R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

If $a,b,c\in(0,1],x,y>0$ ***then:***

$$\frac{3}{2}\log(x^2+y^2)>(a+b+c)\log x+(3-a-b-c)\log y$$

Proposed by Daniel Sitaru – Romania

Solution by Khanh Hung Vu-Ho Chi Minh-Vietnam

If $a,b,c\in(0;1],x,y>0$ ***then*** $\frac{3}{2}\log(x^2+y^2)>(a+b+c)\log x+(3-a-b-c)\log y$ ***(1)***

Case 1. $\log\left(\frac{x}{y}\right)>0$

We have (1) $\Rightarrow(a+b+c-3)\cdot(\log x-\log y)+3\log x<\frac{3}{2}\log(x^2+y^2)\Rightarrow$

$$\Rightarrow(a+b+c-3)\cdot\log\left(\frac{x}{y}\right)+3\log x<\frac{3}{2}\log(x^2+y^2)$$

We have $\log\left(\frac{x}{y}\right)>0$ ***and*** $a+b+c-3\le 0$ ***so*** $(a+b+c-3)\cdot\log\left(\frac{x}{y}\right)\le 0$

$$\Rightarrow(a+b+c-3)\cdot\log\left(\frac{x}{y}\right)+3\log x\le 3\log x$$

On the other hand, we have $\frac{3}{2}\log(x^2+y^2)>\frac{3}{2}\log(x^2)=3\log x$***. So,***

$(a+b+c-3)\cdot\log\left(\frac{x}{y}\right)+3\log x<\frac{3}{2}\log(x^2+y^2)\Rightarrow$ ***(1) true***

Case 2. $\log\left(\frac{x}{y}\right)<0$

We have (1) $\Rightarrow(a+b+c)\cdot(\log x-\log y)+3\log y<\frac{3}{2}\log(x^2+y^2)\Rightarrow$

$$\Rightarrow(a+b+c)\cdot\log\left(\frac{x}{y}\right)+3\log y<\frac{3}{2}\log(x^2+y^2)$$

We have $\log\left(\frac{x}{y}\right)<0$ ***and*** $a+b+c>0$ ***so,*** $(a+b+c)\cdot\log\left(\frac{x}{y}\right)<0\Rightarrow$

$$\Rightarrow (a+b+c)\cdot\log\left(\frac{x}{y}\right)+3\log y<3\log y$$

On the other hand, we have $\frac{3}{2}\log(x^2+y^2)>\frac{3}{2}\log(y^2)=3\log y$

So $(a+b+c)\cdot\log\left(\frac{x}{y}\right)+3\log y<\frac{3}{2}\log(x^2+y^2)\Rightarrow$ ***(1) true***

Therefore, we have QED.