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$$\int_0^1\int_0^1 \ln\Gamma\,(x+y+1)\,dx\,dy$$

Proposed by Shafiqur Rahman-Bangladesh

Solution by Togrul Ehmedov-Baku-Azerbaidian

$$\int_0^1\int_0^1 \ln\Gamma\,(x+y+1)\,dx\,dy = \int_0^1\int_{x+1}^{x+2} \ln\Gamma\,(u)\,du\,dx$$

$$= \left[x\int_{x+1}^{x+2}\ln\Gamma(u)\,du\right]_0^1 - \int_0^1 x\ln\frac{\Gamma(x+2)}{\Gamma(x+1)}\,dx = \int_0^2 \ln\Gamma(u)\,du - \int_0^1 x\ln(x+1)\,dx$$

$$I_1 = \int_1^2 \ln\Gamma(u)\,du$$

$$I_1(a) = \int_a^{a+1} \ln\Gamma(u)\,du$$

$$I_1'(a) = \ln\Gamma(a+1) - \ln\Gamma(a) = \ln a\,, \qquad I_1(a) = a\ln a - a + C$$

$$I_1(0) = \int_0^1 \ln\Gamma\,(u)\,du = \ln\left(\sqrt{2\pi}\right) = C\,, \qquad I_1(a) = a\ln a - a + \ln\left(\sqrt{2\pi}\right)$$

$$I_1 = -1 + \ln\left(\sqrt{2\pi}\right)\,, \qquad I_2 = \int_0^1 x\ln(x+1)\,dx = \frac{1}{4}$$

$$I = \frac{3}{4} + \ln\left(\sqrt{2\pi}\right)$$