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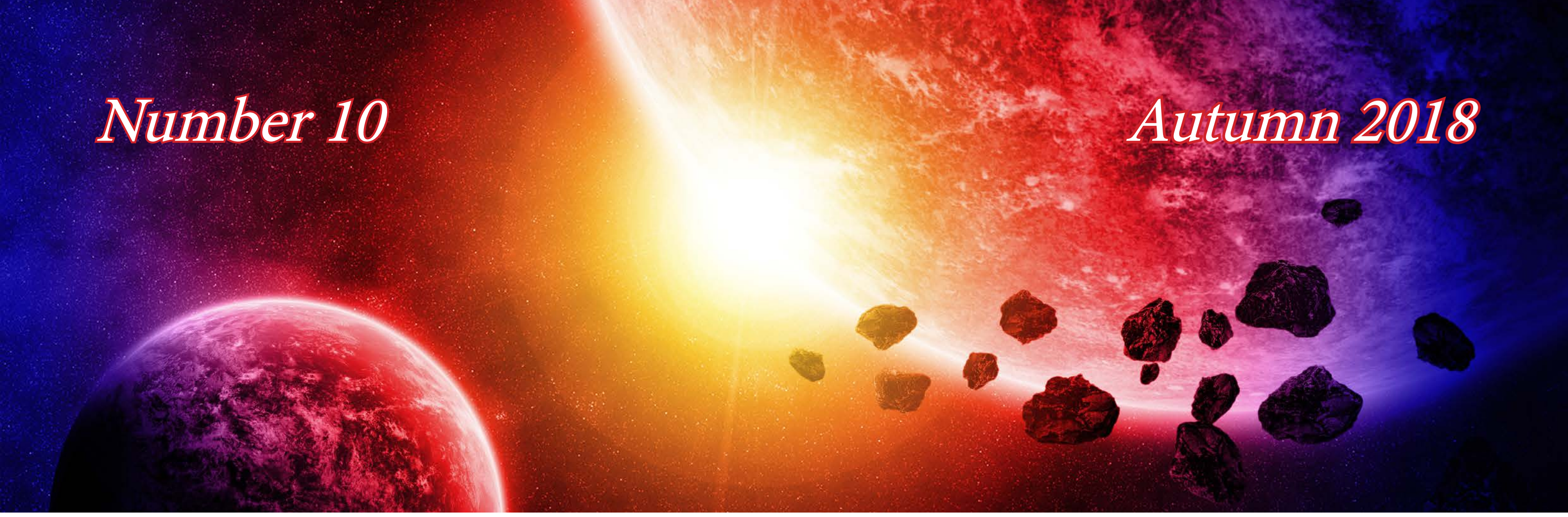

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PROBLEMS FOR JUNIORS

JP.136. Let x, y, z be positive real numbers such that: $xyz = 1$. Find the maximum of the expression:

$$Q = \frac{1}{\sqrt[3]{2x^5 + y^4 - x^2 + 4}} + \frac{1}{\sqrt[3]{2y^5 + z^4 - y^2 + 4}} + \frac{1}{\sqrt[3]{2z^5 + x^4 - z^2 + 4}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

JP.137. Let $x, y \geq 1$. Prove that:

$$\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \geq 2 + \frac{4(x-y)^2}{(2x + xy + 1)(2y + xy + 1)}$$

Proposed by Andrei Ştefan Mihalcea - Romania

JP.138. Let $a, b, c > 0$, with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$. Prove that:

$$(4a - 3)(4b - 3)(4c - 3) \geq 243\sqrt[3]{abc}$$

Proposed by Andrei Ştefan Mihalcea - Romania

JP.139. Let x, y, z be positive real numbers such that: $x^2 + y^2 + z^2 = 3$. Find the minimum of the expression:

$$P = \frac{x}{\sqrt[4]{\frac{y^8+z^8}{2}} + 3yz} + \frac{y}{\sqrt[4]{\frac{z^8+x^8}{2}} + 3zx} + \frac{z}{\sqrt[4]{\frac{x^8+y^8}{2}} + 3xy}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

