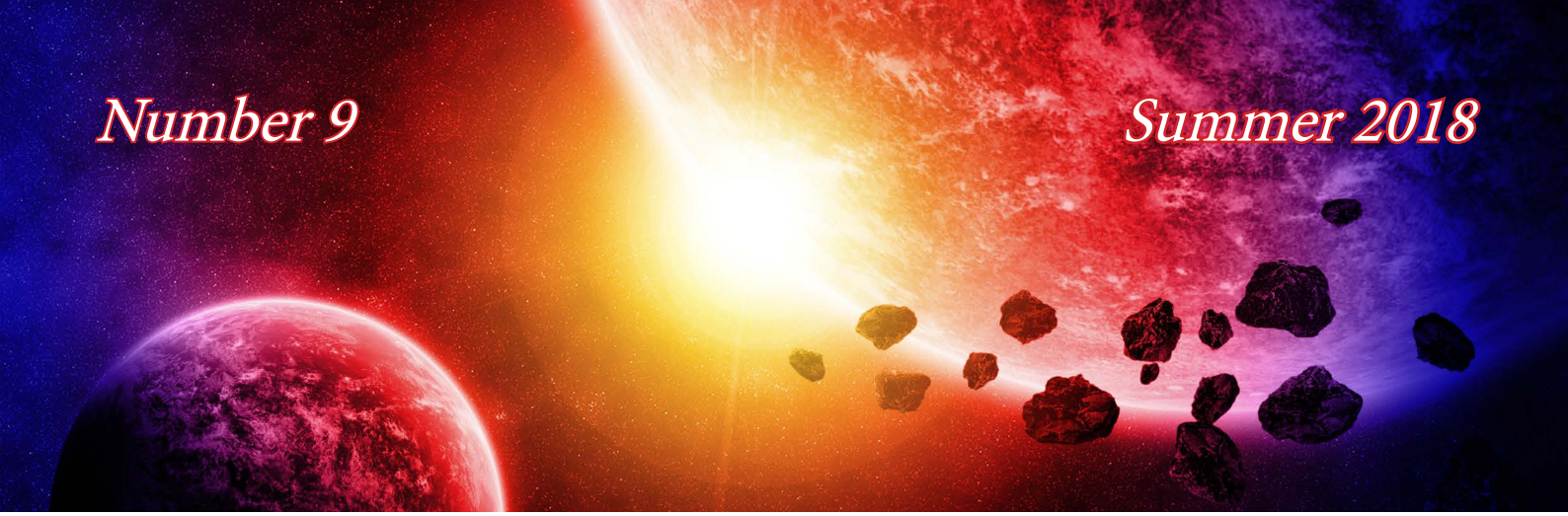
Number 9

Summer 2018



R M M

ROMANIAN MATHEMATICAL MAGAZINE



Founding Editor
DANIEL SITARU

Available online
www.ssmrmh.ro

ISSN-L 2501-0099

PROBLEMS FOR JUNIORS

JP.121. Let ABC be a triangle with inradius r and circumradius R. Prove that:

$$\sum \frac{\sin^3 A + \sin^3 B}{\sin^5 A + \sin^5 B} \leq \left(\frac{R}{r}\right)^2$$

The sum is over all cyclic permutation of (A, B, C).

Proposed by George Apostolopoulos - Messolonghi - Greece

JP.122. Prove that in ΔABC the following relationship holds:

$$\min\left(\frac{a}{s-a}, \frac{b}{s-b}, \frac{c}{s-c}\right) \leq 2\left(\frac{R}{r} - 1\right) \leq \max\left(\frac{a}{s-a}, \frac{b}{s-b}, \frac{c}{s-c}\right)$$

Proposed by Marian Ursărescu - Romania

JP.123. Solve for real numbers:

$$\log_a(b^x + a - b) = \log_b(a^x + b - a), b > a > 1$$

Proposed by Marian Ursărescu - Romania

JP.124. Let a, b, c be the lengths of the sides of a triangle ABC with inradius r. Prove that:

$$\frac{(2r)^{a+b+c}}{a^a \cdot b^b \cdot c^c} \leq \left(\tan\frac{A}{2}\right)^b \cdot \left(\tan\frac{B}{2}\right)^c \cdot \left(\tan\frac{C}{2}\right)^a \leq \left(\frac{\sqrt{3}}{36}\right)^{a+b+c} \cdot \frac{a^{2b} \cdot b^{2c} \cdot c^{2a}}{r^{2(a+b+c)}}.$$

