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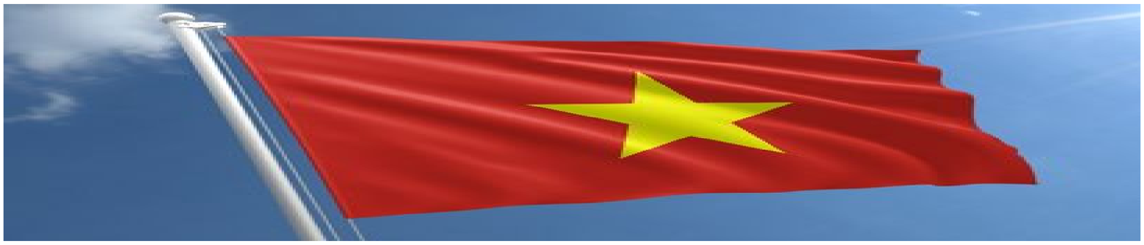
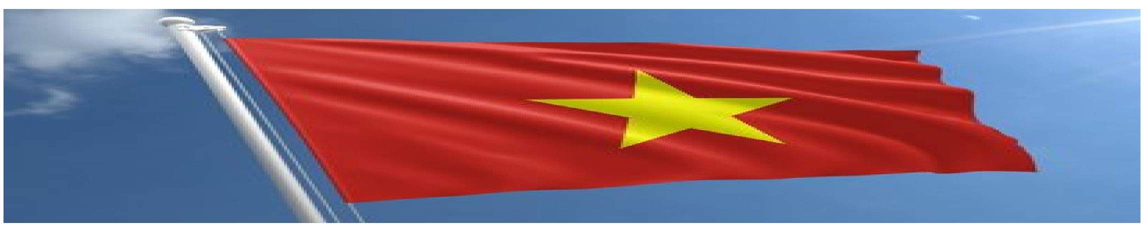


Let x,y,z be positive real numbers such that: $x+y+z=3$. Find the minimum of the expression:

$$P=\frac{x^3}{\sqrt[4]{8(y^8+z^8)}+2yz}+\frac{y^3}{\sqrt[4]{8(z^8+x^8)}+2zx}+\frac{x^3}{\sqrt[4]{8(x^8+y^8)}+2xy}$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

Solution by Do Huu Duc Thinh-Ho Chi Minh-Vietnam



Lemma: $a+b\le\sqrt[4]{8(a^4+b^4)}\ \forall a,b>0$***, so we get:***

$$P\ge\sum\frac{x^3}{\sqrt[4]{8[8(y^8+z^8)+16y^4z^4]}}=\sum\frac{x^3}{2\sqrt{2(y^4+z^4)}}$$

We have: $\sqrt{2(y^4+z^4)}=\sqrt{2[(y^2+z^2)^2-2y^2z^2]}=$

$$=\sqrt{(2+\sqrt{2})(y^2+z^2-yz\sqrt{2})(2-\sqrt{2})(y^2+z^2+yz\sqrt{2})}$$

$$\le\frac{(2+\sqrt{2})(y^2+z^2-yz\sqrt{2})+(2-\sqrt{2})(y^2+z^2+yz\sqrt{2})}{2}=$$

$$=2(y^2-yz+z^2)\Rightarrow P\ge\frac{1}{4}\sum\frac{x^3}{y^2-yz+z^2}$$

We will prove that: $\sum\frac{x^3}{y^2-yz+z^2}\ge x+y+z=3\cdot\sum\frac{x^3}{y^2-yz+z^2}=$

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$$=\sum\left(\frac{x^3}{y^2-yz+z^2}+y+z\right)-2(x+y+z)$$

$$=(x^3+y^3+z^3)\sum\frac{1}{y^2-yz+z^2}-2(x+y+z)$$

By Cauchy-Schwarz: $\sum\frac{1}{y^2-yz+z^2}\geq\frac{9}{2(x^2+y^2+z^2)-(xy+yz+zx)}$***. So just need to prove:***

$$3(x^3+y^3+z^3)\geq(x+y+z)\cdot\left[2\sum x^2-\sum xy\right]$$

$$\Leftrightarrow 3(x^3+y^3+z^3)\geq x^3+y^3+z^3-3xyz+(x+y+z)(x^2+y^2+z^2)$$

$\Leftrightarrow x^3+y^3+z^3+3xyz\geq\sum xy(x+y)$ ***(true because this is Schur deg*** 3***)***

Therefore: $P\geq\frac{1}{4}\sum\frac{x^3}{y^2-yz+z^2}\geq\frac{3}{4}$***. Hence*** $P_{Min}=\frac{3}{4}\Leftrightarrow x=y=z=1$***.***