JP.140. Let $a, b, c > 0$. Prove that:

$$\sum \frac{\sqrt{a+b}}{a} \leq \left(\sum \frac{1}{a}\right)\sqrt{\sum a - \frac{\sum ab}{\sum a}}$$

Proposed by Andrei Ştefan Mihalcea - Romania

JP.141. Let $a, b, c > 0$. Prove that:

$$\left(\sum \sqrt{\frac{b+c}{a}}\right)^2 \leq \frac{2(\sum ab)^3}{3a^2b^2c^2}$$

Proposed by Andrei Ştefan Mihalcea - Romania

JP.142. Let $a, b, c \geq 1$. Prove that:

$$\sum \sqrt{\frac{a-1}{bc}} \leq \Big(\sum \frac{1}{ab}\Big)\sqrt{abc - \frac{\sum a}{3}}$$

Proposed by Andrei Ştefan Mihalcea - Romania

JP.143. In any ABC triangle the following relationship holds:

$$\frac{w_a^2}{h_b \cdot h_c} + \frac{w_b^2}{h_c \cdot h_a} + \frac{w_c^2}{h_a \cdot h_b} \leq \Big(\frac{R}{r}\Big)^2 - 1$$

all notations are usual sense.

Proposed by Mehmet Şahin - Ankara - Turkey

JP.144. In any ABC triangle the following relationship holds:

$$\frac{a}{w_a} + \frac{b}{w_b} + \frac{c}{w_c} \geq \frac{4s}{3R}$$

Proposed by Mehmet Şahin - Ankara - Turkey

JP.145. In any ABC triangle the following relationship holds:

$$\frac{m_a}{r + r_a} + \frac{m_b}{r + r_b} + \frac{m_c}{r + r_c} \leq \frac{s}{2r}$$

all notations are usual sense.

Proposed by Mehmet Şahin - Ankara - Turkey

JP.146. Let x, y, z be positive real numbers such that: $xyz = 1$. Find the maximum of the expression:

$$P = \frac{1}{\sqrt[3]{2(x^5 - x^3 + 4)}} + \frac{1}{\sqrt[3]{2(y^5 - y^3 + 4)}} + \frac{1}{\sqrt[3]{2(z^5 - z^3 + 4)}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

JP.147. Let a, b, c be positive real numbers such that: $a^2 + b^2 + c^2 = 3abc$. Find the minimum of the expression:

$$P = \frac{a^2}{\sqrt[3]{4(b^3 + c^3)}} + \frac{b^2}{\sqrt[3]{4(c^3 + a^3)}} + \frac{c^2}{\sqrt[3]{4(a^3 + b^3)}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

JP.148. Let a, b, c be positive real numbers such that: $ab + bc + ca = 12$. Prove that:

$$\frac{a^3 + b^3}{2b^2 - bc + 2c^2} + \frac{b^3 + c^3}{2c^2 - ca + 2a^2} + \frac{c^3 + a^3}{2a^2 - ab + 2b^2} \geq 4$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

JP.149. Find all functions: $f:(0,+\infty)\to\mathbb{R}$ which verify the relationship:

$$\ln(xy)\le xf(x)+yf(y)\le xyf(xy),\forall x,y>0$$

Proposed by Marian Ursărescu - Romania

JP.150. Let be $z_1,z_2,z_3\in\mathbb{C}^*$ different in pairs such that $|z_1|=|z_2|=|z_3|$. If $(z_1+z_2)(z_2+z_3)(z_3+z_1)+z_1z_2z_3=0$, then z_1,z_2,z_3 are the affixes of an equilateral triangle.