Proposed by George Apostolopoulos - Messolonghi - Greece

JP.125. Prove that in ΔABC the following relationship holds:

$$\frac{1}{3}(r_a^2 + r_b^2 + r_c^2) \geq 4\sqrt{3}S + (a-b)^2 + (b-c)^2 + (c-a)^2$$

Proposed by Marian Ursărescu - Romania

JP.126. Let ABC be a triangle with inradius r and circumradius R. Prove that:

$$\cot A + \cot B + \cot C \geq \sqrt{\frac{3R}{2r}}$$

Proposed by George Apostolopoulos - Messolonghi - Greece

JP.127. In any ΔABC non-equilateral the following relationship holds:

$$\tan\Big(\frac{1}{2}m(\sphericalangle OIH)\Big) < 3\sqrt{\frac{R+2r}{R-2r}}$$

Proposed by Marian Ursărescu - Romania

JP.128. In ΔABC the following relationship holds:

$$\frac{1}{m_a+m_b}+\frac{1}{m_b+m_c}+\frac{1}{m_c+m_a}\leq\frac{1}{2r}$$

Proposed by Marian Ursărescu - Romania

JP.129. Let a, b and c be the lengths of the sides of a triangle ABC with inradius r and circumradius R. Prove that:

a. $\frac{a^3}{b^2+c^2}+\frac{b^3}{c^2+a^2}+\frac{c^3}{a^2+b^2}\leq\frac{3\sqrt{3}}{2}\cdot\frac{R(R-r)}{r}$

b. $\frac{a}{\sqrt{b^2+c^2}}+\frac{b}{\sqrt{c^2+a^2}}+\frac{c}{\sqrt{a^2+b^2}}\leq\sqrt{\frac{6R-3r}{2r}}$.

Proposed by George Apostolopoulos - Messolonghi - Greece

JP.130. Solve the equation in real number:

$$3\sqrt[3]{x^2-x+1}+\sqrt[4]{\frac{x^8+1}{2}}=2(x^4-3x+4)$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

JP.131. Solve the system of equation in positive real numbers:

$$\begin{cases}3(\sqrt[3]{x^2}+\sqrt[3]{y^2}+\sqrt[3]{z^2})+21=10(xy+yz+zx)\\ x+y+z=3\end{cases}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

JP.132. Let $x, y\in\big(0,\frac{\pi}{2}\big)$. Denote $k=2-\min\{\sin^2 x,\cos^2 x\}$. Prove that:

$$\Big(\frac{\sin^2 x}{1-\cos y}+\frac{\cos^2 x}{1-\sin y}\Big)\Big(\frac{\sin^2 x}{1+\cos y}+\frac{\cos^2 x}{1+\sin y}\Big)\leq\Big(\frac{\pi^2}{4}+\frac{\cos^2 x}{\cos^2 y}\Big)^k.$$

Proposed by Mihalcea Andrei Ştefan - Romania

JP.133. Let a, b, c be positive real numbers. Prove that:

$$\sqrt[8]{\frac{a^8+b^8}{2}}+\sqrt[8]{\frac{b^8+c^8}{2}}+\sqrt[8]{\frac{c^8+a^8}{2}}\leq(a+b+c)^{10}\Big(\frac{1}{9a}+\frac{1}{9b}+\frac{1}{9c}\Big)^9$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

©Daniel Sitaru, ISSN-L 2501-0099

JP.134. Let $a, b, c > 0$ sucht that: $a^2 + b^2 + c^2 = 3abc$. Find the maximum of the expression:

$$P = \frac{ab}{2a^6 - a^5 + b^4 + a^2 + 1} + \frac{bc}{2b^6 - b^5 + c^4 + b^2 + 1} + \\ + \frac{ca}{2c^6 - c^5 + a^4 + c^2 + 1}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

