Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
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USING INEQUALITY TO SOLVE THE EQUATION

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• The equation is a familiar and frequent mathematical formula used in secondary mathematics tests. The solution of the equation includes many methods such as: hidden sub, multiplication, ... In this article, I would like to exploit a method that is: use inequalities.
• Method: Experiment of the equation (usually the only solution). Then use the common inequalities such as Cauchy (AM-GM), Bunhiacopxki, Mincopxki ..... to evaluate the equation and satisfy the '=' equation.
• Following are some examples:

Example 1. Solve the equation : $\sqrt{x}+\sqrt{x^2-x+1}=x^2-x+2$ (1)

My solution. $x \geq 0$. Easy to see : $x=1$ is the solution (1). Therefore, $x^2-x+1=\left(x-\frac{1}{2}\right)^2+\frac{3}{4}>0$, The left-hand side equation contains the square root which should apply the AM-GM inequality to the two nonnegative numbers: $\sqrt{x}+\sqrt{x^2-x+1}=\sqrt{x.1}+\sqrt{(x^2-x+1)1} \leq \frac{x+1}{2}+\frac{(x^2-x+1)+1}{2}=\frac{x^2+3}{2}$

+ Let (1): $\Rightarrow x^2-x+2 \leq \frac{x^2+3}{2} \Leftrightarrow 2x^2-2x+4 \leq x^2+3 \Leftrightarrow x^2-2x+1 \leq 0 \Leftrightarrow (x-1)^2 \leq 0$

$\Rightarrow \begin{cases}(x-1)^2=0 \\ x=1 \\ x^2-x+1=1\end{cases} \Leftrightarrow x=1$ (satisfy) . Therefore $S=\{1\}$ is the only solution of the problem. .

Example 2. Solve the equation :$16x^4+5=6.\sqrt[3]{4x^3+x}$ (2)

My solution. Because $16x^4+5 \geq 5>0$ then let (2): $\Rightarrow 4x^3+x>0 \Leftrightarrow x.(4x^2+1)>0 \Leftrightarrow x>0$.

+ We see: $x=\frac{1}{2}$ is the solution (2). Therefore , by AM-GM inequality for three positive real numbers we have : $6.\sqrt[3]{4x^3+x}=3.\sqrt[4]{2.4x.(4x^2+1)} \leq 2+4x+(4x^2+1)=4x^2+4x+3$

+ Let (3): $\Rightarrow 16x^4+5 \leq 4x^2+4x+3 \Leftrightarrow 16x^4-4x^2-4x+2 \leq 0 \Leftrightarrow 8x^4-2x^2-2x+1 \leq 0$

$\Leftrightarrow 4x^3(2x-1)+2x^2(2x-1)-(2x-1) \leq 0 \Leftrightarrow (2x-1).(4x^3+2x-1) \leq 0 \Leftrightarrow (2x-1)^2.(2x^2+x+1) \leq 0$

$\Leftrightarrow (2x-1)^2 \leq 0 \Rightarrow (2x-1)^2=0$ (because $x>0$ then $2x^2+x+1>0$) . Therefore, equation occurs if

$\begin{cases}(2x-1)^2=0 \\ 2=4x=4x^2+1\end{cases} \Leftrightarrow \begin{cases}2x-1=0 \\ x=\frac{1}{2}\end{cases} \Leftrightarrow x=\frac{1}{2}$ (satisfy)

Therefore $S=\left\{\frac{1}{2}\right\}$ is the only solution of the problem.

Example 3. Solve the equation : $\sqrt{x^3+2x}+\sqrt{3x-1}=\sqrt{x^3+4x^2+4x+1}$ (3)

My solution.: $\begin{cases} x^3+2x\ge 0 \\ 3x-1\ge 0 \\ x^3+4x^2+4x+1\ge 0 \end{cases} \Leftrightarrow \begin{cases} x(x^2+2)\ge 0 \\ x\ge \frac{1}{3} \\ (x+1)(x^2+3x+1)\ge 0 \end{cases} \Leftrightarrow x\ge \frac{1}{3}$.

