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For $n \in \mathbb{N}^* \wedge n \geq 2$. Prove:

$$\int_0^1 \left(\sum_{k=1}^{n} e^{x^k}\right) dx > n + \frac{n}{\left((n+1)!\right)^{\frac{1}{n}}}.$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Lazaros Zachariadis-Thessaloniki-Greece,Solution 2 by Sagar Kumar-Kolkata-India, Solution 3 by Soumitra Mandal-Chandar Nagore-India

Solution 1 by Lazaros Zachariadis-Thessaloniki-Greece

$$\int_0^1 \left(e^x + e^{x^2} + \cdots + e^{x^n}\right) dx \overset{Jensen}{\geq} \int_0^1 n \cdot e^{\frac{x+x^3+\cdots+x^n}{n}} dx \overset{e^x \geq x+1}{\geq}$$

$$\geq \int_0^1 n \cdot \left(\frac{x + x^2 + \cdots + x^n}{n} + 1\right) dx = \int_0^1 n \cdot \frac{x + \cdots + x^n + n}{n} dx =$$

$$= \int_0^1 (x + x^2 + \cdots + x^n + n)\, dx = \left[\frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^{n+1}}{n+1} + nx\right]_0^1 =$$

$$= \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1} + n \geq n \cdot \sqrt[n]{\frac{1}{2} \cdot \frac{1}{3} \cdot \ldots \cdot \frac{1}{n+1}} + n = n \cdot \frac{1}{[(n+1)!]^{\frac{1}{n}}} + n$$

Solution 2 by Sagar Kumar-Kolkata-India

$$\int_0^1 \left(\sum_{k=1}^{n} e^{x^k}\right) dx \geq \int_0^1 \sum_{k=1}^{n} \left(1 + x^k\right) dx \geq \sum_{k=1}^{n} \left(1 + \frac{1}{k+1}\right) \geq n + \sum_{k=1}^{n} \frac{1}{k+1}$$

Now, $\sum_{k=1}^{n} \frac{1}{k+1} \geq n \left(\frac{1}{(n+1)!}\right)^{\frac{1}{n}}$

$$\int_0^1 \left(\sum_{k=1}^{n} e^{x}\right) dx \geq n + n \left(\frac{1}{(n+1)!}\right)^{\frac{1}{n}}$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$e^m \geq 1 + m$ ***for all*** $m \geq 0$***, now,*** $\sum_{k=1}^{n} e^{x^k} \geq \sum_{k=1}^{n} \left(1 + x^k\right) = n + \sum_{k=1}^{n} x^k$

$$\int_0^1 \left(\sum_{k=1}^{n} e^{x^k}\right) dx \geq n \int_0^1 dx + \sum_{k=1}^{n} \int_0^1 x^k \, dx = n + \sum_{k=1}^{n} \frac{1}{k+1} \overset{A.M \geq G.M}{\geq}$$

$$\geq n + n \sqrt[n]{\prod_{k=1}^{n} \left(\frac{1}{k+1}\right)} = n + \frac{n}{\sqrt[n]{(n+1)!}}$$

(Proved)