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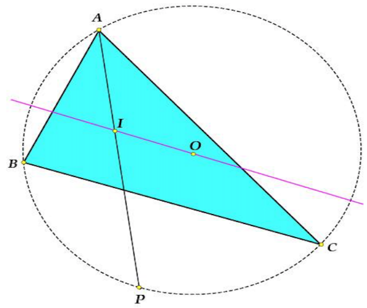


I – incenter of ABC, O – circumcenter, P – point of intersection of AI with circumcircle, R – circumradius, r – inradius, $IO \parallel BC$. Prove:

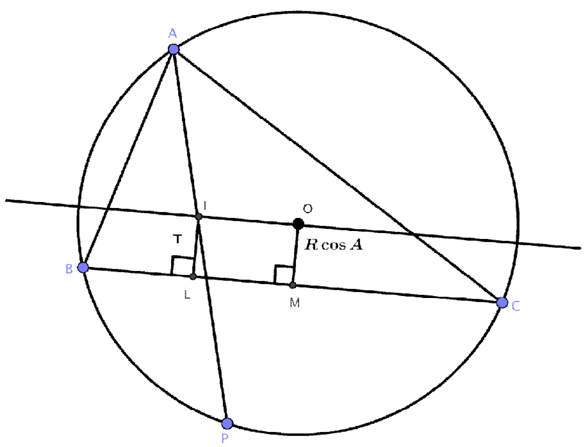
1. $\frac{AI}{AP} = \cos A,\ AI \cdot IP = 2Rr = \sqrt{R \cdot AI \cdot BI \cdot CI},\ \sin^2\frac{B}{2} + \sin^2\frac{C}{2} = \frac{1}{2}$

2. $\cos^2\frac{B}{2} + \cos^2\frac{C}{2} = \frac{3}{2},\ \tan^2\frac{A}{2} = \frac{R-r}{R+r}$

Reference: L. Bankoff

Designed by Abdilkadir Altintas-Afyonkarashisar-Turkey

Solution by Rajsekhar Azaad-India



Distance of BC ***from*** I ***and*** O, $IL = \gamma$ ***and*** $OM = R\cos A$. ***But:*** $OI \parallel BC \Rightarrow \gamma = R\cos A \Rightarrow GA = \frac{\gamma}{R}$ ***(i). Now,*** $\frac{AI}{O} = \frac{\gamma}{\sin\frac{A}{2}}$ ***and*** $IP = 2R\sin\frac{A}{2}$

$$\frac{AI}{IP} = \frac{\gamma}{\sin\frac{A}{2}} \times \frac{1}{2R\sin\frac{A}{2}} = \frac{\cos A}{2\sin^2\frac{A}{2}}\ \textit{(using (1))}$$

$$\frac{AI}{IP} = \frac{\cos A}{1-\cos P} \Rightarrow \frac{IP}{IA} = \frac{L\cos A}{\cos A} \Rightarrow \frac{AP}{IA} = \frac{1}{\cos A} \Rightarrow \frac{AI}{AP} = \cos A\ \textit{(proved)}$$

$$\textit{(2)}\ AI \cdot IP = \frac{\gamma}{\sin\frac{A}{2}} \times 2R\sin\frac{A}{2} = 2R\gamma\ \ \textit{(ii)}$$

Again $R\cdot AI\cdot BI\cdot CI=\frac{R\cdot\gamma^3}{\pi\sin\frac{A}{2}}=4R^2\gamma^2\quad\left\{\because\pi\sin\frac{A}{2}=\frac{\gamma}{4R}\right\}$

$$\Rightarrow\sqrt{R\cdot AI\cdot AI\cdot CI}=2R\gamma\quad\textbf{\textit{(iii)}}$$

From (ii) and (iii): $\left[AI\cdot IP=2R\gamma=\sqrt{R\cdot AI\cdot BI\cdot CI}\right]$ ***(proved)***

(3) we know, that $\sum\cos A=1+\frac{\gamma}{R}\Rightarrow\cos B+\cos C=1\quad\left\{\cos A-\frac{I}{R}\ from\ (i)\right\}$

$$\therefore\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=\frac{1-\cos B}{2}+\frac{1-\cos C}{2}=\frac{2-1}{2}=\frac{1}{2}\quad\textbf{\textit{(proved)}}$$

(4) $\cos^2\frac{B}{2}+\cos^2\frac{C}{2}=\frac{1+\cos B}{2}+\frac{1+\cos C}{2}=\frac{2+\cos B+\cos C}{2}=\frac{2+1}{2}=\frac{3}{2}$ ***(proved)***

(5) $\cos A=\frac{\gamma}{R}$ ***(from (i))***

$$\frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}=\frac{\gamma}{R}\Rightarrow\tan^2\frac{A}{2}=\frac{R-\gamma}{R+r}\quad\textbf{\textit{(proved)}}$$