JP.135. Let x, y, z be positive real numbers such that: $x + y + z = 3$. Prove that:

$$\frac{x^3y^3}{x^4 + y^3 - x + 2} + \frac{y^3z^3}{y^4 + z^3 - y + 2} + \frac{z^3x^3}{z^4 + x^3 - z + 2} \leq \\ \leq \frac{x^4 + y^4 + z^4 + 3xyz}{6}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEMS FOR SENIORS

SP.121. Let x, y, z be a positive real numbers such that: $x + y + z = 3$. Prove that:

$$\frac{x^4}{5 - 3\sqrt[3]{y}} + \frac{y^4}{5 - 3\sqrt[3]{z}} + \frac{z^4}{5 - 3\sqrt[3]{x}} + \frac{\sqrt{x} + \sqrt{y} + \sqrt{z}}{2} \geq 3$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

SP.122. If $z_1, z_2, z_3 \in \mathbb{C}$ are different in pairs and $|z_1| = |z_2| = |z_3| = 1$ then:

$$|z_1 - z_3| + |z_2 - z_3| \leq 3 + |z_1 + z_2|$$

Proposed by Marian Ursărescu - Romania

SP.123. If $A \in M_n(\mathbb{R}), \det A \neq 0, A^T = A$ then:

$$\det(A^2 + A^{-2} + 2I_n) \geq 4^n$$

Proposed by Marian Ursărescu - Romania

SP.124. Let a, b, c be the side lengths of a triangle ABC with inradius r and circumradius R. Prove that:

$$\frac{3}{2} \leq \frac{a^2}{b^2 + c^2} + \frac{b^2}{c^2 + a^2} + \frac{c^2}{a^2 + b^2} \leq \frac{2R - r}{2r}$$

Proposed by George Apostolopoulos - Messolonghi - Greece

©Daniel Sitaru, ISSN-L 2501-0099

SP.125. **Let triangle ABC have exradii r_a, r_b, r_c, altitudes h_a, h_b, h_c and a, b, c be the lenghts of the sides. Prove that:**

$$\Big(\frac{h_a}{r_a}\Big)^2+\Big(\frac{h_b}{r_b}\Big)^2+\Big(\frac{h_c}{r_c}\Big)^2 \le \frac{1}{2}\Big(\frac{a^4+b^4}{c^4}+\frac{b^4+c^4}{a^4}+\frac{c^4+a^4}{b^4}\Big)$$

Proposed by George Apostolopoulos - Messolonghi - Greece

SP.126. **Let m_a, m_b, m_c the lengths of the medians of a triangle ABC with circumradius R. Prove that:**

$$\frac{1}{m_am_b}+\frac{1}{m_bm_c}+\frac{1}{m_cm_a}+\frac{3}{m_am_b+m_bm_c+m_cm_a} \ge \ge 4\cdot\frac{\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}}{m_a+m_b+m_c}$$

Proposed by George Apostolopoulos - Messolonghi - Greece

SP.127. **If $A, B \in M_n(\mathbb{R}), n \in \mathbb{N}, n \ge 2, x \in \mathbb{Z}^*, \det(AB-BA) \ne 0$, $A^2+B^2 = 2\sin\frac{\pi}{x}AB$ then nx is an even number.**

Proposed by Marian Ursărescu - Romania

SP.128. **Let x, y, z be positive real numbers such that: $x+y+z=3$. Find the minimum of the expression:**

$$P = \frac{x^3}{y(\sqrt{2(y^4+z^4)}+yz)^2}+\frac{y^3}{z(\sqrt{2(z^4+x^4)}+zx)^2}+ +\frac{z^3}{x(\sqrt{2(x^4+y^4)}+xy)^2}+\frac{\sqrt[4]{x}+\sqrt[4]{y}+\sqrt[4]{z}}{27}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

SP.129. **Let a, b, c be positive real numbers such that: $a+b+c=3$. Prove that:**

$$\frac{a^3}{b^2\Big(\sqrt[4]{\frac{b^8+c^8}{2}}+5bc\Big)}+\frac{b^3}{c^2\Big(\sqrt[4]{\frac{c^8+a^8}{2}}+5ca\Big)}+\frac{c^3}{a^2\Big(\sqrt[4]{\frac{a^8+b^8}{2}}+5ab\Big)}+ +\frac{ab+bc+ca}{2(a^2+b^2+c^2)} \ge 1$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

