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If in $\Delta\, ABC$, O – circumcenter, I – incenter then:

$$\sqrt{m_a^2 - w_a^2} + \sqrt{m_b^2 - w_b^2} + \sqrt{m_c^2 - w_c^2} \le 2\sqrt{3} \cdot OI$$

Proposed by Rovsen Pirguliyev-Sumgait-Azerbaidian

Solution 1 by Soumava Chakraborty-Kolkata-India, Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{\Delta} \sqrt{\frac{2(b^2+c^2)-a^2}{4} - \frac{bc((b+c)^2-a^2)}{(b+c)^2}} = \sum_{\Delta} \sqrt{\frac{b^2+c^2}{2} - \frac{a^2}{4} - bc + \frac{a^2 \cdot bc}{(b+c)^2}} \overset{Mg \le M_a}{\le}$$

$$\le \sum_{\Delta} \sqrt{\frac{b^2+c^2}{2} - \frac{a^2}{4} - bc + \frac{a^2(b+c)^2}{4(b+c)^2}} = \sum_{\Delta} \sqrt{\frac{b^2+c^2}{2} - bc - \frac{a^2}{4} + \frac{a^2}{4}} \overset{CBS}{\le}$$

$$\le \sqrt{3\left(\sum a^2 - \sum ab\right)} = \sqrt{3(2s^2 - 8Rr - 2r^2 - s^2 - 4Rr - r^2)} =$$

$$= \sqrt{3(s^2 - 12Rr - 3r^2)} \overset{GERRETSEN}{\le} \sqrt{3(4R^2 - 8Rr)} = 2\sqrt{3}\sqrt{R(R-2r)} = 2\sqrt{3} \cdot OI$$

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sqrt{m_a^2 - w_a^2} + \sqrt{m_b^2 - w_b^2} + \sqrt{m_c^2 - w_c^2} \overset{(i)}{\le} 2\sqrt{3}OI$$

$$m_a^2-w_a^2=\frac{2b^2+2c^2-a^2}{4}-\frac{4b^2c^2}{(b+c)^2}\cdot\frac{s(s-a)}{bc}=$$

$$\frac{2b^2+2c^2-a^2}{4}-\frac{bc(b+c+a)(b+c-a)}{(b+c)^2}=$$

$$\frac{2b^2+2c^2-a^2}{4}-\frac{bc\{(b+c)^2-a^2\}}{(b+c)^2}=\frac{2b^2+2c^2-a^2}{4}-$$

$$-bc+\frac{a^2bc}{(b+c)^2}=\frac{2(b-c)^2-a^2}{4}+\frac{a^2bc}{(b+c)^2}=\frac{(b-c)^2}{2}-a^2\left\{\frac{1}{4}-\frac{bc}{(b+c)^2}\right\}=$$

$$=\frac{(b-c)^2}{2}-\frac{a^2}{4(b+c)^2}\{(b+c)^2-4bc\}=\frac{(b-c)^2}{2}=\frac{a^2(b-c)^2}{4(b+c)^2}\overset{(1)}{\le}\frac{(b-c)^2}{2}$$

$\left(\because \frac{a^2(b-c)^2}{4(b+c)^2}\ge 0\right)$. ***Similarly***, $m_b^2-w_b^2\overset{(2)}{\le}\frac{(c-a)^2}{2}$ & $m_c^2-w_c^2\overset{(3)}{\le}\frac{(a-b)^2}{2}$

(1)+(2)+(3)$\Rightarrow \sum(m_a^2-w_a^2)\overset{(a)}{\le}\frac{\sum(a-b)^2}{2}=\sum a^2-\sum ab$. *Now, LHS of (i)* $\overset{CBS}{\le}$

$$\le\sqrt{3}\sqrt{\sum(m_a^2-w_a^2)}\overset{by\ (a)}{\le}\sqrt{3}\sqrt{\sum a^2-\sum ab}\overset{(i)}{\le}2\sqrt{3}OI=2\sqrt{3}\sqrt{R(R-2r)}\Leftrightarrow$$

$\Leftrightarrow s^2-12Rr-3r^2\overset{(i)}{\le}4R(R-2r)\Leftrightarrow s^2\overset{(i)}{\le}4R^2+4Rr+3r^2\to$ *true (Gerretsen)*

(Proved)