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If $a, b > 0, a \neq b$ then:

$$\sqrt{2} < \frac{\sqrt{\frac{a^2+b^2}{2}}-\sqrt{ab}}{\frac{a+b}{2}-\sqrt{ab}} < 2$$

Proposed by Shan He Wu-China

Solution 1 by Khanh Hung Vu-Ho Chi Minh-Vietnam, Solution 2 by Muhammad Alhafi-Aleppo-Syria, Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia, Solution 4 by Soumava Chakraborty-Kolkata-India

Solution 1 by Khanh Hung Vu-Ho Chi Minh-Vietnam

We have $\frac{\sqrt{\frac{a^2+b^2}{2}}-\sqrt{ab}}{\frac{a+b}{2}-\sqrt{ab}} = \frac{\frac{a^2+b^2}{2}-ab}{\left(\sqrt{\frac{a^2+b^2}{2}}+\sqrt{ab}\right)\left(\frac{a+b}{2}-\sqrt{ab}\right)} = \frac{\frac{(a-b)^2}{2}}{\left(\sqrt{\frac{a^2+b^2}{2}}+\sqrt{ab}\right)\cdot\frac{(\sqrt{a}-\sqrt{b})^2}{2}} = \frac{(\sqrt{a}+\sqrt{b})^2}{\sqrt{\frac{a^2+b^2}{2}}+\sqrt{ab}}$

By BCS, we have $a^2+b^2 \geq \frac{(a+b)^2}{2} \Rightarrow \frac{a^2+b^2}{2} \geq \frac{(a+b)^2}{4} \Rightarrow \sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2} \Rightarrow$

$$\Rightarrow \sqrt{\frac{a^2+b^2}{2}}+\sqrt{ab} \geq \frac{a+b}{2}+\sqrt{ab} \Rightarrow \sqrt{\frac{a^2+b^2}{2}}+\sqrt{ab} \geq \frac{(\sqrt{a}+\sqrt{b})^2}{2} \Rightarrow$$

$\Rightarrow \frac{\sqrt{\frac{a^2+b^2}{2}}-\sqrt{ab}}{\frac{a+b}{2}-\sqrt{ab}} \leq 2$. The equality occurs when $a = b$ (Absurd) $\Rightarrow \frac{\sqrt{\frac{a^2+b^2}{2}}-\sqrt{ab}}{\frac{a+b}{2}-\sqrt{ab}} < 2$ **(1)**

On the other hand, we have $a^2+b^2 < (a+b)^2 \Rightarrow \frac{a^2+b^2}{2} < \frac{(a+b)^2}{2} \Rightarrow \sqrt{\frac{a^2+b^2}{2}} < \frac{a+b}{\sqrt{2}} \Rightarrow$

$\Rightarrow \sqrt{\frac{a^2+b^2}{2}}+\sqrt{ab} < \frac{a+b+\sqrt{2ab}}{\sqrt{2}}$. We have $a+b+\sqrt{2ab} < a+b+\sqrt{4ab} \Rightarrow$

$$\Rightarrow a+b+\sqrt{2ab}<\left(\sqrt{a}+\sqrt{b}\right)^2\Rightarrow\sqrt{\frac{a^2+b^2}{2}}+\sqrt{ab}<\frac{\left(\sqrt{a}+\sqrt{b}\right)^2}{\sqrt{2}}\Rightarrow\frac{\sqrt{\frac{a^2+b^2}{2}}-\sqrt{ab}}{\frac{a+b}{2}-\sqrt{ab}}>\sqrt{2}\ \textit{(2)}$$

***(1)* and *(2)* $\Rightarrow$ Q.E.D.**

Solution 2 by Muhammad Alhafi-Aleppo-Syria

Since the inequality is homogenized we may assume that $ab=1$ ***so if*** $S=a+b$ ***then***

$$a^2+b^2=S^2-2 \text{ and the inequality becomes: } \sqrt{2}<\frac{\sqrt{\frac{S^2-2}{2}}-1}{\frac{S}{2}-1}<2\Leftrightarrow$$

$$\Leftrightarrow\left(\frac{S}{\sqrt{2}}-\left(\sqrt{2}-1\right)\right)^2<\frac{S^2-2}{2}<(S-1)^2\Leftrightarrow\frac{S^2}{2}+4-2\sqrt{2}-\left(2-\sqrt{2}\right)S<\frac{S^2}{2}<S^2-2S+$$

2 ***the left inequality is equivalent to:*** $2\left(2-\sqrt{2}\right)<\left(2-\sqrt{2}\right)S$ ***which follows from***

$S>2\sqrt{ab}=2$ ***the right inequality is equivalent to:***

$$S^2-4S+4>0\Leftrightarrow(S-2)^2>0$$

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

1) RHS: $\left(a+b-2\sqrt{ab}\right)^2>0$ ***True***

$$(a+b)^2-4(a+b)\sqrt{ab}+4ab>0\Leftrightarrow a^2+b^2+6ab=4(a+b)\sqrt{ab}>0\Leftrightarrow$$

$$\Leftrightarrow\underbrace{a^2+b^2+2ab-2(a+b)\sqrt{ab}+ab}_{\left(a+b-\sqrt{ab}\right)^2>\frac{a^2+b^2}{2}\Rightarrow a+b-\sqrt{ab}>\sqrt{\frac{a^2+b^2}{2}}\Rightarrow}>\frac{a^2+b^2}{2}$$

$$\Rightarrow 2\left(\frac{a+b}{2}\right)-2\sqrt{ab}>\sqrt{\frac{a^2+b^2}{2}}-\sqrt{ab}\Rightarrow 2>\frac{\sqrt{\frac{a^2+b^2}{2}}-\sqrt{ab}}{\frac{a+b}{2}-\sqrt{ab}}\quad \textit{(RHS)}$$

