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ROMANIAN MATHEMATICAL MAGAZINE
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$$-1 < a,b,c < 1, \Omega(a) = \int_0^{\pi} \frac{\log(1 + a\cos x)}{\cos x} dx$$

Prove that:

$$\frac{1}{\pi^2}\left(\Omega^2(a) + \Omega^2(b) + \Omega^2(c)\right) \ge \sum (\sin^{-1} a \cdot \sin^{-1} b)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursarescu-Romania, Solution 2 by Sagar Kumar-Kolkata-India

Solution 1 by Marian Ursarescu-Romania

Let $f(a) = \frac{\ln(1+a\cos x)}{\cos x}$ is a continuous function in $a \Rightarrow \Omega'(a) = \int_0^{\pi} \frac{1}{1+a\cos x} dx$

$$\left.\begin{matrix} Let\ \tan\frac{x}{2} = t \Rightarrow x = 2\arctan t \Rightarrow dx = \frac{2}{1+t^2}dt \\ x = 0 \Rightarrow t = 0; x = \pi \Rightarrow t = \infty \end{matrix}\right\} \Rightarrow \Omega'(a)$$

$$= \int_0^{\infty} \frac{1}{1 + a\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt$$

$$= 2\int_0^{\infty} \frac{1}{1 + t^2 + a - at^2} dt = 2\int_0^{\infty} \frac{1}{(1-a)t^2 + 1 + a} dt = \frac{2}{1-a}\int_0^{\infty} \frac{1}{t^2 + \left(\sqrt{\frac{1+a}{1-a}}\right)^2} dt =$$

$$=\frac{2}{1-a}\cdot\frac{1}{\sqrt{\frac{1+a}{1-a}}}\arctan\frac{t}{\sqrt{\frac{1+a}{1-a}}}\Bigg|_0^{\infty}=\frac{\pi}{\sqrt{1-a^2}}\Rightarrow$$

$$\left.\begin{array}{c}\Omega(a)=\pi\int\frac{1}{\sqrt{1-a^2}}da=\pi\arcsin a+c\\ But\ \Omega(a)=0\Rightarrow c=0\end{array}\right\}\Rightarrow\Omega(a)=\pi\arcsin a\Rightarrow$$ ***we must show:***

$\sum(\arcsin a)^2\geq\sum\arcsin a\cdot\arcsin b$, ***which its true because*** $\sum x^2\geq\sum xy$

Solution 2 by Sagar Kumar-Kolkata-India

$$I(a)=\int_0^{\pi}\int_0^{a\cos x}\frac{dy}{1+y}\cdot\frac{dx}{(\cos x)}=\int_0^{\pi}\frac{dy}{1+y}\log\left(\frac{\sin(a\cos x)}{\sin(a)}\right)$$

$$I(a)=\int_0^{\pi}\frac{\log(1+a\cos x)}{\cos x}dx$$

$$I'(a)=\int_0^{\pi}\frac{dx}{1+a\cos x}$$

Put $\tan\frac{x}{2}=t;\sin^2\frac{x}{2}dx=2t$

$$I'(a)=2\int_0^{\infty}\frac{dt}{(1+a)+(1-a)t^2}$$

$$I'(a)=\frac{\pi}{\sqrt{1-a^2}};I(a)=\pi\sin^{-1}(a)+C$$

$$I(0)=0\Rightarrow c=0$$

$$I(a)=\pi\sin^{-1}(a)$$

$$\frac{\sum I^2(a)}{\pi^2}=(\sin^{-1}(a))^2+(\sin^{-1}(b))^2+\sin^{-1}(c)^2$$

as $x^2+y^2+z^2\geq xy+yz+zx\Rightarrow\frac{\sum I^2(a)}{\pi^2}\geq\sum\sin^{-1}(a)\sin^{-1}(b)$

(proved)