanca-Romania

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Let $a, b \in \mathbb{R}_+$ $\gamma_n(a,b) = -\ln(a+n) + \sum_{k=1}^{n} \frac{1}{k+b}$ *with* $\lim_{n\to\infty} \gamma_n(a,b) = \gamma(a,b) \in \mathbb{R}$,

calculate: $\lim_{n\to\infty}\left(\ln\frac{e}{n+a} + \sum_{k=1}^{n}\frac{1}{k+b} - \gamma(a,b)\right)^n$

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k+b} - \ln(n+a)\right) = \lim_{n\to\infty}\left(\frac{1}{b+1} + \cdots + \frac{1}{b+n} - \ln(n+a)\right)$$

$$= \lim_{n\to\infty}\left(1 + \cdots + \frac{1}{b+n} - \ln(b+n) + \ln(b+n) - \ln(n+a) - \left(1 + \cdots + \frac{1}{b}\right)\right) =$$

$$\gamma - \left(1 + \cdots + \frac{1}{b}\right) + \lim_{n\to\infty}\ln\frac{b+n}{n+a} = \gamma - 1 - \cdots - \frac{1}{b} = \gamma(a,b)$$

$$\lim_{n\to\infty}\left(\ln\frac{e}{n+a} + \sum_{k=1}^{n}\frac{1}{k+b} - \gamma + 1 + \cdots + \frac{1}{b}\right) =$$

$$= \lim_{n\to\infty}\left(\sum_{k=1}^{b+n}\frac{1}{k} - \ln(b+n) + \ln(b+n) + \ln\frac{e}{n+a} - \gamma\right) = \lim_{n\to\infty}\ln\frac{en+be}{n+a} = 1$$

$$\lim_{n\to\infty}\left(\ln\frac{e}{n+a} + \sum_{k=1}^{b+n}\frac{1}{k} - \gamma\right)^n = \lim_{n\to\infty}\left(\ln\frac{e}{n+a} + \sum_{k=1}^{b+n}\frac{1}{k} - \gamma - 1 + 1\right)^n$$

$$= \lim_{n\to\infty}\left(\ln\frac{e}{n+a} + \sum_{k=1}^{b+n}\frac{1}{k} - \gamma - 1 + 1\right)^{\frac{1}{\ln\frac{e}{n+a}+\sum_{k=1}^{b+n}\frac{1}{k}-\gamma-1}\cdot\left(\ln\frac{e}{n+a}+\sum_{k=1}^{b+n}\frac{1}{k}-\gamma-1\right)n} =$$

$$= \lim_{n\to\infty} e^{n\left(\ln\frac{e}{n+a}+\sum_{k=1}^{b+n}\frac{1}{k}-\gamma-1\right)}$$

$$\lim_{n\to\infty}\frac{\ln\frac{e}{n+a} + \sum_{k}^{b+n}\frac{1}{k} - \gamma - 1}{\frac{1}{n}} \overset{Stolz\ Cesoro}{=} \lim_{n\to\infty}\frac{\ln\frac{n+a}{n+a+1} + \frac{1}{b+n+1}}{-\frac{1}{n(n+1)}}$$

Let $f: \mathbb{R}\setminus\{-1; 0; (-a-1); (b-1-1)\}$. *Such that* $f(x) = \frac{\ln\left(\frac{x+a}{x+a+1}\right)+\frac{1}{b+x+1}}{\frac{-1}{x(x+1)}}$

$$\lim_{n\to\infty} f(x) \overset{\frac{0}{0}}{=} \lim_{n\to\infty}\frac{\frac{x+a+1}{x+a}\cdot\frac{1}{(x+a+1)^2} - \frac{1}{(b+x+1)^2}}{\frac{1}{(x^2+x)^2}(2x+1)} = \lim_{n\to\infty}\frac{x^4+2x^3+x^2}{2x+1}\cdot$$

$$\cdot\frac{b^2+x^2+1+2bx+2b+2x-x^2-a^2-2ax-x-a}{\begin{array}{c}b^2x^2+b^2a^2+2ab^2x+b^2x+b^2a+x^4+x^2a^2+2ax^3+x^3+x^2a+x^2+a^2+2ax+x+\\+a+2bx^3+2bxa^2+4abx^2+2bx^2+2abx+2bx^2+2ba^2+4abx+2bx+2ab+\\+2x^3+2xa^2+4ax^2+2x^2+2ax\end{array}}$$

$$=\frac{2b-2a+1}{2}\Rightarrow L=e^{\frac{2b-2a+1}{2}}$$

UP.144. If $x,y,z\geq 0$ then:

$$\cosh^2 x\cosh^2 y\cosh^2 z\geq 2(1+\cosh(x-y)+\cosh(y-z)+\cosh(z-x))\cdot$$

$$\cdot\sinh\frac{x+y}{2}\sinh\frac{y+z}{2}\sin h\frac{z+x}{2}$$

Proposed by Mihály Bencze – Romania

Solution by proposer

If $a,b\geq 0$ *then:* $(a^2+1)(b^2+1)=(a^2b^2+1)+a^2+b^2\geq a^2+2ab+b^2=(a+b)^2$

$\prod(a^2+1)^2=\prod(a^2+1)(b^2+1)\geq\prod(a+b)^2$ *therefore* $\prod(a^2+1)\geq\prod(a+b)$

We take $a=\sinh x, b=\sinh y, c=\sinh z\Rightarrow\prod\cosh^2 x=\prod(1+\sinh^2 x)\geq$

$$\geq\prod(\sinh x+\sinh y)=\prod 2\sinh\frac{x+y}{2}\cosh\frac{x-y}{2}=$$

$$=2\prod\sinh\frac{x+y}{2}\cdot\left(4\prod\sinh\frac{x-y}{2}\right)=2\prod\sinh\frac{x+y}{2}\left(1+\sum\cosh(x-y)\right)$$

UP.145. Let be $(x_n)_{n\geq 1}, x_n\in\mathbb{R}_+^*, \forall n\in\mathbb{N}^*$, such that exists $\lim_{n\to\infty}(x_{n+1}-x_n)=x\in\mathbb{R}_+^*$. Find:

$$\lim_{n\to\infty}\left(\frac{(n+1)x_{n+1}}{\sqrt[n+1]{(2n+1)!!}}-\frac{nx_n}{\sqrt[n]{(2n-1)!!}}\right)$$

Proposed by Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by Ruanghaw Chaokha-Chiangrai-Thailand

$$\lim_{n\to\infty}(x_{n+1}-x_n)=x\Rightarrow L=\lim_{n\to\infty}\left(\frac{(n+1)x_{n+1}}{\sqrt[n+1]{(2n+1)!!}}-\frac{nx_n}{\sqrt[n]{(2n-1)!!}}\right)=??$$