2) LHS: $\sqrt{a^2+b^2}\ge\sqrt{2ab}$ ***(True)***

$$AM\ge GM$$

$$\left(2-\sqrt{2}\right)\cdot\sqrt{a^2+b^2}\ge\left(2-\sqrt{2}\right)\cdot\sqrt{2ab}\Leftrightarrow\left(2-\sqrt{2}\right)\cdot\sqrt{a^2+b^2}+\left(\sqrt{2}-2\right)\sqrt{2ab}\ge 0$$

$$\left(2-\sqrt{2}\right)\sqrt{a^2+b^2}+\left(2-2\sqrt{2}\right)\sqrt{ab}\ge 0|\cdot 2\sqrt{ab}$$

$$\left(4-2\sqrt{2}\right)\sqrt{ab}\cdot\sqrt{a^2+b^2}+\left(4-4\sqrt{2}\right)ab\ge 0$$

$$2\left(2-\sqrt{2}\right)\cdot\sqrt{ab}\cdot\sqrt{a^2+b^2}+\left(6-4\sqrt{2}\right)ab\ge 2ab$$

$$a^2+b^2+2\left(2-\sqrt{2}\right)\cdot\sqrt{ab}\cdot\sqrt{a^2+b^2}+\left(6-4\sqrt{2}\right)ab\ge a^2+b^2+2ab$$

$$\left(\sqrt{a^2+b^2}+(2-\sqrt{2})\sqrt{ab}\right)^2 \geq (a+b)^2$$

$$\sqrt{a^2+b^2}+(2-\sqrt{2})\cdot\sqrt{ab} \geq a+b$$

$$\sqrt{\frac{a^2+b^2}{2}}+(\sqrt{2}-1)\sqrt{ab} \geq \frac{a+b}{\sqrt{2}} \Rightarrow \sqrt{\frac{a^2+b^2}{2}}-\sqrt{ab} \geq \frac{a+b}{\sqrt{2}}-\sqrt{2}\sqrt{ab}$$

$$\sqrt{\frac{a^2+b^2}{2}}-\sqrt{ab} \geq \sqrt{2}\left(\frac{a+b}{2}-\sqrt{ab}\right) \Leftrightarrow \frac{\sqrt{\frac{a^2+b^2}{2}}-\sqrt{ab}}{\frac{a+b}{2}-\sqrt{ab}} \geq \sqrt{2} \quad \textbf{(LHS)}$$

Solution 4 by Soumava Chakraborty-Kolkata-India

$$\sqrt{2} \overset{(1)}{<} \frac{\sqrt{\frac{a^2+b^2}{2}}-\sqrt{ab}}{\frac{a+b}{2}-\sqrt{ab}} \overset{(2)}{<} 2$$

$$\text{Let } \sqrt{\frac{a^2+b^2}{2}} = Q, \frac{a+b}{2} = A, \sqrt{ab} = G$$

$$\textbf{(2)} \Leftrightarrow \frac{Q-G}{A-G} < 2 \ (\text{of course, } Q > A > G) \Leftrightarrow Q-G < 2A-2G \Leftrightarrow Q+G < 2A \Leftrightarrow$$

$$\Leftrightarrow \sqrt{\frac{a^2+b^2}{2}}+\sqrt{ab} < a+b \Leftrightarrow \frac{a^2+b^2}{2}+ab+\sqrt{2ab(a^2+b^2)} < a^2+b^2+2ab \Leftrightarrow$$

$$\Leftrightarrow (a^2+b^2)+2ab-2\sqrt{2ab(a^2+b^2)} > 0 \Leftrightarrow \left(\sqrt{a^2+b^2}-\sqrt{2ab}\right)^2 > 0 \to \textbf{true} \Rightarrow$$

$$\Rightarrow \textbf{(2) is true} \therefore Q+G < 2A \to \textbf{(2a)}$$

(1)

$$\Leftrightarrow \frac{(Q-G)^2}{(A-G)^2} > 2 \Leftrightarrow Q^2+G^2-2QG > 2(A^2+G^2-2AG) \Leftrightarrow \frac{a^2+b^2}{2}+ab-2\sqrt{\frac{ab(a^2+b^2)}{2}} >$$

$$> 2\cdot\frac{(a+b)^2}{4}+2ab-2(a+b)\sqrt{ab} \Leftrightarrow (a+b)\sqrt{ab}-ab > \sqrt{\frac{ab(a^2+b^2)}{2}} \Leftrightarrow$$

$$\Leftrightarrow (a+b)-\sqrt{ab} > \sqrt{\frac{a^2+b^2}{2}} \Leftrightarrow 2A-G > Q \to \textbf{true by (2a)} \Rightarrow \textbf{(1) is true (Done)}$$