(3) $\Leftrightarrow \sqrt{x(x^2+2)}+\sqrt{3x-1}=\sqrt{(x+1)(x^2+3x+1)}$ (4)

+ We see that the expression on the left side (4) is the sum of two roots of the second order, noting that: $(\sqrt{x})^2+1^2=x+1$ và $(\sqrt{x^2+2})^2+(\sqrt{3x-1})^2=x^2+3x+1$ are 2 terms in the right hand side. Therefore by Bunhiacopxki inequality we have :

$$\left(\sqrt{x(x^2+2)}+1.\sqrt{3x-1}\right)^2\le\left[(\sqrt{x})^2+1^2\right].\left[(\sqrt{x^2+2})^2+(\sqrt{3x-1})^2\right]=(x+1)(x^2+3x+1)$$

$\Rightarrow \sqrt{x(x^2+2)}+\sqrt{3x-1}\le\sqrt{(x+1)(x^2+3x+1)}$ .Let (4), equation occurs when:

$\begin{cases} x\ge \frac{1}{3} \\ \sqrt{x}.\sqrt{3x-1}=1.\sqrt{x^2+2} \end{cases} \Leftrightarrow \begin{cases} x\ge \frac{1}{3} \\ x(3x-1)=x^2+2 \end{cases} \Leftrightarrow \begin{cases} x\ge \frac{1}{3} \\ 2x^2-x-2=0 \end{cases} \Rightarrow x=\frac{1+\sqrt{17}}{4}$ (satisfy)

Therefote $S=\left\{\frac{1+\sqrt{17}}{4}\right\}$ is the only solution of the problem.

Example 4. Solve the equation : $\sqrt[4]{x-2}+\sqrt[4]{4-x}=x^2-6x+11$ (5)

My solution : $\begin{cases} x-2\ge 0 \\ 4-x\ge 0 \end{cases} \Leftrightarrow 2\le x\le 4$:

- We see the expression on the left side as the sum of the two quadratic modes and: $(x-2)+(4-x)=2$. From there, by Bunhiacopxki inequality we have:

$$\left(\sqrt[4]{x-2}+\sqrt[4]{4-x}\right)^4=\left[\left(\sqrt[4]{x-2}+\sqrt[4]{4-x}\right)^2\right]^2\le\left[2\left(\sqrt{x-2}+\sqrt{4-x}\right)\right]^2=4.\left(\sqrt{x-2}+\sqrt{4-x}\right)^2\le$$

$$\le 4.2.[(x-2)+(4-x)]=4.2.2=16$$

$\Rightarrow \sqrt[4]{x-2}+\sqrt[4]{4-x}\le\sqrt[4]{16}=2$

+ Let (5)$\Rightarrow x^2-6x+11\le 2 \Leftrightarrow x^2-6x+9\le 0 \Leftrightarrow (x-3)^2\le 0 \Rightarrow (x-3)^2=0$ and equality if occur simultaneously: $\begin{cases} x-2=4-x \\ x-3=0 \end{cases} \Leftrightarrow x=3$. Therefore $S=\{3\}$ is the only solution of the problem.

Example 5. Solve the equation : $\sqrt{x^2+12x+61}+\sqrt{x^2-14x+113}=\sqrt{338}$ (6)

My solution : $\begin{cases} x^2+12x+61\ge 0 \\ x^2-14x+113\ge 0 \end{cases} \Leftrightarrow \begin{cases} (x+6)^2+25\ge 0 \\ (x-7)^2+64\ge 0 \end{cases}$. So that means consciousness $\forall x\in R$.

(6) $\Leftrightarrow \sqrt{(x+6)^2+5^2}+\sqrt{(7-x)^2+8^2}=\sqrt{338}$ (7)

+ We see, the left side of (7) is the tangent of two roots of form 2 : $\sqrt{a^2+b^2}+\sqrt{c^2+d^2}$, This suggests that Mincopxki's inequality is in the form $\sqrt{a^2+b^2}+\sqrt{c^2+d^2}\ge\sqrt{(a+c)^2+(b+d)^2}$ ('=' sign occurs when : $ad=bc$)

+ Therefore: $\sqrt{(x+6)^2+5^2}+\sqrt{(7-x)^2+8^2}\ge\sqrt{(x+6+7-x)^2+(5+8)^2}=\sqrt{13^2+13^2}=\sqrt{338}$. And let (7), equality occurs if : $(x+6).8=5.(7-x)\Leftrightarrow 8x+48=35-5x\Leftrightarrow 13x=-13\Leftrightarrow x=-1$.

Therefore $S=\{1\}$ is the only solution of the problem.