Stolz-Cesaro; $\lim_{n\to\infty}(a_n-a_{n-1})=\lim_{n\to\infty}\frac{a_n}{n}; a_n=\frac{(n+1)x_{n+1}}{\sqrt[n+1]{(2n+1)!!}}$

$$L=\lim_{n\to\infty}\frac{(n+1)x_{n+1}}{n\cdot\sqrt[n+1]{(2n+1)!!}}=\lim_{n\to\infty}\frac{x_{n+1}}{\sqrt[n+1]{(2n+1)!!}}\overset{n\to n-1}{=}\lim_{n\to\infty}\frac{x_n}{\sqrt[n]{(2n-1)!!}}$$

Again, Stolz – Cesaro; $\lim_{n\to\infty}\frac{x_n}{y_n}=\lim_{n\to\infty}\frac{x_n-x_{n-1}}{y_n-y_{n-1}}$; $y_n=\sqrt[n]{(2n-1)!!}$

$$L=\lim_{n\to\infty}\frac{x_n-x_{n-1}}{\sqrt[n]{(2n-1)!!}-\sqrt[n-1]{(2n-3)!!}}=\frac{\lim_{n\to\infty}(x_n-x_{n-1})}{\lim_{n\to\infty}\left(\sqrt[n]{(2n-1)!!}-\sqrt[n-1]{(2n-3)!!}\right)}=\frac{x}{K}\to *$$

$$a_n=\sqrt[n]{(2n-1)!!}\Rightarrow K=\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{n^n}}$$

And Stolz-Cesaro again; $\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\frac{a_n}{a_{n-1}}$; $a_n=\frac{(2n-1)!!}{n^n}$

$$K=\lim_{n\to\infty}\frac{(2n-1)!!}{n^n}\cdot\frac{(n-1)^{n-1}}{(2n-3)!!}=\lim_{n\to\infty}\frac{(2n-1)}{(n-1)}\cdot\left(\frac{n-1}{n}\right)^n=$$

$$=\lim_{n\to\infty}\frac{(2n-1)}{(n-1)}\cdot\left(1-\frac{1}{n}\right)^n=2e^{-1};\ *;\ \therefore L=\frac{xe}{2}$$

Solution 2 by Nassim Nicholas Taleb-USA

$$f=\frac{(n+1)}{\sqrt[n+1]{(2n+1)!!}}x_{n+1}-\frac{n}{\sqrt[n]{(2n-1)!!}}x_n$$

We write at the limit, with $a_n=\frac{(n+1)}{\sqrt[n+1]{(2n+1)!!}}$ ***and*** $a_{n+1}=\frac{n}{\sqrt[n]{(2n-1)!!}}$

for n ***large,*** $f\to f'=a_{n+1}x+(a_{n+1}-a_n)x_n$

$$\lim_{n\to\infty}f'=\lim_{n\to\infty}a_{n+1}x+\lim_{n\to\infty}(a_{n+1}-a_n)x_n$$

$$\lim_{n\to\infty}(a_{n+1}-a_n)=0$$

$$\lim_{n\to\infty}f=x\lim_{n\to\infty}\frac{n}{((2n-1)!!)^{\frac{1}{n}}}=\frac{x}{2}e$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{(2n-1)!!}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{(2n-1)!!}}\overset{CAUCHY\ D'ALEMBERT}{=}\lim_{n\to\infty}\left(\frac{(n+1)^{n+1}}{(2n+1)!!}\cdot\frac{(2n-1)!!}{n^n}\right)$$

$$=\lim_{n\to\infty}\left(\frac{n+1}{\frac{(2n+2)!}{2^{n+1}\cdot(n+1)!}}\cdot\frac{(2n)!}{2^n\cdot n!}\cdot\left(1+\frac{1}{n}\right)^n\right)=e\lim_{n\to\infty}\frac{1+\frac{1}{n}}{2+\frac{1}{n}}=\frac{e}{2}$$

$$\therefore \lim_{n\to\infty}\frac{x_n}{n}=\lim_{n\to\infty}\frac{x_{n+1}-x_n}{n+1-n}=x, \lim_{n\to\infty}\left(\frac{x_n+1}{x_n}\right)^n=\lim_{n\to\infty}\left(1+\frac{x_{n+1}-x_n}{x_n}\right)^n$$

$$=\lim_{n\to\infty}\left\{\left(1+\frac{x_{n+1}-x_n}{x_n}\right)^{\frac{x_n}{x_{n+1}-x_n}}\right\}^{n\frac{x_{n+1}-x_n}{x_n}}=e^{\frac{1}{a}\cdot a}=e$$

Let $u_n=\frac{(n+1)x_{n+1}}{\sqrt[n+1]{(2n+1)!!}}\cdot\frac{\sqrt[n]{(2n-1)!!}}{nx_n}$ for all $n\in\mathbb{N}$

$$\therefore \lim_{n\to\infty}u_n=\lim_{n\to\infty}\left(\frac{n+1}{\sqrt[n+1]{(2n+1)!!}}\cdot\frac{x_{n+1}}{n+1}\cdot\frac{\sqrt[n]{(2n-1)!!}}{n}\cdot\frac{n}{x_n}\cdot\left(1+\frac{1}{n}\right)\right)=1$$

so, $\frac{u_n-1}{\ln u_n}\to$ as $n\to\infty$

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^n\cdot\left(\frac{x_{n+1}}{x_n}\right)^n\cdot\frac{n+1}{2n+1}\cdot\frac{\sqrt[n+1]{(2n+1)!!}}{n+1}\right)=e$$

$$\therefore \lim_{n\to\infty}\left(\frac{(n+1)x_{n+1}}{\sqrt[n+1]{(2n+1)!!}}-\frac{nx_n}{\sqrt[n]{(2n-1)!!}}\right)=\lim_{n\to\infty}\left(\frac{n}{\sqrt[n]{(2n-1)!!}}\cdot\frac{x_n}{n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n\right)=\frac{ex}{2}$$

