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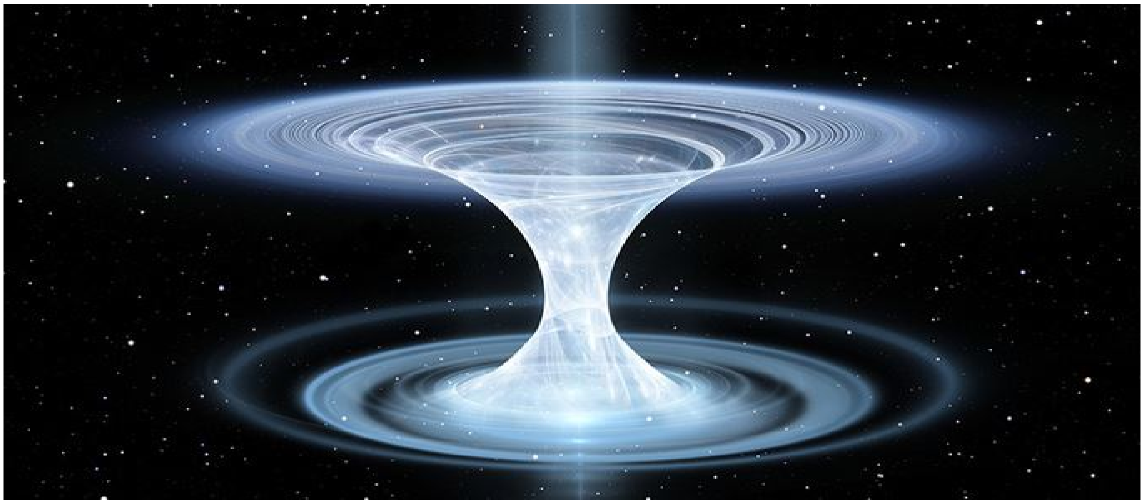

If $x,y,z,t\in\left(0,\frac{\pi}{2}\right)$ then:

$$64\cdot\cos x\cdot\cos z\cdot\sin y\cdot\sin t\cdot\sin(x-y)\cdot\sin(z-t)\leq 1$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Michail Stergioiu-Greece, Solution 2 by Marian Ursarescu-Romania

Solution 1 by Michail Stergioiu-Greece

$$64\cos x\cos z\sin y\sin t\sin(x-y)\sin(z-t)\leq \quad \textbf{\textit{(1)}}$$

By AM-GM: LHS of (1) $\leq 64\left[\frac{\overbrace{\cos x+\cos z+\sin y+\sin t+\sin(x-y)+\sin(z-t)}^{A\,(nominator)}}{6}\right]^6$

It suffices to prove that $64\left[\frac{A}{6}\right]^6\leq 1$ ***or*** $A\leq\left[\frac{1}{2^6}\cdot 2^6\cdot 3^6\right]^{\frac{1}{6}}=3$

As $\cos x=\sin\left(\frac{\pi}{2}-x\right)$, $\cos z=\sin\left(\frac{\pi}{2}-2\right)$ ***and the function***

$f(\vartheta)=\sin\vartheta$ ***is concave over***

$\left(0,\frac{\pi}{2}\right)$ $[f''(\vartheta)=\sin\vartheta<0]$ ***we can apply the Jensen inequality for the variables***

$\frac{\pi}{2}-x,\frac{\pi}{2}-2,y,t,x-y,z-t$ ***of the function*** $f(\vartheta)=\sin\vartheta$***. We get:***

$$A \le 6\sin\left(\frac{\frac{\pi}{2}-x+\frac{\pi}{2}-z+y+t+x-y+z-t}{6}\right)=6\cdot\sin\frac{\pi}{6}=6\cdot\frac{1}{2}=3$$

We are done!

Solution 2 by Marian Ursarescu-Romania

We must show this:

$$\cos x\cos z\cdot\sin y\cdot\sin t\,(\sin x\cos y-\cos x\sin y)(\sin z\cot t-\cos z\sin t)\le\frac{1}{64}\quad (1)$$

We show this: $\cos x\sin y\,(\sin x\cos y-\cos x\sin y)\le\frac{1}{8}$ ***(2)***

$$\left.\begin{array}{c}\cos x=a,\ \sin y=b\ (2)\Leftrightarrow ab\left(\sqrt{(1-a^2)(1-b^2)}-ab\right)\le\frac{1}{8}\\ But\ \sqrt{(1-a^2)(1-b^2)}\le\frac{2-a^2-b^2}{2}\end{array}\right\}\Rightarrow$$

$$ab\left(\frac{2-a^2-b^2}{2}-ab\right)\le\frac{1}{8}\Leftrightarrow ab(2-a^2-b^2-2ab)\le\frac{1}{4}\Leftrightarrow$$

$$4ab(2-(a+b)^2)\le 1\quad (3)$$

But $(a+b)^2\ge 4ab\Rightarrow -(a+b)^2\le -4ab$ ***(4)***

From (3)+(4)$\Rightarrow 4ab(2-4ab)\le 1\Leftrightarrow 8ab-16a^2b^2\le 1\Leftrightarrow$

$$16a^2b^2-8ab+1\ge 0\Leftrightarrow$$

$\Leftrightarrow (4ab-1)^2\ge 0$ ***true (equality for*** $a=b=\frac{1}{2}$***).***

Similarly: $\cos z\sin t\sin(z-t)\le\frac{1}{8}$ ***(5)***

From (2)+(5)$\Rightarrow \cos x\cos z\cdot\sin y\cdot\sin t\cdot\sin(x-y)\sin(z-t)\le 1$***, with equality for*** $x=z=\frac{\pi}{3}$ ***and*** $y=t=\frac{\pi}{6}$***.***