R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

In $\Delta\ ABC$ the following relationship holds:

$$\frac{m_a}{w_a^2}+\frac{m_b}{w_b^2}+\frac{m_c}{w_c^2}\ge\frac{1}{2R}\left(2+\frac{R}{r}\right)$$

Proposed by Bogdan Fustei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\because m_a\ge h_a\ \&\ w_a=\frac{2bc}{b+c}\cos\frac{A}{2}$$

$$\therefore\frac{m_a}{w_a^2}\overset{(1)}{\ge}\frac{\frac{bc}{2R}}{\frac{4b^2c^2}{(b+c)^2}\cdot\frac{s(s-a)}{bc}}=\frac{1}{8Rs}\cdot\frac{(b+c)^2}{s-a}$$

Similarly, $\frac{m_b}{w_b^2}\overset{(2)}{\ge}\frac{1}{8Rs}\cdot\frac{(c+a)^2}{s-b}\ \&\ \frac{m_c}{w_c^2}\overset{(3)}{\ge}\frac{1}{8Rs}\cdot\frac{(a+b)^2}{s-c}$

(1)+(2)+(3) $\Rightarrow LHS\ge\frac{1}{8Rs}\sum\frac{(b+c)^2}{s-a}=\frac{1}{8Rs}\sum\frac{(s+s-a)^2}{s-a}=\frac{1}{8Rs}\sum\frac{s^2+(s-a)^2+2s(s-a)}{s-a}=$

$$=\frac{s}{8R}\sum\frac{1}{s-a}+\frac{1}{8Rs}\sum(s-a)+\frac{3}{4R}=\frac{s^2}{8R}\cdot\frac{\sum(s-b)(s-c)}{r^2s^2}+\frac{7}{8R}$$

$$=\frac{3s^2-4s^2+s^2+4Rr+r^2}{8Rr^2}+\frac{7r}{8Rr}=\frac{4R+r+7r}{8Rr}=$$

$$=\frac{4R+8r}{8Rr}=\frac{R+2r}{2Rr}=\frac{1}{2R}\left(2+\frac{R}{r}\right)$$

(proved)