UP.146. Let $f:(0,\infty)\to(0,\infty)$ be a function with:

$$lim_{x\to\infty}\frac{f(x)}{x}=a\in(0,\infty)\text{ and } t\in\mathbb{R}.\text{ Find:}$$

$$\lim_{n\to\infty}\left((n+1)^{sin^2 t}\cdot\sqrt[n+1]{\left(f(1)f(2)\dots f(n)f(n+1)\right)^{cos^2 t}}-n^{sin^2 t}\cdot\sqrt[n]{\left(f(1)f(2)\dots f(n)\right)^{cos^2 t}}\right)$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$\lim_{x\to\infty}\frac{f(x)}{x}=a\Rightarrow\lim_{\substack{x\to\infty\\ n\in\mathbb{N}}}\frac{f(n)}{n}=a \text{ let } u_n=\frac{(n+1)^{\sin^2 t}\sqrt[n+1]{\left(\prod_{k=1}^{n+1}f(k)\right)^{\cos^2 t}}}{n^{\sin^2 t}\sqrt[n]{\left(\prod_{k=1}^{n}f(k)\right)^{\cos^2 t}}}\text{ for all } n\in\mathbb{N}$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{\left(\prod_{k=1}^{n}f(k)\right)^{\cos^2 t}}}{n^{\cos^2 t}}=\lim_{n\to\infty}\sqrt[n]{\frac{\left(\prod_{k=1}^{n}f(k)\right)^{\cos^2 t}}{n^{n\cos^2 t}}}=$$

$$\overset{CAUCHY}{\underset{D'ALEMBERT}{=}} \lim_{n\to\infty}\left(\left(\frac{f(n+1)}{n+1}\right)^{\cos^2 t}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^{n\cos^2 t}}\right)=\left(\frac{a}{e}\right)^{\cos^2 t}$$

$$\therefore \lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{\frac{\sqrt[n+1]{\left(\prod_{k=1}^{n+1} f(k)\right)^{\cos^2 t}}}{(n+1)^{\cos^2 t}}}{\frac{\sqrt[n]{\left(\prod_{k=1}^{n} f(k)\right)^{\cos^2 t}}}{n^{\cos^2 t}}}\left(1+\frac{1}{n}\right)=1 \text{ so, } \frac{u_n-1}{\ln u_n}\to 1 \text{ as } n\to\infty$$

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty}\left(\left(\frac{f(n+1)}{n+1}\right)^{\cos^2 t}\cdot\frac{(n+1)^{\cos^2 t}}{\sqrt[n+1]{\left(\prod_{k=1}^{n+1} f(k)\right)^{\cos^2 t}}}\right)=e^{\cos^2 t}$$

$$\therefore \lim_{n\to\infty}\left((n+1)^{\sin^2 t}\sqrt[n+1]{\left(\prod_{k=1}^{n+1} f(k)\right)^{\cos^2 t}}-n^{\sin^2 t}\sqrt[n]{\left(\prod_{k=1}^{n} f(k)\right)^{\cos^2 t}}\right)$$

$$=\lim_{n\to\infty}\left(\left(\frac{\sqrt[n]{\left(\prod_{k=1}^{n} f(k)\right)^{\cos^2 t}}}{n^{\cos^2 t}}\right)\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n\right)=\left(\frac{a}{e}\right)^{\cos^2 t}$$

Solution 2 by Remus Florin Stanca-Romania

$$l=\lim_{n\to\infty} n^{\sin^2 t}\sqrt[n]{(f(1)\cdot\ldots\cdot f(n))^{\cos^2 t}}\left(\left(\frac{n+1}{n}\right)^{\sin^2 t}\cdot\left(\frac{\sqrt[n+1]{f(1)\cdot\ldots\cdot f(n+1)}}{\sqrt[n]{f(1)\cdot\ldots\cdot f(n)}}\right)-1\right)$$

$$=\lim_{n\to\infty} n\cdot\sqrt[n]{\left(\frac{f(1)\cdot\ldots\cdot f(n)}{n^n}\right)^{\cos^2 t}}\cdot\left(\left(\left(\frac{n+1}{n}\right)^{\tan^2 t}\cdot\frac{\sqrt[n+1]{f(1)\cdot\ldots\cdot f(n+1)}}{\sqrt[n]{f(1)\cdot\ldots\cdot f(n)}}\right)^{\cos^2 t}-1\right)$$

We know that $\lim_{n\to\infty}\left(\sqrt[n+1]{f(1)\cdot\ldots\cdot f(n+1)}-\sqrt[n]{f(1)\cdot\ldots\cdot f(n)}\right)=\frac{a}{e}$ ***and***

$$\lim_{x\to 1}\frac{x^a-1}{x-1}=a$$

$$\lim_{n\to\infty}\sqrt[n]{\frac{f(1)\cdot\ldots\cdot f(n)}{n^n}}=\lim_{n\to\infty} e^{\frac{\ln\frac{f(1)\cdot\ldots\cdot f(n)}{n^n}}{n}}=\lim_{n\to\infty} e^{\ln\frac{f(n+1)}{(n+)}}=\frac{a}{e}$$

$$\lim_{n\to\infty} n\cdot\left(\left(\frac{n+1}{n}\cdot\frac{\sqrt[n+1]{\prod_{k=1}^{n+1}f(k)}}{\sqrt[n]{\prod_{k=1}^{n}f(k)}}\right)^{\tan^2 t}-1\right)=$$

$$\tan^2 t\cdot\lim_{n\to\infty} n\cdot\left(\frac{n+1}{n}\cdot\frac{\sqrt[n+1]{f(1)\cdot\ldots\cdot f(n+1)}}{\sqrt[n]{f(1)\cdot\ldots\cdot f(n)}}-1\right)=$$

$$\tan^2 t\cdot\lim_{n\to\infty}\frac{(n+1)\sqrt[n+1]{f(1)\cdot\ldots\cdot f(n+1)}-n\sqrt[n]{f(1)\cdot\ldots\cdot f(n)}}{\sqrt[n]{f(1)\cdot\ldots\cdot f(n)}}=$$

$$t=\tan^2 t\cdot\frac{e}{a}\cdot\left(\frac{a}{e}+\frac{a}{e}\right)=2\tan^2 t$$

$$\lim_{n\to\infty} n\left(\left(\frac{\sqrt[n+1]{f(1)\cdot\ldots\cdot f(n+1)}}{\sqrt[n]{f(1)\cdot\ldots\cdot f(n)}}\right)^{\tan^2 t-1}-1\right)=$$

$$(\tan^2 t-1)\lim_{n\to\infty} n\cdot\left(\frac{\sqrt[n+1]{f(1)\cdot\ldots\cdot f(n+1)}}{\sqrt[n]{f(1)\cdot\ldots\cdot f(n)}}-1\right)=(\tan^2 2-1)\cdot\frac{e}{a}\cdot\frac{a}{e}=\tan^2 t-1$$

$$\Rightarrow l_1=\tan^2 t+1\Rightarrow l=\left(\frac{a}{e}\right)^{\cos^2 t}\cdot\cos^2 t\cdot(\tan^2 t+1)=\left(\frac{a}{e}\right)^{\cos^2 t}\Rightarrow l=\left(\frac{a}{e}\right)^{\cos^2 t}$$

