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ROMANIAN MATHEMATICAL MAGAZINE
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Let AK_1, BK_2, CK_3 be diameters in circumcircle of $\Delta\ ABC$ with $I -$ incentre.

Prove that:

$$IK_1^2 + IK_2^2 + IK_3^2 \geq 12r^2$$

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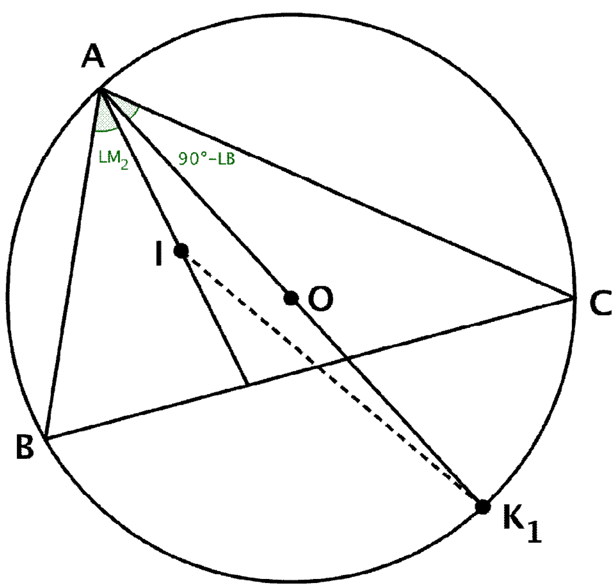
Solution by Rajsekhar Azaad-India



$$\angle K_1AI = \angle CAI - \angle CAK_1 = \angle\frac{A}{2} - (90 - LB) = \angle\frac{A}{2} - \angle\frac{A}{2} - \angle\frac{B}{2} - \angle\frac{C}{2} + LB = \frac{\angle B - \angle C}{2}$$

$\therefore$ **In** $\Delta AIK_1; IK_1^2 = AI^2 + 4R^2 - 4RAI\cos\frac{(B-C)}{2}$

$$= r^2\csc^2\frac{A}{2} + 4R^2 - 4R\cdot r\csc\frac{A}{2}\cdot\csc\frac{(B-C)}{2} = r^2\csc^2\frac{A}{2} + 4R^2 - 4Rr\frac{(\sin B + \sin C)}{\sin A}$$

$$= r^2\csc^2\frac{A}{2} + 4R^2 - 4Rr\frac{(b+c)}{a}$$

$$\therefore LHS = r^2\sum\csc^2\frac{A}{2} + 12R^2 - 4Rr\cdot\sum\frac{b+c}{a}$$

$$= r^2\sum\csc^2\frac{A}{2} + 12R^2 - 4Rr\cdot\frac{\sum bc(b+c)}{abc}$$

$$= r^2\cdot\sum\csc^2\frac{A}{2} + 12R^2 - 4Rr\left[\frac{(a+b+c)(ab+bc+ca) - 3abc}{4Rrs}\right]$$

$$= r^2\sum\csc^2\frac{A}{2} + 12R^2 - 2(s^2 + r^2 + 4Rr) + 12Rr$$

$$= r^2\sum\csc^2\frac{A}{2} + 12R^2 - 2s^2 - 2r^2 + 4Rr \quad (i)$$

Now, $f(x) = \csc^2 x$ *for* $x \in \left[0,\frac{\pi}{2}\right]$, $f''(x) \geq 0$. *Now, using Jensen's inequality*

$$f\left(\frac{a+b+c}{3}\right) \leq \frac{f(a)+f(b)+f(c)}{3} \Rightarrow \sum\csc^2\frac{A}{2} \geq 3\csc^2\left(\frac{A+B+C}{6}\right) = 3\times 2^2 = 12 \quad (ii)$$

Using (i) & (ii): $LHS \geq 12r^2 + 12R^2 - 2s^2 - 2r^2 + 4Rr$

$$10r^2 + 12R^2 + 4Rr - 2\left[4R^2 + 4Rr + 3r^2\right] \quad \textit{(Gerretsen)}$$

$$= 4r^2 + 4R^2 + 4Rr = 4r^2 + 4R(R-r) \geq 4r^2 + 4\cdot 2r(2r-r)\{\because R \geq 2r\} = 12r^2$$