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SP.108. If $a, b, c > 0, a + b + c = abc$ then:

$$\frac{4(a+b)(a+c)}{(b+c)^2} + \frac{4(b+c)(b+a)}{(c+a)^2} + \frac{4(c+a)(c+b)}{(a+b)^2} \le 3 + a^2 + b^2 + c^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Amit Dutta-Jamshedpur-India, Solution 2 by Boris Colakovic-Belgrade-Serbia, Solution 3 by Do Huu Duc Thinh-Ho Chi Minh-Vietnam

Solution 1 by Amit Dutta-Jamshedpur-India

The above expression can be written as

$$\left\{\frac{4(a+b)(a+c)}{(b+c)^2} - 1\right\} + \left\{\frac{4(b+c)(b+a)}{(c+a)^2} - 1\right\} + \left\{\frac{4(c+a)(c+b)}{(a+b)^2} - 1\right\}$$
$$\le a^2 + b^2 + c^2$$

Let us take

$$\frac{4(a+b)(a+c)}{(b+c)^2} - 1 = \frac{4(a^2+ac+ab+bc) - (b+c)^2}{(b+c)^2}$$

$$= \frac{4a^2 + 4ac + 4ab + 4bc - b^2 - c^2 - 2bc}{(b+c)^2} = \frac{4a^2 + 4ac + 4ab + 2bc - b^2 - c^2}{(b+c)^2}$$

$$= \frac{4a^2 + 4ac + 4ab - (b-c)^2}{(b+c)^2} = \frac{4a(a+b+c)}{(b+c)^2} - \left(\frac{b-c}{b+c}\right)^2$$

$$= \frac{4a(abc)}{(b+c)^2} - \left(\frac{b-c}{b+c}\right)^2 \{\because a + b + c = abc\}$$

$$\because b, c > 0, AM - GM$$

$$\frac{b+c}{2} \ge \overline{bc} \Rightarrow (b+c)^2 \ge 4bc$$

$$\le \frac{4a^2(bc)}{(4bc)} - \left(\frac{b-c}{b+c}\right)^2 \le a^2 - \left(\frac{b-c}{b+c}\right)^2 \le a^2$$

$$\therefore \frac{4(a+b)(a+c)}{(b+c)^2} - 1 \le a^2 \Rightarrow \frac{4(a+b)(a+c)}{(b+c)^2} \le a^2 + 1 \ \textit{(i)}$$

Similarly, we have

$$\frac{4(b+c)(b+a)}{(c+a)^2} \le b^2 + 1 \quad \textit{(ii)}$$

$$\textit{and} \frac{4(c+a)(c+b)}{(a+b)^2} \le c^2 + 1 \ \textit{(iii)}$$

Adding (i), (ii), (iii) we get the result

$$\frac{4(a+b)(a+c)}{(b+c)^2} + \frac{4(b+c)(b+a)}{(c+a)^2} + 4\left(\frac{(c+a)(c+b)}{(a+b)^2}\right) \le a^2 + b^2 + c^2 + 3$$

(proved)

Solution 2 by Boris Colakovic-Belgrade-Serbia

$$b + c \overset{AM-GM}{\ge} 2\sqrt{bc} \Rightarrow (b+c)^2 \ge 4bc \Rightarrow \frac{1}{(b+c)^2} \le \frac{1}{4bc}$$

$$\frac{4(a+b)(a+c)}{(b+c)^2} \le \frac{(a+b)(a+c)}{bc} = \frac{a+b}{b} \cdot \frac{a+c}{c} = \left(1 + \frac{a}{b}\right)\left(1 + \frac{a}{c}\right) = 1 + \frac{a^2+ab+ac}{bc} \quad \textit{(1)}$$

Similarly

$$\frac{4(b+c)(b+a)}{(c+a)^2} \le \frac{(b+c)(b+a)}{ca} = 1 + \frac{b^2+ab+bc}{ca} \quad \textit{(2)}$$

$$\frac{4(c+a)(c+b)}{(a+b)^2} \le 1 + \frac{c^2+ac+bc}{ab} \quad \textit{(3)}$$

$$\textit{(1)}+\textit{(2)}+\textit{(3)} \Rightarrow LHS \le 3 + \frac{a^2+ab+ac}{bc} + \frac{b^2+ab+bc}{ca} + \frac{c^2+ac+bc}{ab} =$$

$$= 3 + \frac{(a^3 + a^2b + a^2c) + (b^3 + b^2a + b^2c) + (c^3 + c^2a + c^2b)}{abc} =$$

$$= 3 + \frac{a^2(a+b+c) + b^2(a+b+c) + c^2(a+b+c)}{a+b+c} = 3 + a^2 + b^2 + c^2$$

Solution 3 by Do Huu Duc Thinh-Ho Chi Minh-Vietnam

By Cauchy's inequality we get: $\sum \frac{4(a+b)(a+c)}{(b+c)^2} \le \sum \frac{4(a+b)(a+c)}{4bc} = \sum \frac{a(a+b+c)+bc}{bc}$

$$= \sum \frac{a^2bc+bc}{bc} = \sum(a^2+1) = 3 + a^2 + b^2 + c^2 \Rightarrow \textit{Q.E.D.}$$