Solution 3 by Nassim Nicholas Taleb – USA

We have $\sin(t)^2=1-\cos^2(t)=\beta$, ***with*** $\beta\in[0,1]$

$$g=(n+1)^{\beta}\left(\prod_{i=1}^{n+1}f(i)\right)^{\frac{1-\beta}{n+1}}-n^{\beta}\left(\prod_{i=1}^{n}f(i)\right)^{\frac{1-\beta}{n}}$$

Replacing $f(i)$ ***by*** $a\,i$ ***we have***

$$g=(n+1)^{\beta}\left(a^{n+1}\Gamma(n+2)\right)^{\frac{1-\beta}{n+1}}-n^{\beta}\left(a^{n}\Gamma(n+1)\right)^{\frac{1-\beta}{n}},$$

Where $\Gamma(.)$ ***is the (standard) gamma function.***

By Stirling's formula, as n ***becomes large***

$$\Gamma(n+2)\to\sqrt{2\pi(n+1)}\left(\frac{n+1}{e}\right)^{n+1},\Gamma(n+1)\to\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}$$

$$g\to e^{\beta-1}a^{1-\beta}\left((n+1)^{\frac{-\beta+2n+3}{2n+2}}(2\pi)^{\frac{1-\beta}{2n+2}}-n^{\frac{-\beta+2n+1}{2n}}(2\pi)^{-\frac{\beta-1}{2n}}\right)$$

$$\left((n+1)^{\frac{-\beta+2n+3}{2n+2}}(2\pi)^{\frac{1-\beta}{2n+2}}-n^{\frac{-\beta+2n+1}{2n}}(2\pi)^{-\frac{\beta-1}{2n}}\right)\to 1$$

Hence $\lim_{n\to\infty} g = e^{\beta-1}a^{1-\beta}$, ***with*** $\beta = \sin(t)^2$

Solution 4 by Aaditya Joshi-Mumbai-India

$$f:(0,\infty)\to(0,\infty);\lim_{x\to\infty}\frac{f(x)}{x}=a \text{ s.t. } a\in(0,\infty),t\in\mathbb{R}$$

$$\lim_{n\to\infty}\left\{(n+1)^{\sin^2 t}\sqrt[n+1]{\left(\prod_{i=1}^{n+1}f(i)\right)^{\cos^2 t}}-n^{\sin^2 t}\sqrt[n]{\left(\prod_{i=1}^{n}f(i)\right)^{\cos^2 t}}\right\}$$

$$=\lim_{n\to\infty}\left\{(n+1)^{\sin^2 t}\left(\prod_{i=1}^{n}f(i)\right)^{\frac{\cos^2 t}{n+1}}-n^{\sin^2 t}\left(\prod_{i=1}^{n}f(i)\right)^{\frac{\cos^2 t}{n}}\right\}$$

$$=\lim_{n\to\infty}\left\{\underbrace{n^{\sin^2 t}\left(\prod_{i=1}^{n}f(i)\right)^{\frac{\cos^2 t}{n}}}_{nt_n}\left[\underbrace{\left(\frac{n+1}{n}\right)^{\sin^2 t}\left(\frac{\prod_{i=1}^{n+1}f(i)^{\frac{\cos^2 t}{n+1}}}{\prod_{i=1}^{n}f(i)^{\frac{\cos^2 t}{n}}}\right)}_{u_n}-1\right]\right\}$$

$$=\lim_{n\to\infty}\{n\cdot t_n(u_n-1)\}$$

$$\lim_{n\to\infty}\left\{n^{\sin^2 t}\left(\prod_{i=1}^{n}f(i)\right)^{\frac{\cos^2 t}{n}}\right\}=\lim_{n\to\infty}\left\{n^{1-\cos^2 t}\left(\prod_{i=1}^{n}f(i)\right)^{\frac{\cos^2 t}{n}}\right\}=$$

$$=\lim_{n\to\infty}\left\{\underbrace{n\left(\prod_{i=1}^{n}\frac{f(i)}{n^n}\right)^{\frac{\cos^2 t}{n}}}_{\substack{\parallel\\ \sqrt[n]{\frac{\left(\prod_{i=1}^{n}f(i)\right)^{\cos^2 t}}{n^{n\cos^2 t}}}\Rightarrow t_n}}\right\}$$

$$\lim_{n\to\infty}t_n=\lim_{n\to\infty}\left(\left[\frac{f(n+1)}{n+1}\right]^{\cos^2 t}\times\frac{1}{\left(\frac{n+1}{n}\right)^{n\cos^2 t}}\right)\to Cauchy-D'Alembert$$

$$= \lim_{n\to\infty}\left(\left[\frac{f(n+1)}{n+1}\right]^{\cos^2 t} \times \left(1+\frac{1}{n}\right)^{-n\cos^2 t}\right) = \left(\frac{a}{e}\right)^{\cos^2 t}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left\{\frac{(n+1)^{\sin^2 t}\left(\prod_{i=1}^{n+1} f(i)\right)^{\frac{\cos^2 t}{n+1}}}{n^{\sin^2 t}\left(\prod_{i=1}^{n} f(i)\right)^{\frac{\cos^2 t}{n}}}\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{\sqrt[n+1]{[f(1)f(2)\dots f(n+1)]^{\cos^2 t}} \cdot (n+1)^{1-\cos^2 t}}{\sqrt[n]{[f(1)f(2)\dots f(n)]^{\cos^2 t}} \cdot n^{1-\cos^2 t}}\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{\sqrt[n+1]{[f(1)f(2)\dots f(n)f(n+1)]^{\cos^2 t}}}{\sqrt[n]{[f(1)f(2)\dots f(n)]^{\cos^2 t}}} \times \frac{n^{\cos^2 t-1}}{(n+1)^{\cos^2 t-1}}\right\}$$

$$= \lim_{n\to\infty}\left\{\underbrace{\frac{\left(\prod_{i=1}^{n+1} f(i)\right)^{\cos^2\frac{t}{n+1}}}{\left(\prod_{i=1}^{n} f(i)\right)^{\cos^2\frac{t}{n}}} \times \frac{n^{\cos^2 t}}{(n+1)^{\cos^2 t}}}_{p_n} \times \frac{n+1}{n}\right\}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(p_n \cdot \frac{n+1}{n}\right)$$