Proposed by Marian Ursărescu - Romania

PROBLEMS FOR SENIORS

SP.136. Let x,y,z be positive real numbers such that: $x^4+y^4+z^4=xy+yz+zx$. Find the maximum of the expression:

$$P=\sqrt[3]{\frac{x^6+y^6}{2}}+\sqrt[3]{\frac{y^6+z^6}{2}}+\sqrt[3]{\frac{z^6+x^6}{2}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

SP.137. Let $a,b,c>0$ such that: $a+b+c=3$. Prove that:

$$\frac{a}{\sqrt[3]{4(b^6+c^6)}+7bc}+\frac{b}{\sqrt[3]{4(c^6+a^6)}+7ca}+\frac{c}{\sqrt[3]{4(a^6+b^6)}+7ab}+$$
$$+\frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{12}\ge\frac{7}{12}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

SP.138. Let a,b,c be positive real numbers such that: $a+b+c=3$. Prove that:

$$\frac{a^2}{\sqrt{5(b^4+4)}}+\frac{b^2}{\sqrt{5(c^4+4)}}+\frac{c^2}{\sqrt{5(a^4+4)}}\ge\frac{3}{5}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

SP.139. In ABC triangle the lengths of sides BC,CA,AB are a,b,c. Let h_a,h_b,h_c be the distances from A,B,C to BC,CA,AB; l_a,l_b,l_c are the lengths of the bisectors A,B,C. Prove that:

$$\frac{l_al_b}{l_c}+\frac{l_bl_c}{l_a}+\frac{l_cl_a}{l_b}\ge\frac{h_ah_b}{h_c}+\frac{h_bh_c}{h_a}+\frac{h_ch_a}{h_b}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

SP.140. Let a, b, c be positive real numbers. Prove that:

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\geq\frac{4(a^2+b^2+c^2)}{ab+bc+ca}+\frac{2(ab+bc+ca)}{a^2+b^2+c^2}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

SP.141. Let $a, b, c > 0$ such that: $a+b+c=3$. Prove that:

$$\frac{a^4}{b^4(2ab-\sqrt{c}+2)}+\frac{b^4}{c^4(2bc-\sqrt{a}+2)}+\frac{c^4}{a^4(2ca-\sqrt{b}+2)}\geq$$

$$\geq\frac{a^2+b^2+c^2}{3}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

SP.142. Let a, b, c be positive real numbers such that: $abc=1$. Prove that:

$$\frac{a^2b^2}{a^4-2a+b^2+2}+\frac{b^2c^2}{b^4-2b+c^2+2}+\frac{c^2a^2}{c^4-2c+a^2+2}\leq$$

$$\leq\frac{a^2+b^2+c^2+3}{4}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

SP.143. Let x, y, z be non-negative real numbers. Prove that:

$$x\sqrt{3x^2+yz}+y\sqrt{3y^2+zx}+z\sqrt{3z^2+xy}\geq x^2+y^2+z^2+xy+yz+zx$$

Proposed by Do Quoc Chinh - Ho Chi Minh - Vietnam

SP.144. Let A, B, C be the corners in a triangle ABC. Prove that:

$$\left(\frac{\sin\frac{A}{2}}{\tan\frac{B}{2}}\right)^2+\left(\frac{\sin\frac{B}{2}}{\tan\frac{C}{2}}\right)^2+\left(\frac{\sin\frac{C}{2}}{\tan\frac{A}{2}}\right)^2\geq\frac{9}{4}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

SP.145. If $1 < a \leq b$ then:

$$\int_a^b\int_a^b\int_a^b\frac{dxdydz}{1+\sqrt[3]{xyz}}\leq\log\left(\sqrt[3]{\frac{b+1}{a+1}}\right)^{(b-a)^2}$$

Proposed by Daniel Sitaru - Romania

SP.146. Let be $A, B \in M_3(\mathbb{R})$ such that:

$$AB = \begin{pmatrix} 2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

Find: $\det((BA)^2 - 3I_3)$.

Proposed by Marian Ursărescu - Romania

SP.147. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ having the property:

$$f(x) + 2f(2x) + f(4x) = 25x^2 + 9x + 4, \forall x \in \mathbb{R}.$$

Proposed by Marian Ursărescu - Romania

SP.148. Let be $x_0 > 0$ and $x_{n+1} = \arctan \frac{x_n}{1+x_n}, \forall n \in \mathbb{N}$. Find: $\lim_{n\to\infty} n \cdot x_n$.