 ©Daniel Sitaru, ISSN-L 2501-0099

SP.130. Let a, b, c be positive real numbers such that: $a+b+c=3$. Prove that:

$$\frac{a^3}{b^2(b^2+bc+c^2)}+\frac{b^3}{c^2(c^2+ca+a^2)}+\frac{c^3}{a^2(a^2+ab+b^2)}+$$
$$+\frac{ab+bc+ca}{a^2+b^2+c^2}\geq 2$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

SP.131. Let $a, b, c>0$ with sum 1. Prove that:

$$\sum ab^2\geq\Big(\sum ab\Big)^2-10\sum ab+12abc$$

Proposed by Mihalcea Andrei Ştefan - Romania

SP.132. Let a, b be two positive numbers. Prove that:

$$\frac{(1+ab)(ab-a-b-1)}{(a^2+1)(b^2+1)}+$$
$$+\frac{4(a^2+b^2+a+b)}{(2+a+b)(a^2+b^2+a+b)-2(a+b)^2}\leq 1$$

Proposed by Mihalcea Andrei Ştefan - Romania

SP.133. Let $a, b, c>0$ with sum 1. Show that:

$$2\sqrt{abc}\sum\frac{a}{1+a^2}\leq\frac{1+\sum a^2}{3+\sum a^2}$$

Proposed by Mihalcea Andrei Ştefan - Romania

SP.134. Let $ABCD$ be a cyclic quadrilateral with perimeter 2. Denote $AB=a, BC=b, CD=c, DA=d$. Show that:

$$4\leq\sum\tan\frac{A}{2}<\frac{2(a+c)(b+d)}{\sqrt{\prod(1-a)}}.$$

Proposed by Mihalcea Andrei Ştefan - Romania

SP.135. If $a, b, c>1$ then:

$$\frac{\log_a b}{a+b+c}+\frac{\log_b c}{b+c+d}+\frac{\log_c d}{c+d+a}+\frac{\log_d a}{d+a+b}\geq\frac{16}{3(a+b+c+d)}$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

UNDERGRADUATE PROBLEMS

UP.121. Show that:

$$\int_0^{\infty} x^{p-1}\Big(\frac{2}{1+\sqrt{1+4x}}\Big)^n dx = \Big(\frac{n}{n-p}\Big)B(n-2p,p)$$

Proposed by Shivam Sharma - New Delhi - India

UP.122.

$$a_n = \sum_{k=1}^{n} \tan^{-1}\Big(\frac{1}{k^2+k+1}\Big)$$

Find:

$$\Omega = \lim_{n\to\infty} n\left(a_n^{\frac{\pi}{4}} - \Big(\frac{\pi}{4}\Big)^n\right)$$

Proposed by Marian Ursărescu - Romania

UP.123. Find:

$$\Omega = \lim_{n\to\infty} \prod_{k=1}^{n}\left(\frac{\sqrt{n^4+k^2}+k}{\sqrt{n^4+k^2}-k}\right)$$

Proposed by Marian Ursărescu - Romania

UP.124. Find all continuous functions such that:

$$\int_{e^x}^{e^{2x}} f(t)dt = \int_1^{e^x} f(t)dt$$

Proposed by Marian Ursărescu - Romania

UP.125. Prove that:

$$\frac{4i\pi}{\sqrt{5}}\int_{-1}^{1}\frac{1+x+x^2}{1+x+x^2+x^3+x^4}dx = \frac{1+\sqrt{5}}{2}\int_{-\infty}^{\infty}\frac{(1+x+x^2)\ln(x)}{1+x+x^2+x^3+x^4}dx$$