$$\lim_{n\to\infty}(n\cdot[u_n-1]) = \lim_{n\to\infty}\left(n\cdot\left[p_n\cdot\frac{n+1}{n}-1\right]\right) = \lim_{n\to\infty}(p_n(n+1)-n)$$

$$\lim_{n\to\infty} p_n = \lim_{n\to\infty}\left\{\frac{\left(\prod_{i=1}^{n+1} f(i)\right)^{\cos^2\frac{t}{n+1}}}{\left(\prod_{i=1}^{n} f(i)\right)^{\cos^2\frac{t}{n}}} \times \frac{n^{\cos^2 t}}{(n+1)^{\cos^2 t}}\right\}$$

$$\therefore \lim_{n\to\infty} p_n = 1 \;\; \therefore \lim_{n\to\infty}(p_n(n+1)-n) = 1$$

$$\therefore \lim_{n\to\infty}\left\{(n+1)^{\sin^2 t}\sqrt[n+1]{\prod_{i=1}^{n+1} f(i)^{\cos^2 t}} - n^{\sin^2 t}\sqrt[n]{\prod_{i=1}^{n} f(i)^{\cos^2 t}}\right\}$$

$$= \lim_{n\to\infty}\{n\cdot t_n\cdot(u_n-1)\} = \left(\frac{a}{e}\right)^{\cos^2 t}$$

UP.147. In an ABC triangle let be a, b, c the lengths of BC, CA, AB, and r_a, r_b, r_c exradii.

Prove that:

$$\frac{r_a^2}{\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{r_b^2}{\tan\frac{C}{2}\tan\frac{A}{2}}+\frac{r_c^2}{\tan\frac{A}{2}\tan\frac{B}{2}}\ge\frac{9(a^2+b^2+c^2)}{4}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Hoang Le Nhat Tung-Hanoi-Vietnam

*** Let a, b, c be the lengths BC, CA, AB of ΔABC. We have:**

$$\tan\frac{A}{2}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}=\frac{\sqrt{\frac{1-\cos A}{2}}}{\sqrt{\frac{1+\cos A}{2}}}=\sqrt{\frac{1-\cos A}{1+\cos A}}=\sqrt{\frac{1-\frac{b^2+c^2-a^2}{2bc}}{1+\frac{b^2+c^2-a^2}{2bc}}}=\sqrt{\frac{\frac{a^2-(b-c)^2}{2bc}}{\frac{(b+c)^2-a^2}{2bc}}}=\sqrt{\frac{a^2-(b-c)^2}{(b+c)^2-a^2}}$$

$$\Leftrightarrow\tan\frac{A}{2}=\sqrt{\frac{(a-b+c)(a+b-c)}{(b+c+a)(b+c-a)}}.\ \textit{Similar: }\tan\frac{B}{2}=\sqrt{\frac{(b-c+a)(b+c-a)}{(c+a+b)(c+a-b)}}$$

$$\Rightarrow\tan\frac{A}{2}\tan\frac{B}{2}=\sqrt{\frac{(a-b+c)(a+b-c)}{(b+c+a)(b+c-a)}}\cdot\sqrt{\frac{(b-c+a)(b+c-a)}{(c+a+b)(c+a-b)}}=\frac{a+b-c}{a+b+c}\quad(2)$$

- Other, let $p=\frac{a+b+c}{2}$ is half circumference of ΔABC and S its area. By Heron's formula we have: $S=\sqrt{p(p-a)(p-b)(p-c)}\Leftrightarrow 4S=\sqrt{2p(2p-2a)(2p-2b)(2p-2c)}\Leftrightarrow$

$$\Leftrightarrow 4S=\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}\Leftrightarrow 4S^2=\frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4}$$

- Therefore (2), by the radius formula of the circle in a triangle:

$$\Rightarrow\frac{r_c^2}{\tan\frac{A}{2}\tan\frac{B}{2}}=\frac{\left(\frac{2S}{a+b-c}\right)^2}{\frac{a+b-c}{a+b+c}}=\frac{\frac{4S^2}{(a+b-c)^2}}{\frac{a+b-c}{a+b+c}}=\frac{\frac{\frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4}}{(a+b-c)^2}}{\frac{a+b-c}{a+b+c}}$$

$$\Leftrightarrow\frac{r_c^2}{\tan\frac{A}{2}\tan\frac{B}{2}}=\frac{(a+b+c)^2(b+c-a)(c+a-b)}{4(a+b-c)^2}\quad(3)$$

+ Let: $\begin{cases}b+c-a=2x\\c+a-b=2y\\a+b-c=2z\end{cases};(x,y,z>0)\Leftrightarrow\begin{cases}a=y+z\\b=z+x\\c=x+y\end{cases}\Rightarrow a+b+c=2(x+y+z)$

- Let (3): $\Rightarrow\frac{r_c^2}{\tan\frac{A}{2}\tan\frac{B}{2}}=\frac{(2(x+y+z))^2\cdot 2x\cdot 2y}{4(2z)^2}=\frac{(x+y+z)^2\cdot xy}{z^2}\quad$ ***(4)***

+ Similar: $\frac{r_b^2}{\tan\frac{C}{2}\tan\frac{A}{2}}=\frac{(x+y+z)^2\cdot zx}{y^2};\ \frac{r_a^2}{\tan\frac{B}{2}\tan\frac{C}{2}}=\frac{(x+y+z)^2\cdot yz}{x^2}\quad$ ***(5)***

- Let (4), (5):

$$\Rightarrow \frac{r_a^2}{\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{r_b^2}{\tan\frac{C}{2}\tan\frac{A}{2}}+\frac{r_c^2}{\tan\frac{A}{2}\tan\frac{B}{2}}=\frac{(x+y+z)^2\cdot yz}{x^2}+\frac{(x+y+z)^2\cdot zx}{y^2}+\frac{(x+y+z)^2\cdot xy}{z^2}\quad (6)$$

- Other: $\frac{9(a^2+b^2+c^2)}{4}=\frac{9}{4}((y+z)^2+(z+x)^2+(x+y)^2)=\frac{9(x^2+y^2+z^2+xy+yz+zx)}{2}$ **(7)**

*** Let (1), (6), (7). We need to prove that:**

$$\frac{(x+y+z)^2\cdot yz}{x^2}+\frac{(x+y+z)^2\cdot zx}{y^2}+\frac{(x+y+z)^2\cdot xy}{z^2}\ge\frac{9(x^2+y^2+z^2+xy+yz+zx)}{2}\quad (8)$$

