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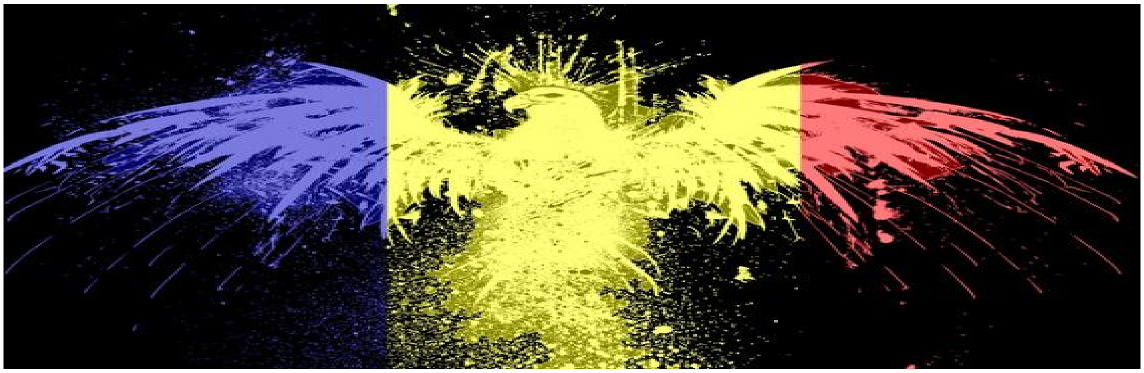

SP.120. In $\Delta\ ABC$ the following relationship holds:

$$\sqrt[3]{a^2B}+\sqrt[3]{b^2C}+\sqrt[3]{c^2A}\le\sqrt[3]{4\pi s^2}$$

s – semiperimeter; a,b,c – length's sides; A,B,C – angled's measures

Proposed by Daniel Sitaru – Romania

Solution 1 by Marian Ursarescu-Romania, Solution 2 by Soumitra Mandal-Chandar Nagore-India

Solution 1 by Marian Ursarescu-Romania

$$B\sqrt[3]{\frac{a^2}{B^2}}+C\sqrt[3]{\frac{b^2}{C^2}}+A\sqrt[3]{\frac{c^2}{A^2}}\le\sqrt[3]{\pi(a+b+c)^2}\Leftrightarrow$$

$$\frac{B}{\pi}\sqrt[3]{\frac{a^2}{B^2}}+\frac{C}{\pi}\sqrt[3]{\frac{b^2}{C^2}}+\frac{A}{\pi}\sqrt[3]{\frac{c^2}{A^2}}\le\sqrt[3]{\frac{(a+b+c)^2}{\pi^2}}\quad (1)$$

Let $f:(0,\infty)\to\mathbb{R};\ f(x)=\sqrt[3]{x^2};\ f'(x)=\left(x^{\frac{2}{3}}\right)'=\frac{2}{3}x^{-\frac{1}{3}};\ f''(x)=-\frac{2}{9}x^{-\frac{4}{3}}<0\Rightarrow$

from Jensen's inequality $\Rightarrow$

$$p_1f(x_1)+p_2f(x_2)+p_3f(x_3)\le f(p_1x_1+p_2x_2+p_3x_3)\ \textbf{with}\ p_1+p_2+p_3=1$$

$$p_1=\frac{B}{\pi},p_2=\frac{C}{\pi},p_3=\frac{A}{\pi}$$

$$x_1=\frac{a}{B},x_2=\frac{b}{C},x_3=\frac{c}{A}\Rightarrow$$

$$\frac{B}{\pi}\sqrt[3]{\left(\frac{a}{B}\right)^2}+\frac{C}{\pi}\sqrt[3]{\left(\frac{b}{c}\right)^2}+\frac{A}{\pi}\sqrt[3]{\left(\frac{c}{A}\right)^2}\le\sqrt[3]{\left(\frac{a+b+c}{A+B+C}\right)^2}\Rightarrow$$

$$\frac{B}{\pi}\sqrt[3]{\left(\frac{a}{B}\right)^2}+\frac{C}{\pi}\sqrt[3]{\left(\frac{b}{C}\right)^2}+\frac{A}{\pi}\sqrt[3]{\left(\frac{c}{A}\right)^2}\le\sqrt[3]{\frac{(a+b+c)^2}{\pi^2}}$$

then (1) is true

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Applying Hölder's Inequality

$$(a+b+c)^2(A+B+C)\ge\left(\sqrt[3]{a^2B}+\sqrt[3]{b^2C}+\sqrt[3]{c^2A}\right)^3$$

$$\sqrt[3]{4\pi p^2}\ge\sqrt[3]{a^2B}+\sqrt[3]{b^2C}+\sqrt[3]{c^2A}$$

(proved)