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In acute $\Delta\ ABC$ the following relationship holds:

$$a^2+b^2+c^2\ge 6abc\sqrt{\frac{6\cos A\cos B\cos C}{a^2+b^2+c^2}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Serban George Florin-Romania, Solution 2 by Marian Ursarescu-Romania, Solution 3 by Soumava Chakraborty-Kolkata-India, Solution 4 by Myagmarsuren Yadamsuren-Darkhan-Mongolia, Solution 5 by Soumitra Mandal-Chandar Nagore-India, Solution 6 by Sanong Huayrerai-Nakon Pathom-Thailand

Solution 1 by Serban George Florin-Romania

$$a^2+b^2+c^2\ge 6abc\left.\sqrt{\frac{6\cos A\cos B\cos C}{a^2+b^2+c^2}}\right|^2$$

$$(a^2+b^2+c^2)^2\ge\frac{6^2a^2b^2c^2\cdot 6\cos A\cos B\cos C}{a^2+b^2+c^2}$$

$$(a^2+b^2+c^2)^3\ge 6^3a^2b^2c^2\cos A\cos B\cos C$$

$$(a^2+b^2+c^2)^3\overset{Ma\ge Mg}{\ge}\left(3\sqrt[3]{a^2b^2c^2}\right)^3=3^3a^2b^2c^2\ge 6^3a^2b^2c^2\cos A\cos B\cos C$$

$$\Rightarrow\cos A\cos B\cos C\le\frac{1}{8},\ \sqrt[3]{\cos A\cos B\cos C}\le\frac{1}{2}$$

If ΔABC *is obtuse triangle* $\Rightarrow\cos A\cos B\cos C<0<\frac{1}{8}$ *(A)*

If $\Delta\ ABC$ *is acute-angled.*

$$\sqrt[3]{\cos A\cos B\cos C}\overset{(Mg\le Ma)}{\le}\frac{\cos A+\cos B+\cos C}{3}\le\frac{1}{2}\Rightarrow\cos A+\cos B+\cos C\le\frac{3}{2}\Rightarrow$$

$$\Rightarrow 1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\le\frac{3}{2}\Rightarrow\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\le\frac{1}{8}$$

$$\sin\frac{A}{2}\le\frac{a}{2\sqrt{bc}},\prod\sin\frac{A}{2}\le\prod\frac{a}{2\sqrt{bc}}=\frac{1}{8}\quad (A)$$

Solution 2 by Marian Ursarescu-Romania

Inequality $\Leftrightarrow\sqrt{(a^2+b^2+c^2)^3}\ge 6abc\sqrt{6\cos A\cos B\cos C}\Leftrightarrow$

$$(a^2+b^2+c^2)^3\ge 6^3\cdot a^2b^2c^2\cdot\cos A\cos B\cos C\quad (1)$$

$$a^2+b^2+c^2=2(s^2-r^2-4Rr)\ (2)\ abc=4sRr\quad (3)$$

$$\cos A\cos B\cos C=\frac{s^2-(2R+r)^2}{4R^2}=\frac{s^2-4R^2-4Rr-r^2}{4R^2}\quad (4)$$

From (1) + (2) + (3) + (4) we must show:

$$8(s^2-r^2-4Rr)^3\ge 6^3\cdot 16s^2R^2r^2\cdot\frac{s^2-4R^2-4Rr-r^2}{4R^2}\Leftrightarrow$$

$$\Leftrightarrow (s^2-r^2-4Rr)^3\ge 108s^2r^2(s^2-4R^2-4Rr-r^2)\quad (5)$$

Now, use the Gerretsen inequality $\Rightarrow$

$$16Rr-5r^2\le s^2\le 4R^2+4Rr+3r^2\quad (6)$$

From (5) + (6) we must show: $(2R-r)^3\ge s^2r\quad (7)$

Now, from Gergonne's inequality: $s^2\le\frac{R(4R+r)^2}{2(2R-r)}\quad (8)$

From (7) + (8) we must show:

$$(2R-r)^3\ge\frac{R(4R+r)^2}{2(2R-r)}\Leftrightarrow(2R-r)^4\ge\frac{Rr}{2}(4R+r)^2\quad (9)$$

