

**A dissection proof of the law of cosines,
replacing Cuoco-McConnell's rectangles with congruent triangles.**

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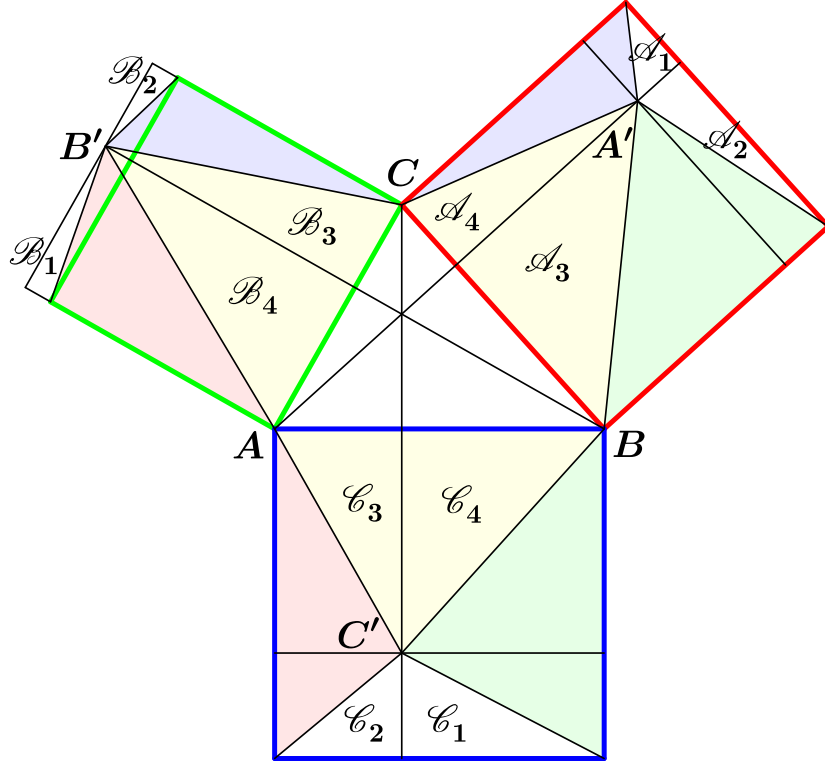
Taking up the challenge McConnell laid down at the end of his proof of the law of cosines ([6]), we give a completely visual dissection proof of this theorem, which applies to any triangle. In order to avoid the trigonometric expressions of Cuoco-McConnell's proof, we replaced the equal-area rectangles with congruent triangles. As a matter of fact, trigonometric expressions are implicitly based on the similarity of two right triangles with a common non-right angle. So they are conceptually less simple than our congruent triangles which are, moreover, easy to visualize. This makes our proof the only dissection proof and the simplest proof of its family, and thus one of the best options for a course of geometry.

Cuoco-McConnell's proof was published in [6], [3], [5], [8]. In [8], the proof is written in an algebraic way, whereas in [6], [3], [5], it is written in an equivalent geometric way involving three pairs of rectangles with the same area. A similar proof, involving two pairs of rectangles with the same area, can be found in [7]. Unfortunately, in these references, the areas of all the rectangles had to be expressed as trigonometric functions, in order to show that they were pairwise equal. In the particular case of a right triangle (where a pair of rectangles have zero area), and as in our proof, Euclid had already managed to avoid this by considering, for every pair of rectangles, a pair of congruent triangles with half the area of one of the rectangles ([4]). A generalization of Euclid's proof to the law of cosines can be found in [2] (where every triangle is replaced with a parallelogram made of this triangle and its symmetric). Each triangle of Euclid, as well as each triangle of our proof, has two sides in common with the triangle ABC of the theorem. However, denoting by x the angle formed by these sides, the corresponding angle in Euclid's triangle is $\pi/2 + x$, whereas the corresponding oriented angle in our triangle is $\pi/2 - x$. As a consequence of this, and unlike Euclid's triangles and the parallelograms of [2], two triangles of a same pair of our proof do not have intersection. This helps us in finding a visual dissection argument (cutting each rectangle into a set of congruent pieces) in order to prove that the area of the triangle is half the area of the corresponding rectangle. In both our proof and [2], trigonometry is only used to compute the inevitable area of the triangles/parallelograms corresponding to the term $2CA \cdot CB \cos(BCA)$ of the identity.