- By inequality: $(m+n+p)^2\ge 3(mn+np+pm)$**. We have:**

$$\frac{(x+y+z)^2\cdot yz}{x^2}+\frac{(x+y+z)^2\cdot zx}{y^2}+\frac{(x+y+z)^2\cdot xy}{z^2}\ge$$

$$\ge\frac{3(xy+yz+zx)\cdot yz}{x^2}+\frac{3(xy+yz+zx)\cdot zx}{y^2}+\frac{3(xy+yz+zx)\cdot xy}{z^2}$$

$$\Leftrightarrow\frac{(x+y+z)^2\cdot yz}{x^2}+\frac{(x+y+z)^2\cdot zx}{y^2}+\frac{(x+y+z)^2\cdot xy}{z^2}\ge 3\left(\frac{x^2y^2}{z^2}+\frac{y^2z^2}{x^2}+\frac{z^2x^2}{y^2}+\frac{xy(x+y)}{z}+\frac{yz(y+z)}{x}+\frac{zx(z+x)}{y}\right)\quad (9)$$

- By AM-GM inequality for 2 positive real numbers:

$$\frac{x^2y^2}{z^2}+\frac{y^2z^2}{x^2}+\frac{z^2x^2}{y^2}+\frac{xy(x+y)}{z}+\frac{yz(y+z)}{x}+\frac{zx(z+x)}{y}=$$

$$=\frac{x^2\left(\frac{y^2}{z^2}+\frac{z^2}{y^2}\right)}{2}+\frac{y^2\left(\frac{z^2}{x^2}+\frac{x^2}{z^2}\right)}{2}+\frac{z^2\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}\right)}{2}+x^2\left(\frac{y}{z}+\frac{z}{y}\right)+y^2\left(\frac{z}{x}+\frac{x}{z}\right)+z^2\left(\frac{x}{y}+\frac{y}{x}\right)\ge$$

$$\ge\frac{x^2\cdot 2\sqrt{\frac{y^2}{z^2}\cdot\frac{z^2}{y^2}}}{2}+\frac{y^2\cdot 2\sqrt{\frac{z^2}{x^2}\cdot\frac{x^2}{z^2}}}{2}+\frac{z^2\cdot 2\sqrt{\frac{x^2}{y^2}\cdot\frac{y^2}{x^2}}}{2}+x^2\cdot 2\sqrt{\frac{y}{z}\cdot\frac{y}{z}}+y^2\cdot 2\sqrt{\frac{z}{x}\cdot\frac{x}{z}}+z^2\cdot 2\sqrt{\frac{x}{y}\cdot\frac{y}{x}}$$

$$=x^2+y^2+z^2+2x^2+2y^2+2z^2=3(x^2+y^2+z^2)$$

$$\Leftrightarrow\frac{x^2y^2}{z^2}+\frac{y^2z^2}{x^2}+\frac{z^2x^2}{y^2}+\frac{xy(x+y)}{zx}+\frac{yz(y+z)}{x}+\frac{zx(z+x)}{y}\ge 3(x^2+y^2+z^2)\quad (10)$$

- Let (9), (10):

$$\Rightarrow\frac{(x+y+z)^2\cdot xy}{z^2}+\frac{(x+y+z)^2\cdot zx}{y^2}+\frac{(x+y+z)^2\cdot yz}{x^2}\ge 3\cdot 3(x^2+y^2+z^2)=9(x^2+y^2+z^2)\quad (11)$$

- Other: $m^2+n^2+p^2\ge mn+np+pm$**. Therefore:**

$$x^2+y^2+z^2=\frac{(x^2+y^2+z^2)+(x^2+y^2+z^2)}{2}\ge\frac{x^2+y^2+z^2+xy+yz+zx}{2}\quad (12)$$

- Let (11), (12):

$$\Rightarrow \frac{(x+y+z)^2\cdot xy}{z^2}+\frac{(x+y+z)^2\cdot zx}{y^2}+\frac{(x+y+z)^2\cdot yz}{x^2}\geq\frac{9(x^2+y^2+z^2+xy+yz+zx)}{2}$$

$\Rightarrow$ ***Inequality (8) True. Therefore (1) true and we get the result.***

+ Equality occurs if: $x=y=z\Leftrightarrow b+c-a=c+a-b=a+b-c\Leftrightarrow a=b=c\Leftrightarrow\Delta ABC$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum\frac{r_a^2}{\tan\frac{B}{2}\tan\frac{C}{2}}=\sum\frac{r_a^3}{s\prod\tan\frac{A}{2}}=\frac{\sum r_a^3}{s\left(\frac{r}{4R}\right)\left(\frac{4R}{s}\right)}=\frac{\sum r_a^3}{r}\geq\frac{9\sum a^2}{4}\Leftrightarrow 4\sum r_a^3\overset{(1)}{\geq}9r\sum a^2$$

Now, $\sum r_a^3=(\sum r_a)^3-3\prod(r_a+r_b)=(4R+r)^3-3\left(2r_ar_br_c+\sum r_ar_b\left(\sum r_a-r_c\right)\right)=$

$$=(4R+r)^3-3\left((4R+r)s^2-rs^2\right)\overset{(2)}{=}(4R+r)^3-12Rs^2$$

Now, RHS of (1) $\overset{Leibniz}{\leq}81R^2r\overset{?}{\leq}4\sum r_a^3\Leftrightarrow 4(4R+r)^3-48Rs^2\overset{?}{\geq}81R^2r$ ***(by (2))***

$\Leftrightarrow 4(4R+r)^3-81R^2r\underset{(3)}{\overset{?}{\geq}}48s^2$***. Now, RHS of (3)*** $\overset{Gerretsen}{\leq}48R(4R^2+4Rr+3r^2)$

$$\overset{?}{\leq}4(4R+r)^3-81R^2r\Leftrightarrow 64t^3-81t^2-96t+4\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)(64t^2+47t-2)\overset{?}{\geq}0\to\textit{true}\because t\overset{Euler}{\geq}2\ \textit{(Proved)}$$

UP.148. Let a,b,c be positive real numbers such that: $a+b+c=3$.

