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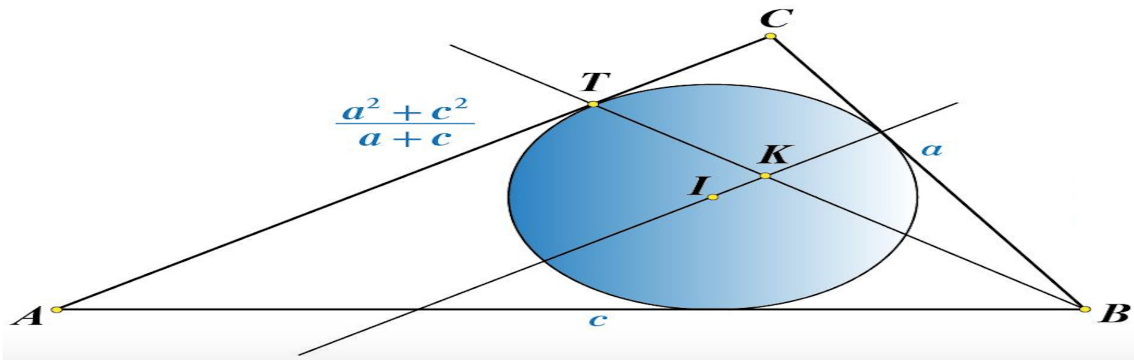

I-Incenter of ABC, K-symmedian point of ABC, T-tangency point with incircle

AC = Contraharmonic Mean $\{a, c\} = \frac{a^2+c^2}{a+c}$. Prove:

1. $IK \parallel AC$ 2. B, K, T are collinear

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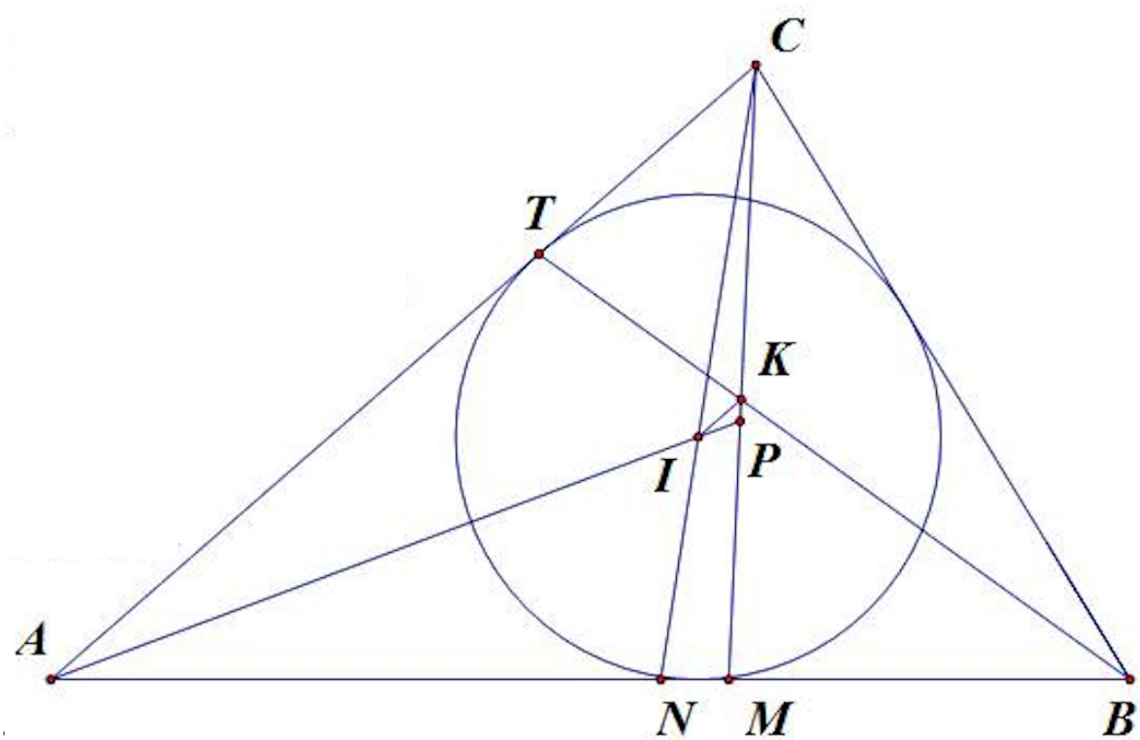
Solution by Khanh Hung Vu-Ho Chi Minh-Vietnam



