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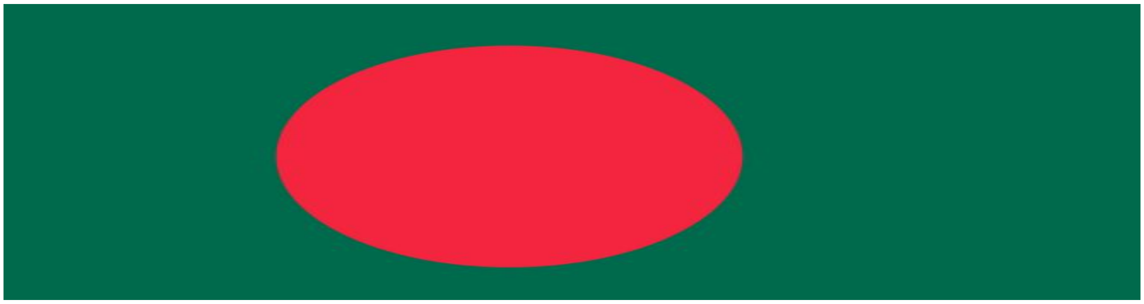

Find:

$$\Omega = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \ln(\sin(x+y))\,dx\,dy$$

Proposed by Shafiqur Rahman-Bangladesh

Solution by Togrul Ehmedov-Baku-Azerbaidian

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \ln(\sin(x+y))\,dx\,dy = \int_0^{\frac{\pi}{2}}\int_x^{\frac{\pi}{2}+x} \ln(\sin(u))\,du\,dx$$

$$= \left[x\int_x^{\frac{\pi}{2}+x} \ln(\sin(u))\,du\right]_x^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} x\left[\ln\left(\sin\left(\frac{\pi}{2}+x\right)\right) - \ln(\sin(x))\right]dx$$

$$= \frac{\pi}{2}\int_{\frac{\pi}{2}}^{\pi} \ln(\sin(u))\,du - \int_0^{\frac{\pi}{2}} x[\ln(\cot(x))]\,dx$$

$$I_1 = \int_{\frac{\pi}{2}}^{\pi} \ln(\sin(u))\,du\Bigg|_{\pi-u=z} = \int_0^{\frac{\pi}{2}} \ln(\sin(z))\,dz = -\frac{\pi}{2}\ln 2$$

$$I_2 = \int_0^{\frac{\pi}{2}} x[\ln(\cot(x))]\,dx = \int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2}-x\right)\int_0^{\frac{\pi}{2}} x[\ln(\tan(x))]\,dx$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \ln(\tan(x))\,dx - \int_0^{\frac{\pi}{2}} x[\ln(\tan(x))]\,dx = \int_0^{\frac{\pi}{2}} x[\ln(\tan(x))]\,dx$$

$$= \int_0^{\frac{\pi}{2}} x\left[\sum_{k=1}^{\infty}\left(\frac{(-1)^k - 1}{k}\right)\cos(2kx)\right]dx$$

$$= \sum_{k=1}^{\infty}\left(\frac{(-1)^k - 1}{k}\right)\int_0^{\frac{\pi}{2}} x\cos(2kx)\,dx$$

$$= \sum_{k=1}^{\infty}\left(\frac{(-1)^k - 1}{k}\right)\left[\left[\frac{x\sin(2kx)}{2k}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{\sin(2kx)}{2k}\,dx\right]$$

$$= \sum_{k=1}^{\infty}\left(\frac{(-1)^k - 1}{k}\right)\left[-\frac{1}{2k}\int_0^{\frac{\pi}{2}} \sin(2kx)\,dx\right]$$

$$= \sum_{k=1}^{\infty}\left(\frac{(-1)^k - 1}{k}\right)\left[\frac{\cos(\pi k) - 1}{4k^2}\right] = \sum_{k=1}^{\infty}\frac{\left((-1)^k - 1\right)^2}{4k^3}$$

$$= \frac{1}{4}\sum_{k=1}^{\infty}\frac{2 - 2(-1)^k}{k^3} = \frac{1}{2}\sum_{k=1}^{\infty}\frac{1 - (-1)^k}{k^3} = \frac{1}{2}\left[\sum_{k=1}^{\infty}\frac{1}{k^3} + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3}\right]$$

$$= \frac{1}{2}\left[\zeta(3) + \left(1 - \frac{1}{4}\right)\zeta(3)\right] = \frac{7}{8}\zeta(3)$$

$$I = \frac{\pi}{2}\left(-\frac{\pi}{2}\ln 2\right) - \frac{7}{8}\zeta(3) = -\frac{\pi^2}{4}\ln 2 - \frac{7}{8}\zeta(3)$$