Proposed by K. Srinivasa Raghava - AIRMC - India

UP.126. Prove that:

$$\int_0^1\int_0^1 \frac{\sqrt{x+y}+\sqrt{xy}}{(x\sqrt{y}+y\sqrt{x})}dxdy = \ln(17-12\sqrt{2}) + \sqrt{2}\ln(17+12\sqrt{2}) -$$

$$-\frac{4}{3}(1-3\sqrt{2}+2\ln(2))$$

Proposed by K. Srinivasa Raghava - AIRMC - India

©Daniel Sitaru, ISSN-L 2501-0099

UP.127. Prove that:

$$\sum_{k=1}^{\infty}\int_0^1 \frac{1}{(\cos(\pi x)^2+ik\sin(\pi x)^2)^3}dx =$$

$$= \frac{e^{\frac{3\pi i}{4}}}{8}\left(3\zeta\left(\frac{5}{2}\right)+2i\zeta\left(\frac{3}{2}\right)-3\zeta\left(\frac{1}{2}\right)\right)$$

Proposed by K. Srinivasa Raghava - AIRMC - India

UP.128. Prove that:

$$\int_0^{\pi}\operatorname{arccot}(\cot(x)^5)\cot(x)dx = \frac{\pi}{5}(4\ln(2)-3\ln(5)+\ln(25+11\sqrt{5}))$$

Proposed by K. Srinivasa Raghava - AIRMC - India

UP.129. Let $\mathbb{R}_+^*$ be the set of real positive numbers, let $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ be two sequences of real positive numbers with $\lim_{n\to\infty}(a_{n+1}-a_n)=a\in\mathbb{R}_+^*, \lim_{n\to\infty}(b_{n+1}-b_n)=b\in\mathbb{R}_+^*$, let $P_n=\sqrt{\frac{a_1^2+a_2^2+\ldots+a_n^2}{n}}$ and we denote $P_n!=P_1P_2\ldots P_n$, for any positive integer n. If $u,v\in\mathbb{R}$ with $u+v=1$. Evaluate:

$$\lim_{n\to\infty}\left(b_{n+1}^u\sqrt[n+1]{(P_{n+1}!)^v}-b_n^u\sqrt[n]{(P_n!)^v}\right).$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

UP.130. If $a,b,c,d\in[1,\infty); x\in\mathbb{R}$ then:

$$(ac)^{\sin x}\cdot(bd)^{\cos x}\leq 2^{\sqrt{2(\log_2^2 a+\log_2^2 b+\log_2^2 c+\log_2^2 d)}}$$

Proposed by Daniel Sitaru - Romania

UP.131. Prove that in any triangle ABC the following relationship holds:

$$2^{\sin A}+2^{\sin B}+2^{\sin C}+2^{\cos A}+2^{\cos B}+2^{\cos C}>\frac{6}{(\sqrt{2})^{\sqrt{2}}}$$

Proposed by Daniel Sitaru - Romania

UP.132. If $a,b>0; m,n\geq 1$ then:

$$\int_0^b\left(\int_0^a(\sin x\sin y)^{2n}(\cos x\cos y)^{2m}dx\right)dy\leq\frac{ab}{4^{m+n}}\left(\frac{m}{n}\right)^{m-n}$$

Proposed by Daniel Sitaru - Romania

©Daniel Sitaru, ISSN-L 2501-0099

UP.133. Prove that if: $0 \leq b \leq a \leq \frac{\pi}{4}$ then:

$$\int_b^a \left(\int_b^a \Big(\frac{\sin^2(x+y) + \sin^2(x-y) - 1}{1 + 2\sin x \cos y} \Big) dx \right) dy \geq$$

$$\geq (a-b)(\sin^2 a - \sin^2 b + b - a)$$

Proposed by Daniel Sitaru - Romania

UP.134. In ΔABC the following relationship holds:

$$\frac{a^2(m_a + m_b)}{h_c} + \frac{b^2(m_b + m_c)}{h_a} + \frac{c^2(m_c + m_a)}{h_b} \geq 8\sqrt{3}S$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

UP.135. If $a, b, c > 0$ then:

$$\frac{a + c^2}{xb + yc} + \frac{b + a^2}{zc + ya} + \frac{c + b^2}{xa + yb} \geq \frac{3}{x + y} + \frac{a + b + c}{x + y}$$

Proposed by D.M. Bătineţu-Giurgiu, Neculai Stanciu - Romania

Mathematics Department, "Theodor Costescu" National Economic, College Drobeta Turnu - Severin, MEHEDINTI, ROMANIA

 ©Daniel Sitaru, ISSN-L 2501-0099