Example 6. Solve the equation : $\sqrt{8x^2-16x+10}+\sqrt{2x^2-4x+4}=\sqrt{7-x^2+2x}$ (8)

My solution. $\begin{cases}8x^2-16x+10\ge 0\\2x^2-4x+4\ge 0\\7-x^2+2x\ge 0\end{cases}\Leftrightarrow\begin{cases}8(x-1)^2+2\ge 0\\2(x-1)^2+2\ge 0\Leftrightarrow 1-2\sqrt{2}\le x\le 1+2\sqrt{2}\\(x-1)^2\le 8\end{cases}$.

$$(8)\Leftrightarrow\sqrt{(4x^2-4x+1)+(4x^2-12x+9)}+\sqrt{(x^2-4x+4)+x^2}=\sqrt{8-(x^2-2x+1)}$$

$$\Leftrightarrow\sqrt{(2x-1)^2+(3-2x)^2}+\sqrt{(2-x)^2+x^2}=\sqrt{8-(x-1)^2}\qquad(9)$$

+ The left side of (9) is the tangent of two roots of type 2 $\sqrt{a^2+b^2}+\sqrt{c^2+d^2}$, by Mincopxki inequality we have :

$$\sqrt{(2x-1)^2+(3-2x)^2}+\sqrt{(2-x)^2+x^2}\ge\sqrt{(2x-1+2-x)^2+(3-2x+x)^2}$$

$$=\sqrt{(x+1)^2+(3-x)^2}=\sqrt{2x^2-4x+10}=\sqrt{2(x-1)^2+8}\ge\sqrt{8}$$

+ Let (9) $\Rightarrow\sqrt{8-(x-1)^2}\ge\sqrt{8}\Leftrightarrow 8-(x-1)^2\ge 8\Leftrightarrow(x-1)^2\le 0\Rightarrow(x-1)^2=0$ and equality occurs if

$$\begin{cases}x-1=0\\(2x-1).x=(3-2x).(2-x)\end{cases}\Leftrightarrow\begin{cases}x=1\\2x^2-x=2x^2-7x+6\end{cases}\Leftrightarrow\begin{cases}x=1\\6x=6\end{cases}\Leftrightarrow x=1.$$

Therefore $S=\{1\}$ is the only solution of the problem.

Example 7. Solve the equation : $2\sqrt{x^3+1}+\sqrt{\frac{x^4+16}{2}}=3x^2-6x+10$ (10)

My solution. $x^3+1\ge 0\Leftrightarrow x\ge -1$. We see: $x=2$ is the solution (10), The left-hand side equation contains square root.. By AM-GM inequality and Bunhiacopxki we have :

$$\begin{cases}2\sqrt{x^3+1}=2.\sqrt{(x+1).(x^2-x+1)}\le(x+1)+(x^2-x+1)=x^2+2\\\left(\sqrt{\frac{x^4+16}{2}}+2x\right)^2\le(1^2+1^2).\left(\frac{x^4+16}{2}+(2x)^2\right)=x^4+8x^2+16=(x^2+4)^2\end{cases}$$

$$\Rightarrow\begin{cases}2\sqrt{x^3+1}\le x^2+2\\\sqrt{\frac{x^4+16}{2}}+2x\le x^2+4\end{cases}\Leftrightarrow\begin{cases}2\sqrt{x^3+1}\le x^2+2\\\sqrt{\frac{x^4+16}{2}}\le x^2-2x+4\end{cases}\Rightarrow 2\sqrt{x^3+1}+\sqrt{\frac{x^4+16}{2}}\le 2x^2-2x+6$$

+ Let (10) $\Rightarrow 3x^2-6x+10\le 2x^2-2x+6 \Leftrightarrow x^2-4x+4\le 0 \Leftrightarrow (x-2)^2\le 0 \Rightarrow (x-2)^2=0$ and

equality occurs if $\begin{cases} x+1=x^2-x+1 \\ \sqrt{\dfrac{x^4+16}{2}}=2x \\ x-2=0 \end{cases} \Leftrightarrow \begin{cases} x.(x-2)=0 \\ (x^2-4)^2=0 \\ x=2 \end{cases} \Leftrightarrow x=2$ (satisfy)

Therefore $S=\{2\}$ is the only solution of the equation.

* **Application exercises**. Solve the equations :

a) $2\sqrt{2x^3-x}=3x^2-3x+2$

b) $\sqrt{\dfrac{x^4+16}{2}}+\sqrt{2(x^2+4)}=3x+2$

c) $\sqrt{2x^3-2x^2+x}+2.\sqrt[4]{3x-2x^2}=x^4-x^3+3$

d) $\sqrt{2(x^4+1)}+3.\sqrt[3]{x}=x^2+4$

e) $3.\sqrt[3]{x^2-x+1}+\sqrt[4]{\dfrac{x^8+1}{2}}=2(x^4-3x+4)$

f) $\sqrt{2x^2+2x+5}+\sqrt{2x^2-10x+17}=6$