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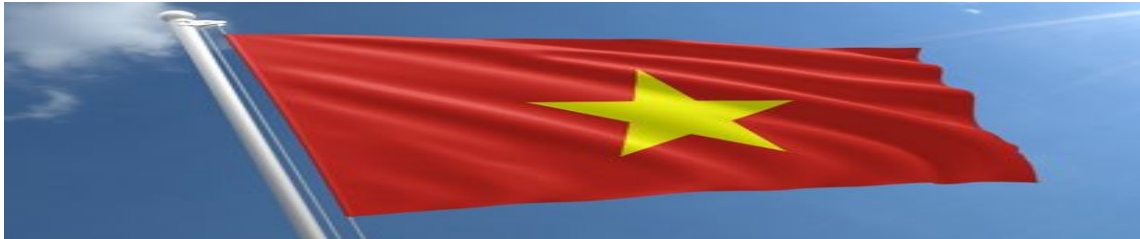


Find all continuous functions $f: \mathbb{R} \rightarrow \mathbb{R}$ such that:

$$f(x) + f(3x) + f(9x) = 91x^2 + 26x + 3$$

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$$f(x) + f(3x) + f(9x) = 91x^2 + 26x + 3 \quad (3)$$

Put $g(x) = f(x) - x^2 - 2x - 1$. We have

$$\begin{aligned} (1) \Rightarrow g(x) + x^2 + 2x + 1 + g(3x) + 9x^2 + 6x + 1 + g(9x) + 81x^2 + 18x + 1 &= \\ = 91x^2 + 26x + 3 \Rightarrow g(x) + g(3x) + g(9x) &= 0 \quad (2) \end{aligned}$$

$$\text{Put } x \rightarrow 3x, \text{ we have } (2) \Rightarrow g(3x) + g(9x) + g(27x) = 0 \quad (3)$$

$$(2) \text{ and } (3) \Rightarrow g(x) = g(27x) \quad (4)$$

$$\text{Put } x \rightarrow \frac{x}{27}, \text{ we have } (4) \Rightarrow g(x) = g\left(\frac{x}{27}\right) \quad (5)$$

$$\text{Put } x \rightarrow \frac{x}{27}, \text{ we have } (5) \Rightarrow g\left(\frac{x}{27}\right) = g\left(\frac{x}{27^2}\right)$$

$$\text{Similarly, we have } g(x) = g\left(\frac{x}{27}\right) = g\left(\frac{x}{27^2}\right) = \dots = g\left(\frac{x}{27^n}\right) \quad \forall n \in \mathbb{N}.$$

$$\text{The sequence } (u_n) \text{ such that } u_0 = x, u_{n+1} = \frac{x}{27^n}.$$

$$\text{We have } \lim_{n \rightarrow +\infty} u_n = 0$$

$$\text{We have } g(u_0) = g(u_1) = \dots = g(u_n) = g(u_{n+1}) = \dots = g(\lim_{n \rightarrow \infty} u_n) = g(0)$$

$$\text{Put } x \rightarrow 0, \text{ we have } (2) \Rightarrow 3g(0) = 0 \Rightarrow g(0) = 0 \Rightarrow g(x) = 0 \quad \forall x \in \mathbb{R}$$

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$$\text{So, } f(x) = x^2 + 2x + 1 \quad \forall x \in \mathbb{R}$$

$$\text{We have (1)} \Rightarrow x^2 + 2x + 1 + 9x^2 + 6x + 1 + 81x^2 + 18x + 1 = 91x^2 + 26x + 3$$

(True)

$$\text{Therefore } f(x) = x^2 + 2x + 1 \quad \forall x \in \mathbb{R}$$