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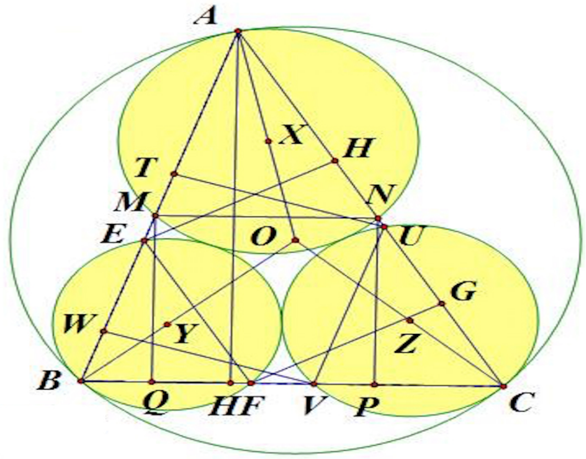
Choose a point inside circumcircle of ΔABC, $AB = c$, $BC = a$, $CA = b$. Find the probability that this point its not in yellow area in terms of a, b, c.



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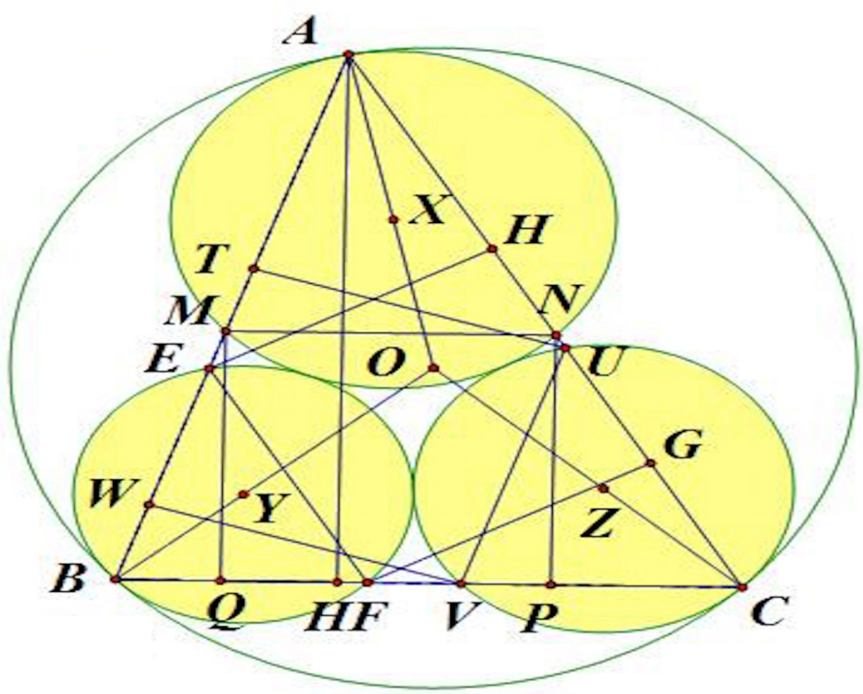

Construct the squares $MNPQ$, $TUVW$ and $EFGH$ as shown.

Put X,Y,Z is the center of $(AMN),(EBF)$ and (UCV)

Put R_1,R_2,R_3,R is the radii of $(AMN),(EBF)$ and $(UCV),(ABC)$

1) Prove that $(X),(Y),(Z)$ are internally tangent circles with (O)

Since $MN\parallel BC$ so $\exists V_A: M\to B, N\to C \Rightarrow V_A:\Delta AMN\to\Delta ABC$

On the other hand, X and O is the center of ΔAMN and $\Delta ABC\Rightarrow \overline{A,X,O}$. Similarly, we have $\overline{B,Y,O}$ and $\overline{C,Z,O}\Rightarrow (X),(Y),(Z)$ are internally tangent circles with (O)

2) Prove that $MN=\frac{2RS}{a^2+2S}, EF=\frac{2RS}{b^2+2S}$ and $UV=\frac{2RS}{c^2+2S}$

Put $AH\perp BC$. Put $MN=x$. By Thales theorem for $MN\parallel BC\Rightarrow\frac{MN}{BC}=\frac{AM}{AB}$

Similarly, we have $\frac{BM}{AB}=\frac{MQ}{AH}\Rightarrow\frac{MN}{BC}+\frac{MQ}{AH}=1\Rightarrow\frac{x}{a}+\frac{x}{h_a}=1\Rightarrow x=MN=\frac{ah_a}{a+h_a}$

Similarly, we have $EF=\frac{bh_b}{b+h_b}$ and $UV=\frac{ch_c}{b+h_c}$

2) Prove that $(X),(Y),(Z)$ are 3 externally tangent circles

Since $V_A:\Delta AMN\to\Delta ABC$ and X and O is the center of $\Delta\ AMN$ and $\Delta\ ABC\Rightarrow$

$$\Rightarrow\frac{AX}{AO}=\frac{MN}{BC}\Rightarrow\frac{R_1}{R}=\frac{x}{a}=\frac{h_a}{a+h_a}\Rightarrow R_1=\frac{Rh_a}{a+h_a}\Rightarrow R_1=\frac{2RS}{a^2+2S}$$

Similarly, we have $R_2=\frac{2RS}{b^2+2S}$ and $R_3=\frac{2RS}{c^2+2S}$.

