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Let $a,b,c \in (0;+\infty) \wedge ab+bc+ca=3$. Prove:

$$\frac{1}{\sqrt[6]{a^2+3}}+\frac{1}{\sqrt[6]{b^2+3}}+\frac{1}{\sqrt[6]{c^2+3}} \le \frac{\sqrt[3]{36}}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^{\frac{1}{3}}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

Solution 1 by Marian Ursarescu-Romania, Solution 2 by Soumava Chakraborty-Kolkata-India, Solution 3 by Soumitra Mandal-Chandar Nagore-India

Solution 1 by Marian Ursarescu-Romania

Because $ab+bc+ac=3 \Rightarrow \exists x,y,z>0$ *such that:*

$a=\frac{\sqrt{3}x}{\sqrt{xy+xz+yt}}, b=\frac{\sqrt{3}y}{\sqrt{xy+xt+yt}}, c=\frac{\sqrt{3}z}{\sqrt{xy+xz+yt}}$. *Inequality becomes:*

$$\sum \frac{\sqrt[6]{xy+xz+yt}}{\sqrt[6]{3}\sqrt[6]{x^2+xy+xz+yt}} \le \frac{\sqrt[3]{36}}{2}\cdot\frac{\sqrt[6]{xy+xz+yz}}{3}\left(\sqrt[3]{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}\right) \Leftrightarrow$$

$$\Leftrightarrow \sum \frac{1}{\sqrt[6]{(x+y)(x+z)}} \le \frac{\sqrt[3]{36}}{2}\left(\sqrt[6]{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}\right) \quad (1)$$

$$(1) \Leftrightarrow \left(\sum \frac{1}{\sqrt[6]{(x+y)(x+z)}}\right)^3 \le \frac{9}{2}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \quad (2)$$

From Hölder's inequality $\Rightarrow \left(\sum \frac{1}{\sqrt[6]{(x+y)(x+z)}}\right)^3 \le 9\sum \frac{1}{\sqrt{(x+y)(x+z)}}$ *(3)*

From (2)+(3) we must show: $\sum\frac{1}{\sqrt{(x+y)(x+z)}}\le\frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$ *(4)*

But $\begin{matrix}x+y\ge2\sqrt{xy}\\x+z\ge2\sqrt{xz}\end{matrix}\Rightarrow(x+y)(x+z)\ge4x\sqrt{yz}\Rightarrow\frac{1}{\sqrt{(x+y)(x+z)}}\le\frac{1}{2\sqrt{x\sqrt{yz}}}\Rightarrow$

$$\Rightarrow\sum\frac{1}{\sqrt{(x+y)(x+z)}}\le\frac{1}{2}\sum\frac{1}{\sqrt{x\sqrt{yz}}}\quad(5)$$

From (4)+(5) we must show: $\sum\frac{1}{\sqrt{x\sqrt{yz}}}\le\sum\frac{1}{x}$ *(6)*

Now use $\alpha^2+\beta^2+\gamma^2\ge\alpha\beta+\alpha\gamma+\beta\gamma\Rightarrow\sum\frac{1}{x}=\sum\frac{1}{(\sqrt{x})^2}\ge\sum\frac{1}{\sqrt{xy}}=\sum\frac{1}{(\sqrt{\sqrt{xy}})^2}\ge$

$\ge\sum\frac{1}{\sqrt{x\sqrt{yz}}}\Rightarrow$ *(6) its true.*

Solution 2 by Soumava Chakraborty-Kolkata-India

Firstly, $\sum x^3\overset{Chebyshev}{\ge}\frac{1}{9}\left(\sum x\right)^3\Rightarrow\left(\sum x\right)^3\le9\sum x^3\Rightarrow\sum x\overset{(1)}{\le}\sqrt[3]{9}\sqrt[3]{\sum x^3}$

Now, $a^2+3=a^2+ab+bc+ca\overset{(2)}{=}(c+a)(a+b)$.

Similarly, $b^2+3\overset{(3)}{=}(b+c)(a+b)$

$\&\ c^2+3\overset{(4)}{=}(b+c)(c+a)$; *(2), (3), (4)* $\Rightarrow LHS=\sum\frac{1}{\sqrt[6]{(a+b)(b+c)}}\overset{CBS}{\le}\sqrt{\sum\frac{1}{\sqrt[3]{a+b}}}\sqrt{\sum\frac{1}{\sqrt[3]{b+c}}}=$

$$=\sum\frac{1}{\sqrt[3]{a+b}}\overset{A-G}{\le}\sum\frac{1}{\sqrt[3]{2\sqrt{ab}}}=\frac{1}{\sqrt[3]{2}}\sum\frac{1}{\sqrt[6]{ab}}\overset{CBS}{\le}\frac{1}{\sqrt[3]{2}}\sqrt{\sum\frac{1}{\sqrt[3]{a}}}\sqrt{\sum\frac{1}{\sqrt[3]{a}}}=\frac{1}{\sqrt[3]{2}}\sum\frac{1}{\sqrt[3]{a}}=$$

$$=\frac{1}{\sqrt[3]{2}\sqrt[3]{abc}}\left(\sum\sqrt[3]{ab}\right)\overset{by\ (1)}{\le}\frac{1}{\sqrt[3]{2}\sqrt[3]{abc}}\sqrt[3]{9}\sqrt[3]{\sum ab}=\sqrt[3]{\frac{9}{2}}\frac{1}{\sqrt[3]{abc}}\left(\sum ab\right)^{\frac{1}{3}}=$$

$$=\sqrt[3]{\frac{36}{8}}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^{\frac{1}{3}}=\frac{\sqrt[3]{36}}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^{\frac{1}{3}}$$

(Proved)

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$ab+bc+ca=3\text{ and }\prod_{cyc}(a+b)\ge\frac{9}{8}(a+b+c)(ab+bc+ca)$$

$$\sum_{cyc}\frac{1}{\sqrt[6]{a^2+3}}=\sum_{cyc}\frac{1}{\sqrt[6]{a^2+ab+bc+ca}}=\sum_{cyc}\frac{1}{\sqrt[6]{(a+b)(a+c)}}=$$

$$=\frac{1}{\sqrt[6]{\prod_{cyc}(a+b)}}\sum_{cyc}\sqrt[6]{a+b}\le\frac{1}{\sqrt[6]{\prod_{cyc}(a+b)}}\sqrt{3^5\cdot 2(a+b+c)}$$

$$\le\frac{\sqrt[6]{3^5\cdot 2(a+b+c)}}{\sqrt[6]{\frac{8}{9}(a+b+c)(ab+bc+ca)}}=\sqrt[6]{\frac{3^7}{4(ab+bc+ca)}}\le\sqrt[6]{\frac{3^9}{4}\sum_{cyc}\frac{1}{ab}}\le$$

$$\le\sqrt[6]{\frac{3^8}{4}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}=\frac{\sqrt[3]{36}}{2}\sqrt[3]{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$$

(proved)