1) Put $b = \frac{a^2+c^2}{a+c}$ **and** $P = AI \cap CK, M = CP \cap AB, N = CI \cap AB$

We have $\frac{AM}{MB} = \frac{b^2}{a^2}$ **and** $AM + MB = AB = c \Rightarrow AM = \frac{cb^2}{a^2+b^2}$

ΔMAC **has** AP **is a bisector of** $\angle A \Rightarrow \frac{MP}{PC} = \frac{AM}{AC} = \frac{\frac{cb^2}{a^2+b^2}}{b} = \frac{bc}{a^2+b^2}$

On the other hand, we have $MP + PC = MC = \frac{ab\sqrt{2a^2+2b^2-c^2}}{a^2+b^2}$

$$\Rightarrow CP = \frac{ab\sqrt{2a^2+2b^2-c^2}\cdot(a^2+b^2)}{(a^2+b^2)(a^2+b^2+bc)} = \frac{ab\sqrt{2a^2+2b^2-c^2}}{a^2+b^2+bc} \quad (1)$$

We have $\frac{CK}{KM} = \frac{a^2+b^2}{c^2}$ **and** $CK + KM = CM = \frac{ab\sqrt{2a^2+2b^2-c^2}}{a^2+b^2}$

$$\Rightarrow CK = \frac{ab\sqrt{2a^2+2b^2-c^2}\cdot(a^2+b^2)}{(a^2+b^2)(a^2+b^2+c^2)} = \frac{ab\sqrt{2a^2+2b^2-c^2}}{a^2+b^2+c^2} \quad (2)$$

(1) and (2) $\Rightarrow \frac{CK}{CP} = \frac{ab^2+b^2+c^2}{a^2+b^2+bc}$ **(3)**

On the other hand, ΔANC **has** AI **is a bisector of** $\angle A \Rightarrow AI = \frac{2AN\cdot AC\cdot\cos\frac{A}{2}}{AN+AC}$

Similarly, we have $AP = \frac{2AM\cdot AC\cdot\cos\frac{A}{2}}{AM+AC} \Rightarrow \frac{AI}{AP} = \frac{AN(AM+AC)}{AM(AN+AC)} = \frac{\frac{bc}{b+c}\left(\frac{cb^2}{a^2+b^2}+b\right)}{\frac{cb^2}{a^2+b^2}\left(\frac{bc}{b+c}+b\right)} = \frac{a^2+b^2+bc}{b(a+b+c)}$ **(4)**

Since $b = \frac{a^2+c^2}{a+c} \Rightarrow a^2 + b^2 + c^2 = b^2 + b(a+c) = b(a+b+c)$

(3) and (4) $\Rightarrow \frac{CK}{CP} = \frac{AI}{AP} \Rightarrow IK \parallel AC$ **(Thales theorem)**

2) We need to prove that $\frac{AT}{TC} = \frac{c^2}{a^2}$ **(5)**

$$\Rightarrow \frac{p-a}{p-c} = \frac{c^2}{a^2} \Rightarrow \frac{b+c-a}{a+b-c} = \frac{c^2}{a^2} \Rightarrow ba^2 + ca^2 - a^3 = ac^2 + bc^2 - c^3$$

$$\Rightarrow b = \frac{ac^2-a^2c+a^3-c^3}{a^2-c^2} \Rightarrow b = \frac{(a-c)(a^2+c^2)}{(a-c)(a+c)} \Rightarrow b = \frac{a^2+c^2}{a+c} \quad \textbf{(True)}$$

(5) $\Rightarrow T$ **in symmedian line from** $B \Rightarrow T \in BK \Rightarrow B, K, T$ **are collinear**