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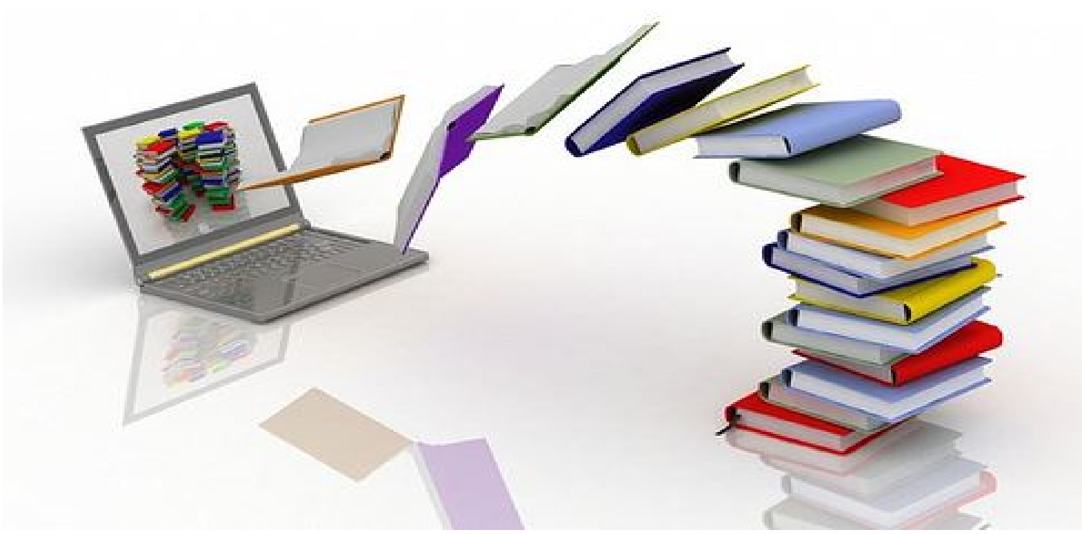

In $\Delta\ ABC$ the following relationship holds:

$$15r_a^2 + 10r_b^2 + 7r_c^2 > 270r^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursărescu – Romania, Solution 2 by Soumava Chakraborty-Kolkata-India

Solution 1 by Marian Ursărescu – Romania

From Cauchy's inequality $\Rightarrow 15r_a^2 + 10r_b^2 + 7r_c^2 > \frac{\left(\sqrt{15}r_a + \sqrt{10}r_b + \sqrt{7}r_c\right)^2}{3}$ **(1)**

From (1) the inequality becomes: $\left(\sqrt{15}r_a + \sqrt{10}r_b + \sqrt{7}r_c\right)^2 > 3 \cdot 270r^2 \Leftrightarrow$

$$\Leftrightarrow \sqrt{15}r_a + \sqrt{10}r_b + \sqrt{7}r_c > 9\sqrt{10}r \quad (2)$$

We have: $r_a = s\tan\frac{A}{2}$ and $r = s\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$ **(3)**

From (2)+(3) we must show this:

$$\sqrt{15}s\tan\frac{A}{2} + \sqrt{10}s\tan\frac{B}{2} + \sqrt{7}s\tan\frac{C}{2} > 9\sqrt{10}s\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} \Leftrightarrow$$

$$\sqrt{15}\tan\frac{A}{2} + \sqrt{10}\tan\frac{B}{2} + \sqrt{7}\tan\frac{C}{2} > 9\sqrt{10}\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} \quad (4)$$

But $\sqrt{15}\tan\frac{A}{2} + \sqrt{10}\tan\frac{B}{2} + \sqrt{7}\tan\frac{C}{2} > 3\sqrt[3]{5\sqrt{42}\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}$ **(5)**

From (4)+(5) we must show:

$$3\sqrt[3]{5\sqrt{42}\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}} > 9\sqrt{10}\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} \Leftrightarrow$$

$$\Leftrightarrow \sqrt[3]{5\sqrt{42}\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}} > 3\sqrt{10}\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} \Leftrightarrow$$

$$5\sqrt{42}\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} > 3\sqrt{3}\cdot 10\sqrt{10}\tan^3\frac{A}{2}\tan^3\frac{B}{2}\tan^3\frac{C}{2} \Leftrightarrow$$

$$\sqrt{7}\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} > 6\sqrt{5}\tan^3\frac{A}{2}\tan^3\frac{B}{2}\tan^3\frac{C}{2} \Leftrightarrow$$

$$\Leftrightarrow \tan^2\frac{A}{2}\tan^2\frac{B}{2}\tan^2\frac{C}{2} < \frac{\sqrt{7}}{6\sqrt{5}} \quad \textit{(6)}$$

But in any $\Delta\, ABC$ ***we have relation:*** $\sum\tan\frac{A}{2}\tan\frac{B}{2} = 1 \Rightarrow$

$$\Rightarrow 1 = \sum\tan\frac{A}{2}\tan\frac{B}{2} \geq 3\sqrt[3]{\tan^2\frac{A}{2}\tan^2\frac{B}{2}\tan^2\frac{C}{2}} \Rightarrow$$

$$\Rightarrow \sqrt[3]{\tan^2\frac{A}{2}\tan^2\frac{B}{2}\tan\frac{C}{2}} \leq \frac{1}{3} \Leftrightarrow \tan^2\frac{A}{2}\tan^2\frac{B}{2}\tan^2\frac{C}{2} \leq \frac{1}{27} \Leftrightarrow \tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} \leq$$

$$\leq \frac{1}{3\sqrt{3}} < \frac{\sqrt{7}}{6\sqrt{5}} \Rightarrow (20 < 21)$$

6 *its true. Remark:* $s = \frac{a+b+c}{2}$

Solution 2 by Soumava Chakraborty-Kolkata-India

Given inequality $\Leftrightarrow s^2\left\{\frac{15}{(s-a)^2}+\frac{10}{(s-b)^2}+\frac{7}{(s-c)^2}\right\} > \frac{270s^2}{s^2} \Leftrightarrow \frac{15}{(s-a)^2}+\frac{10}{(s-b)^2}+\frac{7}{(s-c)^2} \overset{(1)}{>} \frac{270}{s^2}$

Now, LHS of (1) $= \frac{\left(\sqrt[3]{15}\right)^3}{(s-a)^2}+\frac{\left(\sqrt[3]{10}\right)^2}{(s-b)^2}+\frac{\left(\sqrt[3]{7}\right)^2}{(s-c)^2} \overset{Radon}{>} \frac{\left(\sqrt[3]{15}+\sqrt[3]{10}+\sqrt[3]{7}\right)^3}{\{\sum(s-a)\}^2} > \frac{278}{s^2} > \frac{270}{s^2}$

$\Rightarrow$ ***(1) is true (Proved)***