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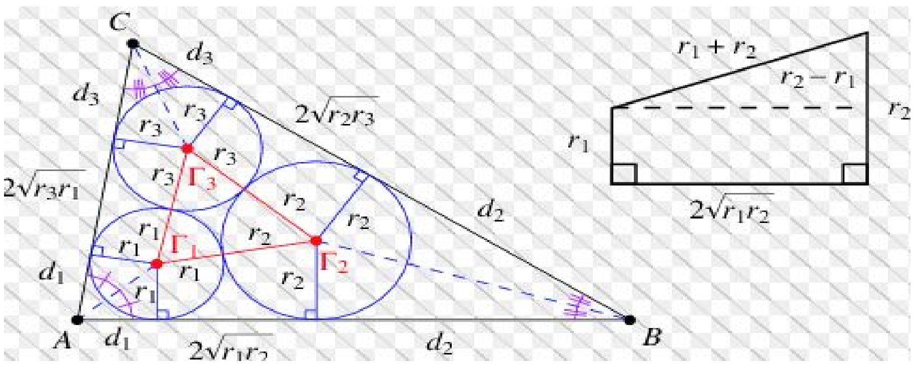
If in $\Delta\ ABC, r_1, r_2, r_3$ are radii of Malfatti's circles then:

$$\frac{a^2}{r_a}+\frac{b^2}{r_b}+\frac{c^2}{r_c}\ge\frac{48\sqrt{r_1r_2r_3}}{3+r_1+r_2+r_3-2\sqrt{r_1+r_2+r_3}}$$



Proposed by Daniel Sitaru – Romania

Solution 1 by Rajsekhar Azaad-India, Solution 2 by Soumava Chakraborty-Kolkata-India

Solution 1 by Rajsekhar Azaad-India

From Malfatti circles,

$$r=\frac{2\sqrt{r_1r_2r_3}}{\sqrt{r_1}+\sqrt{r_2}+\sqrt{r_3}-\sqrt{r_1+r_2+r_3}}\ge\frac{2\sqrt{r_1r_2r_3}}{\frac{1+r_1}{2}+\frac{1+r_2}{2}+\frac{1+r_3}{3}-8\sqrt{r_1+r_2+r_3}}$$

$$=\frac{4\sqrt{r_1r_2r_3}}{3+r_1+r_2+r_3-2\sqrt{r_1+r_2+r_3}}\quad (i)$$

$$\text{Now, }\frac{a^2}{r_a}+\frac{b^2}{r_b}+\frac{c^2}{r_c}\ge 3\sqrt[3]{\frac{a^2b^2c^2}{r_ar_br_c}}$$

$$=3\sqrt[3]{\frac{16R^2\Delta^2r^2}{\Delta^2}}\qquad \begin{cases}\because\ \Delta=\frac{abc}{4R}\\ \Delta=\sqrt{rr_ar_br_c}\end{cases}$$

$$=3\sqrt[3]{16R^2r^2}=3\sqrt[3]{64r^3}\qquad (\because R\ge 2r)=12r$$

$$\ge\frac{12\times 4\sqrt{r_1r_2r_3}}{3+r_1+r_2+r_3-2\sqrt{r_1+r_2+r_3}}=\frac{4s\sqrt{r_1+r_2+r_3}}{3+r_1+r_2+r_3-2\sqrt{r_1+r_2+3}}$$

(proved) from (i)

Solution 2 by Soumava Chakraborty-Kolkata-India

We have: $\frac{\sum_{i=1}^{3}\sqrt{r_i}-\sqrt{\sum_{i=1}^{3}r_i}}{\sqrt{r_1r_2r_3}}\overset{(1)}{=}\frac{2}{r}$

We observe that the given inequality transforms to:

$$\frac{\left(\sum\frac{a^2}{r_a}\right)\left(3+\sum_{i=1}^{3}r_i-2\sqrt{\sum_{i=1}^{3}r_i}\right)}{48\sqrt{r_1r_2r_3}}\overset{(2)}{\geq}1$$

Now, $3+\sum_{i=1}^{3}r_i-2\sqrt{\sum_{i=1}^{3}r_i}=\sum_{i=1}^{3}(1+r_i)-2\sqrt{\sum_i^3 r_i}$

$$\underset{(3}{\overset{A-G}{\geq}}2\left(\sum_{i=1}^{3}\sqrt{r_i}-\sqrt{\sum_{i=1}^{3}r_i}\right)$$

(3) $\Rightarrow$ *LHS of (2)* $\geq\frac{1}{2^4}\left(\sum\frac{a^2}{r_a}\right)\left(\frac{\sum_{i=1}^{3}\sqrt{r_i}-\sqrt{\sum_{i=1}^{3}r_i}}{\sqrt{r_1r_2r_3}}\right)$

$$\overset{by\,(1)}{=}\frac{1}{12r}\left(\sum\frac{a^2}{r_a}\right)\overset{Bergstrom}{\geq}\frac{4s^2}{12(4R+r)r}\overset{?}{\geq}1$$

$$\Leftrightarrow s^2\overset{(4)}{\geq}12Rr+3r^2$$

Now, LHS of (4) $\overset{Gerretsen}{\geq}16Rr-5r^2\overset{?}{\geq}12Rr+3r^2$

$\Leftrightarrow 4Rr\overset{?}{\geq}8r^2\Leftrightarrow R\overset{?}{\geq}2r\rightarrow$ *true by Euler* $\Rightarrow$ *(2) is true (Proved)*