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For $a, b \in \mathbb{R}, a \neq b$ solve the system:

$$\begin{cases} 3x + z = 2y + a + b \\ 3x^2 + 3xz = y^2 + 2(a+b)y + ab \\ x^3 + 3x^2z = (a+b)y^2 + 2aby \end{cases}$$

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$$\begin{cases} 3x + z = 2y + a + b & (1) \\ 3x^2 + 3xz = y^2 + 2(a+b)y + ab & (2) \\ x^3 + 3x^2z = (a+b)y^2 + 2aby & (3) \end{cases}$$

We have (1) $\Rightarrow a + b = 3x + z - 2y$ ***(4)***

Similarly, we have (2) $\Rightarrow ab = 3x^2 + 3xz - y^2 - 2(a+b)y$ ***(5) and (3)*** $\Rightarrow$

$\Rightarrow 2aby = x^3 + 3x^2 - (a+b)y^2$ ***(6)***

(4) and (5) $\Rightarrow ab = 3x^2 + 3xz - y^2 - 2y(3x + z - 2y)$ ***(7)***

(4) and (6) $\Rightarrow 2aby = x^3 + 3x^2 - y^2(3x + z - 2y)$ ***(8)***

(7) and (8) $\Rightarrow 2y[3x^2 + 3xz - y^2 - 2y(3x + z - 2y)] = x^3 + 3x^2 - y^2(3x + z - 2y) \Rightarrow$

$\Rightarrow x^3 + 3x^2z + 9xy^2 + 3y^2z - 6x^2y - 6xyz - 4y^3 = 0$

$\Rightarrow (x-y)^2(x - 4y + 3z) = 0 \Rightarrow x = y$ ***or*** $x - 4y + 3z = 0$

1) $x = y$

We have (1) $\Rightarrow 3x + z = 2x + a + b \Rightarrow x = a + b - z$

We have (2) $\Rightarrow 3x^2 + 3x(a + b - x) = x^2 + 2(a+b)x + ab \Rightarrow$

$\Rightarrow x^2 - (a+b)x + ab = 0 \Rightarrow (x-a)(x-b) = 0 \Rightarrow x = a$ ***or*** $x = b$

1.1) $x = a \Rightarrow x = y = z$

We have (1) $\Rightarrow z = a + b - x \Rightarrow z = a + b - a = b \Rightarrow (x, y, z) = (a, a, b)$

1.2) $x = b \Rightarrow x = y = b$

We have (1) $\Rightarrow z = a + b - x \Rightarrow z = a + b - b = a \Rightarrow (x, y, z) = (b, b, a)$

2) $x - 4y + 3z = 0 \Rightarrow z = \frac{4y-x}{3}$

We have (1) $\Rightarrow 3x + \frac{4y-x}{3} = 2y + a + b \Rightarrow x = \frac{2y+3(a+b)}{8}$

We have (2) $\Rightarrow 3x^2 + x(4y - x) = y^2 + 2(a + b) + ab \Rightarrow 2x^2 + 4xy =$

$$= y^2 + 2(a + b)y + ab \Rightarrow 2\left(\frac{2y + 3(a + b)}{8}\right)^2 + 4y \cdot \frac{2y + 3(a + b)}{8} =$$

$$= y^2 + 2(a + b)y + ab \Rightarrow \frac{1}{8}y^2 - \frac{1}{8}y(a + b) + \frac{9}{32}(a + b)^2 - ab = 0$$

$$\Rightarrow \frac{1}{8}y^2 - \frac{1}{8}y(a + b) + \frac{1}{32}(a + b)^2 + \frac{1}{4}(a + b)^2 - ab = 0 \Rightarrow$$

$\Rightarrow \frac{1}{8}\left(y - \frac{a+b}{2}\right)^2 + \frac{(a-b)^2}{4} = 0 \Rightarrow a = b$ ***(Absurd)***

So the system has 2 roots: $(x, y, z) = (a, a, b)$ ***and*** $(x, y, z) = (b, b, a)$