Prove that:

$$2(a^2+b^2+c^2)+3\geq 3\sqrt{3abc(a^3b+b^3c+c^3a)}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Marian Ursarescu-Romania

We use Vasc's inequality: $\forall a,b,c\in\mathbb{R}\Rightarrow(a^2+b^2+c^2)^2\geq 3(a^3b+b^3c+c^2a)\Rightarrow$

$\Rightarrow\sqrt{3(a^3b+b^3c+c^3a)}\leq a^2+b^2+c^2\Rightarrow$ ***inequality becomes:***

$$2(a^2+b^2+c^2)+3\geq 3\sqrt{abc}(a^2+b^2+c^2)\ \textit{(1)}$$

$$\left.\begin{array}{l}\sqrt{abc}\leq\frac{a+bc}{2}\Rightarrow\sqrt{abc}(a^2+b^2+c^2)\leq\frac{(a+bc)}{2}(a^2+b^2+c^2)\\ \sqrt{abc}\leq\frac{b+ac}{2}\Rightarrow\sqrt{abc}(a^2+b^2+c^2)\leq\frac{(b+ac)}{2}(a^2+b^2+c^2)\end{array}\right\}\Rightarrow$$

$$\sqrt{abc} \le \frac{c+ab}{2} \Rightarrow \sqrt{abc}(a^2+b^2+c^2) \le \frac{(c+ab)}{2}(a^2+b^2+c^2)$$

$$3\sqrt{abc}(a^2+b^2+c^2) \le \frac{(a^2+b^2+c^2)}{3}(3+ab+ac+bc) \quad (2)$$

From (1)+(2) we must show: $2(a^2+b^2+c^2)+3 \ge \frac{a^2+b^2+c^2}{2}(3+ab+ac+bc) \Leftrightarrow$

$$4(a^2+b^2+c^2)+6 \ge 3(a^2+b^2+c^2)+(a^2+b^2+c^2)(ab+ac+bc) \Leftrightarrow$$

$$a^2+b^2+c^2+6 \ge (a^2+b^2+c^2)(ab+ac+bc) \quad (3)$$

Because $a,b,c>0$ ***such that*** $a+b+c=3 \Rightarrow \exists x,y,z>0$ ***such that:*** $a=\frac{3x}{x+y+t}$, $b=\frac{3y}{x+y+t}$, $c=\frac{3z}{x+y+t}$***. Inequality (3) becomes:*** $9\left(\frac{x^2+y^2+z^2}{(x+y+t)^2}\right)+6 \ge \frac{9(x^2+y^2+z^2)}{(x+y+z)^2}\cdot\frac{9(xy+xz+yt)}{(x+y+z)^2}$

$$\Leftrightarrow \frac{3(x^2+y^2+z^2)}{(x+y+z)^2}+2 \ge \frac{27(x^2+y^2+z^2)(xy+xz+yz)}{(x+y+z)^4} \Leftrightarrow$$

$$\Leftrightarrow 3(x^2+y^2+z^2)(x+y+z)^2+2(x+y+z)^4 \ge 27(x^2+y^2+z^2)(xy+xz+yz) \quad (4)$$

Now, using Cartoaje's theorem: If $f_4(x,y,z)$ ***is an symmetric polynomial function of degree*** $n=4$ ***then:*** $f_4(x,y,z)\ge 0, \forall x,y,z \ge 0 \Leftrightarrow x=0$ ***and*** $y=z$

(if and only if)

In our case let $f_4(x,y,z)=3(x^2+y^2+z^2)(x+y+z)^2+2(x+y+z)^4=$

$$=2t(x^2+y^2+z^2)(xy+zx+yz)$$

$$f_4(0,y,y)=3\cdot 2y^2\cdot 4y^2+2\cdot 2^4y^4-27\cdot 2y^2\cdot y^2=$$

$24y^4+32y^4-54y^4=2y^4\ge 0 \Rightarrow$ ***inequality (4) its true.***

Solution 2 by Michael Sterghiou-Greece

$$2\left(\sum_{cyc} a^2\right)+3 \ge 3\left(3abc\cdot\sum_{cyc} a^3b\right)^{\frac{1}{2}} \quad (1)$$

Let $\left(\sum_{cyc} a, \sum_{cyc} ab, abc\right)=(p,q).\ p=3, \sum_{cyc} a^2=9-2q$

$r \le \left(\frac{a}{3}\right)^{\frac{3}{2}} \to \sqrt{r} \le \left(\frac{a}{3}\right)^{\frac{3}{4}}$***. From Vasc's inequality:***

$\left(\sum_{cyc} a^2\right)^2 \ge 3\sum_{cyc} a^3b$ ***(1)*** $\to f(q)=2\sqrt[4]{3}q^{\frac{7}{4}}-9\sqrt[4]{3}a^{\frac{3}{4}}-4q+21\ge 0$ ***(2)***

Where in the original $21-4q-3\sqrt{r}(9-2q)\ge 0$ ***we've replaced*** $\sqrt{r}$ ***by*** $\left(\frac{a}{3}\right)^{\frac{3}{4}}$ ***to get the stronger inequality (2). Note that*** $9-2q\ge 0, q\le 3$***. Now*** $f(3)=0$,

$f'(q)=\frac{7}{2}\sqrt[4]{3}q^{\frac{3}{4}}-\frac{27\sqrt[4]{3}}{4\sqrt[4]{q}}-4$, $f''(q)=\frac{3^{\frac{1}{4}}3(14q+9)}{16q^{\frac{5}{4}}}\geq 0$. So, $f'(q)\uparrow\to f'(q)<f'(3)=-\frac{1}{4}<$

$<0\to f(q)\downarrow\to f(q)>f(3)=0$ as $q\leq 3$. Done.

UP.149. Prove that:

$$\sum_{k=-l}^{l}\left[(-1)^k\binom{2l}{l+k}\binom{2m}{m+k}\binom{2n}{n+k}\right]=$$

$$=\frac{(l+m+n)!\,(2l)!\,(2m)!\,(2n)!}{(l+m)!\,(l+n)!\,(m+n)!\,(l)!\,(m)!\,(n)!}$$

Proposed by Shivam Sharma – New Delhi – India

Solution by proposer

$$S=\sum_{k=-l}^{l}\left[(-1)^k\binom{2l}{l+k}\binom{2m}{m+k}\binom{2n}{n+k}\right]$$

Assuming that, $l=\min(l,m,n)$. *This reduces to the series,*

$$\frac{(-1)^l(2m)!\,(2n)!}{(m-l)!\,(m+l)!\,(n-l)!\,(n+l)!}\,{}_3f_2\left(\begin{matrix}-2l,-m-l,-n-l\\ m-l+1,n-l+1\end{matrix};1\right)$$

Now, Applying Dixon's formula, we get $\Rightarrow {}_3f_2\left(\begin{matrix}-2l-2\varepsilon,-m-l-\varepsilon,-n-l-\varepsilon\\ m-l-\varepsilon+1,n-l-\varepsilon+1\end{matrix};1\right)$

$$=\frac{\Gamma(1-l-\varepsilon)\Gamma(1+m-l-\varepsilon)\Gamma(1+m+n+l+\varepsilon)}{\Gamma(1-2l-2\varepsilon)\Gamma(1+m)\Gamma(1+n)\Gamma(1+m+n)}$$