We need to prove that $XY=R_1+R_2$. By Cos's law for ΔOXY, we have

$$\cos\angle AOB=\frac{OX^2+OY^2-XY^2}{2OX\cdot OY}$$

Similarly, we have $\cos\angle AOB=\frac{OA^2+OB^2-AB^2}{2OA\cdot OA}$. So, $\frac{OX^2+OY^2-XY^2}{2OX\cdot OY}=\frac{OA^2+OB^2-AB^2}{2OA\cdot OB}$

$$\Rightarrow XY^2=OX^2+OY^2-2\cdot OX\cdot OY\cdot\frac{OA^2+OB^2-AB^2}{2OA\cdot OB}$$

$$\Rightarrow(R_1+R_2)^2=(R-R_1)^2+(R-R_2)^2-\frac{2(R-R_1)(R-R_2)}{2R^2}(R^2+R^2-c^2)$$

$$\Rightarrow R_1^2+R_2^2+2R_1R_2=2R^2-2R(R_1+R_2)+R_1^2+R_2^2-\frac{(R-R_1)(R-R_2)(2R^2-c^2)}{R^2}$$

$$\Rightarrow 2R_1R_2=2R^2-2R(R_1+R_2)-\frac{(R-R_1)(R-R_2)(2R^2-c^2)}{R^2}$$

$$\Rightarrow 2\cdot\frac{2RS}{a^2+2S}\cdot\frac{2RS}{b^2+2S}=2R^2-2R\left(\frac{2RS}{a^2+2S}+\frac{2RS}{b^2+2S}\right)-\frac{\left(R-\frac{2RS}{a^2+2S}\right)\left(R-\frac{2RS}{b^2+2S}\right)(2R^2-c^2)}{R^2}$$

$$\Rightarrow 2R^2\cdot\frac{4S^2}{(a^2+2S)(b^2+2S)}=2R^2-2R^2\left(\frac{2S}{a^2+2S}+\frac{2S}{b^2+2S}\right)-\frac{R^2\cdot\frac{a^2}{a^2+2S}\cdot\frac{b^2}{b^2+2S}(2R^2-c^2)}{R^2}$$

$$\Rightarrow\frac{4S^2}{(2S+a^2)(2S+b^2)}=1-\frac{2S(4S+a^2+b^2)}{(2S+a^2)(2S+b^2)}-\frac{a^2b^2(2R^2-c^2)}{2R^2(2S+a^2)(2S+b^2)}$$

We have LHS $=\frac{a^2b^2-4S^2}{(2S+a^2)(2S+b^2)}-\frac{a^2b^2(2R^2-c^2)}{2R^2(2S+a^2)(2S+b^2)}=\frac{a^2b^2c^2-8S^2R^2}{2R^2(2S+a^2)(2S+b^2)}$

$$\Rightarrow\frac{4S^2}{(2S+a^2)(2S+b^2)}=\frac{a^2b^2c^2-8S^2R^2}{2R^2(2S+a^2)(2S+b^2)}\Rightarrow$$

$\Rightarrow 8R^2S^2=a^2b^2c^2-8S^2R^2\Rightarrow a^2b^2c^2=16S^2R^2\Rightarrow abc=4RS\Rightarrow S=\frac{abc}{4R}$ ***(True)***

So, $XY=R_1+R_2$

$\Rightarrow (X),(Y)$ ***are externally tangent circles. Similarly, we have*** $(Z),(Y)$ ***and*** $(X),(Z)$ ***are externally tangent circles. So*** $(X),(Y),(Z)$ ***are 3 externally tangent circles.***

3) Put X***: "This point its not in yellow area in terms of*** a,b,c***". Find*** $P(X)$

We have $\frac{S_{yellow}}{S_{(O)}}=\frac{S_{(X)}+S_{(Y)}+S_{(Z)}}{S_{(O)}}=\frac{\left[\left(\frac{2RS}{2S+a^2}\right)^2+\left(\frac{2RS}{2S+b^2}\right)^2+\left(\frac{2RS}{2S+c^2}\right)^2\right]\cdot\pi}{\pi R^2}$

$$\Rightarrow\frac{S_{yellow}}{S_{(O)}}=\left(\frac{2S}{2S+a^2}\right)^2+\left(\frac{2S}{2S+b^2}\right)^2+\left(\frac{2S}{2S+c^2}\right)^2$$

$$\Rightarrow P(x)=1-\left(\frac{2S}{2S+a^2}\right)^2+\left(\frac{2S}{2S+b^2}\right)^2+\left(\frac{2S}{2S+c^2}\right)^2$$

So $P(x)=1-\left(\frac{2S}{2S+a^2}\right)^2+\left(\frac{2S}{2S+b^2}\right)^2+\left(\frac{2S}{2S+c^2}\right)^2$ ***vs*** $S=\frac{\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}}{4}$