Proposed by Marian Ursărescu - Romania

SP.149. Let be the sequence $(x_n)_{n\in\mathbb{N}} : x_0 > 1$ and $x_{n+1} = 1 + \ln\left(\frac{2x_n}{1+x_n}\right), \forall n \in \mathbb{N}$. Find: $\lim_{n\to\infty} n \ln x_n$.

Proposed by Marian Ursărescu - Romania

SP.150. Let be $f \in \mathbb{Z}, f = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, such that $a_1, a_2, \ldots, a_n \in \{\pm 1, \pm 2, \ldots, \pm n\}$. If a_0 is a prime number, $a_0 > n^2$ then f is irreducible over $\mathbb{Z}$.

Proposed by Marian Ursărescu - Romania

UNDERGRADUATE PROBLEMS

UP.136. Prove that:

$$\sum_{k=0}^{n} T_{4k}(x) = \frac{1}{4}\left[\frac{2 + U_{4n+2}(x)}{x\sqrt{1-x^2}}\right]$$

where, $T_n(x)$ and $U_n(x)$ denotes the Chebyshev Polynomials of first and second kind.

Proposed by Shivam Sharma - New Delhi - India

UP.137. Let $f, g : \mathbb{R}_+^* \to \mathbb{R}_+^*$ be functions such that:

$$\lim_{x\to\infty}(f(x+1) - f(x)) = a \in \mathbb{R}_+^*, \lim_{x\to\infty}\frac{g(x+1)}{xg(x)} = b \in \mathbb{R}_+^* \text{ and}$$

exists $\lim_{x\to\infty}\frac{f(x)}{x}$ and $\lim_{x\to\infty}\frac{(g(x))^{\frac{1}{x}}}{x}$. For $t \in \mathbb{R}$ calculate the limit:

$$\lim_{x\to\infty}(f(x))^{\cos^2 t}\Big((g(x))^{\frac{\sin^2 t}{x+1}} - (g(x))^{\frac{\sin^2 t}{x}}\Big)$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

UP.138. Let $f, g : \mathbb{R}_+^* \to \mathbb{R}_+^*$ such that:

$$\lim_{x\to\infty}(f(x+1)-f(x)) = a \in \mathbb{R}_+^*, \lim_{x\to\infty}\frac{g(x+1)}{xg(x)} = b \in \mathbb{R}_+^* \text{ and exists}$$

$$\lim_{x\to\infty}\frac{f(x)}{x}, \lim_{x\to\infty}\frac{(g(x))^{\frac{1}{x}}}{x}. \text{ For } t \in \mathbb{R}, \text{ calculate:}$$

$$\lim_{x\to\infty}(f(x))^{\sin^2 t}\Big((g(x))^{\frac{\cos^2 t}{x+1}} - (g(x))^{\frac{\cos^2 t}{x}}\Big)$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

UP.139. Calculate:

$$\lim_{x\to\infty}\left(x^{\cosh^2 t}\left((\Gamma(x+1))^{-\frac{\sinh^2 t}{x}} - ((\Gamma(x+2))^{-\frac{\sinh^2 t}{x+1}}\right)\right)$$

where $t \in \mathbb{R}$ and Γ is the Gamma function (Euler integral of the second kind).

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

UP.140. Calculate:

$$\lim_{x\to\infty}\left(x^{\sin^2 t}\left(((\Gamma(x+2))^{\frac{\cos^2 t}{x+1}} - ((\Gamma(x+1))^{\frac{\cos^2 t}{x}}\right)\right)$$

where $t \in \mathbb{R}$ and Γ is the Gamma function (Euler integral of the second kind).