When one of the points A' , B' , C' of our proof (symmetric of A , B , C with respect to BC , CA , AB) is located outside its square, some areas need to be interpreted in an oriented

sense, and counted as negative. This already happened in [6], [3], [5], when the triangle was obtuse. Here, it can also happen when an altitude of the triangle ABC is longer than its perpendicular side. We give the proof in the acute case, when the altitude passing through B is longer than AC (thus, the point B' is outside its square), the other cases can be studied in a similar way.

It is worth mentioning that, in the particular case where the three points A' , B' , C' are located inside their squares, our proof becomes equivalent to Anderson's ([1]), where both small triangles adjacent to a same blue/green/red triangle of the figure are cut and pasted, so that the three triangles (or equivalently, the blue/green/red triangle and its symmetric) make a parallelogram. Thus, applying the same transformation (replacing a triangle with a parallelogram made of it and its symmetric), we can obtain the parallelograms of [2] from Euclid's triangles, and Anderson's parallelograms from the triangles of the proof of this note. As every triangle of this proof has two sides in common with a triangle of Euclid, with a corresponding angle $\pi - \alpha = \pi/2 - x$ instead of $\alpha = \pi/2 + x$, every parallelogram of Anderson and its corresponding parallelogram of [2] are congruent.



Here is our proof:

$$\begin{aligned}
 BC^2 + CA^2 - AB^2 &= [\text{red square}] + [\text{green square}] - [\text{blue square}] \\
 &= (2\mathcal{A}_3 + 2\mathcal{A}_2 + 2\mathcal{A}_4 + 2\mathcal{A}_1) + ((2\mathcal{B}_3 - 2\mathcal{B}_2) + (2\mathcal{B}_4 - 2\mathcal{B}_1)) - (2\mathcal{C}_3 + 2\mathcal{C}_2 + 2\mathcal{C}_4 + 2\mathcal{C}_1) \\
 &= 2((\mathcal{A}_3 + \mathcal{A}_2) + (\mathcal{A}_4 + \mathcal{A}_1) + (\mathcal{B}_3 - \mathcal{B}_2) + (\mathcal{B}_4 - \mathcal{B}_1) - (\mathcal{C}_3 + \mathcal{C}_2) - (\mathcal{C}_4 + \mathcal{C}_1))
 \end{aligned}$$

$$\begin{aligned}
&= 2([\text{green triangle}] + [\text{blue triangle}] + [\text{blue triangle}] + [\text{red triangle}] - [\text{red triangle}] - [\text{green triangle}]) \\
&= 4[\text{blue triangle}] = 2CA \cdot CB \sin(\pi/2 - BCA) = 2CA \cdot CB \cos(BCA).
\end{aligned}$$

References.

- [1] S. Anderson et al. Comments about McConnell's "Illustrated Law of Cosines". Website "Interactive Mathematics Miscellany and Puzzles" (A. Bogomolny):
<https://www.cut-the-knot.org/htdocs/dcforum/DCForumID11/90.shtml>
- [2] I. Boyadzhiev. The Law of Cosines. Website "Discover Math with GeoGebra":
<https://www.geogebra.org/m/ScsxfNx3>
- [3] A. Cuoco, A. Baccaglini-Frank, J. Benson, N. Antonellis D'Amato, D. Erman, B. Harvey, K. Waterman. CME Project: Geometry. Pearson Education (2009).
- [4] Euclid. Elements. Book I, proposition 47. Can be found in:
<https://mathcs.clarku.edu/~djoyce/java/elements/bookI/propI47.html>
- [5] R. E. Howe. The Cuoco Configuration. Amer. Math. Monthly 120 (2013), no. 10, 916-923.
- [6] D. McConnell. The Illustrated Law of Cosines. Website "Interactive Mathematics Miscellany and Puzzles" (A. Bogomolny):
<https://www.cut-the-knot.org/pythagoras/DonMcConnell.shtml>
- [7] J. Molokach. Law of Cosines-A Proof Without Words. Amer. Math. Monthly 121 (2014), no. 8, 722.
- [8] Entry "Law of cosines" (3.2 Proofs, using trigonometry). Website "Wikipedia":
https://en.wikipedia.org/wiki/Law_of_cosines#Using_trigonometry

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