Now, apply Euler's reflection formula, we get,

$$\frac{\sin\pi(2l+2\varepsilon)}{\sin\pi\,(l+\varepsilon)}\cdot\frac{\Gamma(2l+2\varepsilon)}{\Gamma(l+\varepsilon)}\cdot\frac{\Gamma(1+m-l-\varepsilon)\Gamma(1+n-l-\varepsilon)\Gamma(1+m+n+l+\varepsilon)}{\Gamma(1+m)\Gamma(1+n)\Gamma(1+m+n)}$$

As the limit $\varepsilon\to 0$, *this expression gives* $\Rightarrow 2(-1)^l\frac{(2l-1)!}{(l-1)!}\cdot\frac{(m-l)!(n-l)!(m+n+l)!}{m!n!(m+n)!}$

(or) $S=\frac{(l+m+n)!(2l)!(2m)!(2n)!}{(l+m)!(l+n!)(m+n)!l!m!n!}$ *(Answer)*

UP.150. Determine all continuos functions $f:\mathbb{R}\to\mathbb{R}$ such that

$f(x+y)=f(x)+f(y)+xy$ **and** $f(1)=1$ **for all** $x,y\in\mathbb{R}$.

Proposed by Mihály Bencze – Romania

Solution 1 by proposer

If $x = y = 0$ ***then*** $f(0) = 0$. ***By induction holds*** $f(nr) = nf(r) + \frac{n(n-1)}{2}r^2$

for all $r \in \mathbb{R}$. ***If*** $r = 1$ ***then*** $f(n) = \frac{n(n+1)}{2}$ ***for all*** $n \in \mathbb{N}$. ***If*** $r = -1$ ***then***

$$0 = f(0) = f(1-1) = f(1) + f(-1) - 1 \Rightarrow f(-1) = 0 \Rightarrow f(-n) = \frac{(-n)(-n+1)}{2}$$

for all $n \in \mathbb{N}$. ***If*** $r = \frac{p}{q} \in \mathbb{Q}$ ***then*** $f(qr) = f\left(q \cdot \frac{p}{q}\right) = f(1) = qf\left(\frac{p}{q}\right) + \frac{q(q-1)}{2}\left(\frac{p}{q}\right)^2$

$(p, q \in \mathbb{N}^*)$ ***but*** $f(p) = \frac{p(r+1)}{2} \Rightarrow f\left(\frac{p}{q}\right) = \frac{1}{2} \cdot \frac{p}{q}\left(\frac{p}{q} + 1\right)$. ***Let*** $x_n \to r \in \mathbb{R} \setminus \mathbb{Q}, x_n \in \mathbb{Q}$

f ***is continuosly*** $\Rightarrow f(x_n) = \frac{1}{2}x_n(x_{n+1})$

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \frac{1}{2}x_n(x_n + 1) \Rightarrow f(r) = \frac{r(r+1)}{2}$$

And finally $f(x) = \frac{x(x+1)}{2}, \forall x \in \mathbb{R}$.

Solution 2 by Rovsen Pirguliyev-Sumgait-Azerbaijan

$$f(x+y) = f(x) + f(y) + xy; y = 1 \overset{f(1)=1}{\Rightarrow} f(x+1) - f(x) = x + 1 \quad (1)$$

$$x = 1, 2 \dots n-1 \text{ we have} \begin{cases} f(2) - f(1) = 2 \\ f(3) - f(2) = 3 \\ \dots \\ f(n) - f(n-1) = n \end{cases} \oplus \Rightarrow f(n) = \frac{n(n+1)}{2}$$

then the function $f(x) = \frac{x(x+1)}{2}$ ***is a solution of equation*** $f(x+y) = f(x) + f(y) + xy$ ***(2)***

denote $g(x) = f(x) - f_1(x)$, ***then using (2), we have:*** $g(x+y) = f(x+y) - f_1(x+y) =$

$$= f(x) + f(y) + xy - f_1(x) - f_1(y) - xy = \underbrace{\left(f(x) - f_1(x)\right)}_{g(x)} + \underbrace{\left(f(y) - f_1(y)\right)}_{g(y)} = g(x) + g(y) \quad (3)$$

the function $g(x)$ ***is continuos, since*** $f(x)$ ***is continuos.***

$$g(x+y) = g(x) + g(y) \overset{Cauchy\ func.equation}{\Rightarrow} g(x) = kx, g(x) = f(x) - f_1(x) \Rightarrow$$

$$\Rightarrow kx = f(x) - \frac{x(x+1)}{2} \Rightarrow f(x) = \frac{x^2}{2} + \frac{x}{2} + kx = \frac{x^2}{2} + \left(k + \frac{1}{2}\right)x$$

using $f(1) = 1 \Rightarrow k = 0$. ***Hence*** $f(x) = \frac{x^2+x}{2}$

Solution 3 by Tran Hong-Vietnam

$$x = y \Rightarrow f(2x) = 2f(x) + x^2\ (\forall \in \mathbb{R}) \Leftrightarrow f(2x) - \frac{2x(2x+1)}{2} = 2\left[f(x) - \frac{x(x+1)}{2}\right]$$

Let $g(x) = f(x) - \frac{x(x+1)}{2} \Rightarrow g(1) = f(1) - 1 = 0$***. We have:*** $g(2x) = 2g(x)\ (\forall x \in \mathbb{R})$

***** $\forall x \neq 0 \Rightarrow \frac{g(2x)}{2x} = \frac{g(x)}{x}$***. Let*** $h(x) = \frac{g(x)}{x}\ \ (\forall x \neq 0)$

$$\Rightarrow h(2x) = h(x) \Rightarrow h(x) = h\left(\frac{x}{2}\right) = \cdots = h\left(\frac{x}{2^n}\right)\ \ (n \in \mathbb{N})$$

$$\Rightarrow h(x) = \lim_{n\to\infty} h(x) = h\left(\lim_{n\to\infty}\frac{x}{2^n}\right) = h(0) \Rightarrow g(x) = h(0)x\ \ (\forall x \neq 0)$$

More $g(1) = 0 \Rightarrow h(0) \Rightarrow g(x) = 0\ \ (\forall x \neq 0)$ ***(1)***

***** $x = 0 \Rightarrow g(2.0) = 2g(0) \Rightarrow g(0) = 2$ ***(2)***

From (1) and (2) $\Rightarrow g(x) = 0\ \forall x \in \mathbb{R} \Rightarrow f(x) = \frac{x(x+1)}{2}\ \ (\forall x \in \mathbb{R})$