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

UP.141. For $\{a_n\}_{n\geq 0}, a_n = \frac{(n+2)^{n+1}}{(n+1)^n}, x \in (-\infty, \infty), \{b_n(x)\}_{n\geq 1}$, $b_n(x) = n^{\sin^2 x}\Big(a_{n+1}^{\cos^2 x} - a_n^{\cos^2 x}\Big)$, find $\lim_{n\to\infty} b_n(x)$.

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

UP.142. Let $(x_n)_{n\geq 1}$ be a sequence which satisfy:

$$-\ln(mn+x_n)+\sum_{k=1}^{mn}\frac{1}{k}=\gamma$$

where m is positive integer and γ is Euler-Mascheroni's constant. Compute: $\lim_{n\to\infty} x_n$.

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

UP.143. Let $a,b\in\mathbb{R}_+,\gamma_n(a,b)=-\ln(n+a)+\sum_{k=1}^{n}\frac{1}{k+b}$ with $\lim_{n\to\infty}\gamma_n(a,b)=\gamma(a,b)\in\mathbb{R}$. Calculate:

$$\lim_{n\to\infty}\Big(\ln\frac{e}{n+a}+\sum_{k=1}^{n}\frac{1}{k+b}-\gamma(a,b)\Big)^n.$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

UP.144. If $x,y,z\geq 0$ then:

$$\cosh^2 x\cosh^2 y\cosh^2 z\geq 2(1+\cosh(x-y)+\cosh(y-z)+\cosh(z-x))\cdot$$
$$\cdot\sinh\frac{x+y}{2}\sinh\frac{y+z}{2}\sinh\frac{z+x}{2}$$

Proposed by Mihály Bencze - Romania

UP.145. Let be $(x_n)_{n\geq 1},x_n\in\mathbb{R}_+^*,\forall n\in\mathbb{N}^*$, such that exists

$$\lim_{n\to\infty}(x_{n+1}-x_n)=x\in\mathbb{R}_+^*.\ \textbf{Find:}$$

$$\lim_{n\to\infty}\left(\frac{(n+1)x_{n+1}}{\sqrt[n+1]{(2n+1)!!}}-\frac{nx_n}{\sqrt[n]{(2n-1)!!}}\right)$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

UP.146. Let $f:(0,\infty)\to(0,\infty)$ be a function with $\lim_{x\to\infty}\frac{f(x)}{x}=a\in(0,\infty)$ and $t\in\mathbb{R}$. Find:

$$\lim_{n\to\infty}\Big((n+1)^{\sin^2 t}\cdot\sqrt[n+1]{(f(1)f(2)\dots f(n)f(n+1))^{\cos^2 t}}-$$
$$-n^{\sin^2 t}\cdot\sqrt[n]{(f(1)f(2)\dots f(n))^{\cos^2 t}}\Big)$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

UP.147. In an ABC triangle let be a,b,c the lengths of BC,CA,AB, and r_a,r_b,r_c exradii. Prove that:

$$\frac{r_a^2}{\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{r_b^2}{\tan\frac{C}{2}\tan\frac{A}{2}}+\frac{r_c^2}{\tan\frac{A}{2}\tan\frac{B}{2}}\geq\frac{9(a^2+b^2+c^2)}{4}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

UP.148. Let a, b, c be positive real numbers such that: $a+b+c=3$. Prove that: $2(a^2+b^2+c^2)+3 \geq 3\sqrt{3abc(a^3b+b^3c+c^3a)}$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

UP.149. Prove that:

$$\sum_{k=-l}^{l}\left[(-1)^k\binom{2l}{l+k}\binom{2m}{m+k}\binom{2n}{n+k}\right]=$$

$$=\frac{(l+m+n)!(2l)!(2m)!(2n)!}{(l+m)!(l+n)!(m+n)!(l)!(m)!(n)!}$$

Proposed by Shivam Sharma - New Delhi - India

UP.150. Determine all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(x+y)=f(x)+f(y)+xy$ and $f(1)=1$ for all $x, y\in\mathbb{R}$.

Proposed by Mihály Bencze - Romania

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