From Euler $R\ge 2r$ *(10). From (9) + (10)* $\Leftrightarrow(2R-r)^4\ge r^2(4R+r)^2\Leftrightarrow$

$$\Leftrightarrow(2Rr-r)^2\ge r(4R+r)\Leftrightarrow R\ge 2r\quad \textit{(true)}$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\because\prod\cos A\le\frac{1}{8},\therefore RHS\le 6abc\sqrt{\frac{3}{4\sum a^2}}\overset{?}{\le}\sum a^2\Leftrightarrow\left(\sum a^2\right)^2\overset{?}{\ge}36a^2b^2c^2\cdot\frac{3}{4\sum a^2}\Leftrightarrow$$

$$\Leftrightarrow(\sum a^2)^3\ge 27a^2b^2c^2\rightarrow\textit{true (AM-GM) (Done)}$$

Solution 4 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$m_a \geq \sqrt{s(s-a)}, m_b \geq \sqrt{s(s-b)}$ **and** $m_c \geq \sqrt{s(s-c)}$ **and** $abc = 4Rrs$ **then**

$$\sum_{cyc} \frac{m_a}{bc} \geq 3\sqrt[3]{\frac{m_a m_b m_c}{a^2 b^2 c^2}} \geq 3\sqrt[3]{\frac{s\Delta}{16R^2 r^2 s^2}} = 3\sqrt[3]{\frac{s^2 r}{16R^2 r^2 s^2}} \geq 3\sqrt[3]{\frac{1}{8R^2 \cdot R}} \ [\because R \geq 2r]$$

$$= \frac{3}{2R} \text{ (proved)}$$

Solution 5 by Soumitra Mandal-Chandar Nagore-India

$$a^2 + b^2 + c^2 \geq 6abc\sqrt{\frac{6\cos A\cos B\cos C}{a^2 + b^2 + c^2}}$$

$$(a^2 + b^2 + c^2)^3 \geq 27a^2 b^2 c^2 \ \text{(true)}$$

$$(a^2 + b^2 + c^2)^2 \geq \frac{27a^2 b^2 c^2}{a^2 + b^2 + c^2}$$

$$a^2 + b^2 + c^2 \geq 3abc\sqrt{\frac{3}{a^2 + b^2 + c^2}} = 6abc \cdot \sqrt{\frac{\frac{6}{27} \cdot \left(\frac{3}{2}\right)^3}{a^2 + b^2 + c^2}} =$$

$$= 6abc\sqrt{\frac{6\left(\frac{1 + \frac{1}{2}}{3}\right)^3}{a^2 + b^2 + c^2}} \overset{Euler}{\geq} 6abc\sqrt{\frac{6\left(\frac{1 + \frac{r}{R}}{3}\right)^3}{a^2 + b^2 + c^2}} =$$

$$= 6abc\sqrt{\frac{6\left(\frac{\cos A + \cos B + \cos C}{3}\right)^3}{a^2 + b^2 + c^2}} \overset{Ma \geq Mg}{\geq} 6abc\sqrt{\frac{6\cos A\cos B\cos C}{a^2 + b^2 + c^2}}$$

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R}; \frac{1}{2} \geq \frac{r}{R} \text{ - Euler}$$

Solution 6 by Sanong Huayrerai-Nakon Pathom-Thailand

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = c^2 + a^2 - 2ca\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$a^2 + b^2 + c^2 = 2(bc\cos A + ca\cos B + ab\cos C)$$

$$(a^2+b^2+c^2)^3=\left(2(bc\cos A+ca\cos B+ab\cos C)\right)^3\overset{AM-GM}{\geq}$$

$$\geq 8\cdot 27abc\cdot abc\cos A\cos B\cos C=6^3(abc)^2\cos A\cos B\cos C$$

$$(a^2+b^2+c^2)^2\geq\frac{6^3(abc)^2\cos A\cos B\cos C}{a^2+b^2+c^2}$$

$$a^2+b^2+c^2\geq 6abc\sqrt{\frac{6\cos A\cos B\cos C}{a^2+b^2+c^2}}$$