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If $a, b > 0, a \neq b$ then:

$$0 < \frac{\frac{a-b}{\ln a - \ln b} - \sqrt{ab}}{\frac{a+b}{2} - \sqrt{ab}} < \frac{1}{3}$$

Proposed by B.G.Carlson-USA

Solution 1 by Omran Kouba-Damascus-Syria, Solution 2 by Khanh Hung Vu-Ho Chi Minh-Vietnam

Solution 1 by Omran Kouba-Damascus-Syria

First note that

$$\cosh t + 2 - 3\frac{\sinh t}{t} = \sum_{n=2}^{\infty} \frac{2(n-1)}{(2n+1)!} t^{2n} \geq 0$$

With equality if and only if $t = 0$. This is equivalent to

$$\frac{\frac{\sinh t}{t} - 1}{\cosh t - 1} < \frac{1}{3} \text{ for } t \neq 0$$

Now, setting $t = \ln\sqrt{\frac{a}{b}}$, yields the upper inequality. The lower inequality is trivial since it follows in the same way from $\frac{\sinh t}{t} > 1$ for $t \neq 0$.

Solution 2 by Khanh Hung Vu-Ho Chi Minh-Vietnam

Put $A = \frac{\frac{a-b}{\ln a - \ln b} - \sqrt{ab}}{\frac{a+b}{2} - \sqrt{ab}}$. We need to prove that $0 < A < \frac{1}{3}$

1) LEMMA: $\frac{a-b}{\ln a - \ln b} > \sqrt{ab}$ when $a, b > 0$ and $a \neq b$

We have $\frac{a-b}{\ln a - \ln b} > \sqrt{ab} \Rightarrow \frac{\ln a - \ln b}{a-b} < \frac{1}{\sqrt{ab}}$

$$\Rightarrow \frac{\ln\left(\frac{a}{b}\right)}{\frac{a}{b}-1} < \sqrt{\frac{b}{a}} \quad (1)$$

Put $\frac{a}{b} = t\ (t > 0, t \neq 1)$ *, we have (1)* $\Rightarrow \frac{\ln t}{t-1} < \frac{1}{\sqrt{t}}$ *(2)*

Put $f(t) = \ln t - \frac{t-1}{\sqrt{t}}$

$f'(t) = \frac{-(\sqrt{t}-1)^2}{2\sqrt{t^3}} < 0 \Rightarrow f(t)$ *is decreasing function* $\Rightarrow f(t) < f(1)$ *when* $t > 1$ *and* $f(t) > f(1)$ *when* $t < 1 \Rightarrow f(t) < 0$ *when* $t > 1$ *and* $f(t) > 0$ *when* $t < 1$.

1.1.) If $t > 1$. *We have (2)* $\Rightarrow \ln t < \frac{t-1}{\sqrt{t}}$ *(True)*

1.2) If $t < 1$. *We have (2)* $\Rightarrow \ln t > \frac{t-1}{\sqrt{t}}$ *(True)*

$\Rightarrow$ *(1) true* $\Rightarrow \frac{\ln a - \ln b}{a-b} < \frac{1}{\sqrt{ab}}$

Applying the lemma $\Rightarrow \frac{a-b}{\ln a - \ln b} > \sqrt{ab}$ *(since* $0 < \frac{\ln a - \ln b}{a-b} < \frac{1}{\sqrt{ab}}$*)*

On the other hand, by AM-GM inequality, we have $\frac{a+b}{2} - \sqrt{ab} > 0$ *(since* $a \neq b$*)*

2) We need to prove that $A < \frac{1}{3} \Rightarrow$

$$\Rightarrow \frac{3(a-b)}{\ln a - \ln b} - 3\sqrt{ab} < \frac{a+b}{2} - \sqrt{ab} \Rightarrow \frac{3(a-b)}{\ln a - \ln b} < \frac{a+b}{2} + 2\sqrt{ab}$$

$$\Rightarrow \frac{3\left(\frac{a}{b}-1\right)}{\ln\left(\frac{a}{b}\right)} < \frac{\frac{a}{b}+1}{2} + 2\sqrt{\frac{a}{b}} \quad (3)$$

Put $\frac{a}{b} = t\ (t > 0, t \neq 1)$*, we have (3)* $\Rightarrow \frac{3(t-1)}{\ln t} < \frac{t+1}{2} + 2\sqrt{t}$ *(4)*

Put $g(t) = \frac{t+1}{2} + 2\sqrt{t} - \frac{3(t-1)}{\ln t}$

$$g'(t) = \frac{1}{\sqrt{t}} + \frac{1}{2} + \frac{3(t-1) - 3t \cdot \ln t}{t \cdot \ln^2 t} = \frac{2\sqrt{t} \cdot \ln^2 t + t \cdot \ln^2 t + 6(t-1) - 6t \cdot \ln t}{2t \cdot \ln^2 t}$$

Put $h(t) = 2\sqrt{t} \cdot \ln^2 t + t \cdot \ln^2 t + 6(t-1) - 6t \cdot \ln t$

$$h'(t) = \frac{\ln t \cdot \left(-4\sqrt{t} + 4 + \left(\sqrt{t}+1\right) \cdot \ln t\right)}{\sqrt{t}}$$

$h'(t) = 0 \Rightarrow \ln t = 0$ *(5) or* $-4\sqrt{t} + 4 + \left(\sqrt{t}+1\right) \cdot \ln t = 0$ *(6)*

(5): $\ln t = 0 \Rightarrow t = 1$

(6): $-4\sqrt{t}+4+(\sqrt{t}+1)\cdot\ln t=0\Rightarrow \ln t=\frac{4(\sqrt{t}-1)}{\sqrt{t}+1}$

Put $y(t)=\ln t-\frac{4(\sqrt{t}-1)}{\sqrt{t}+1}$

$y'(t)=\frac{(\sqrt{t}-1)^2}{t(\sqrt{t}+1)^2}>0\Rightarrow y(x)$ ***is increasing function*** $\Rightarrow y(x)=0$ ***has at most 1 root***

On the other hand, we have $y(1)=0\Rightarrow t=1$ ***is the root of (6)***

So $h'(t)=0\Rightarrow t=1$

So we have

2.1) $g'(t)<0$ ***when*** $t<1$

So when $t<1\Rightarrow g(t)$ ***is decreasing function*** $\Rightarrow g(t)>\lim_{t\to 1^+} g(t)\Rightarrow g(t)>0$

2.2) $g'(t)>0$ ***when*** $t>1$

So when $t<1\Rightarrow g(t)$ ***is an increasing function*** $\Rightarrow g(t)>\lim_{t\to 1^+} g(t)$

So, $g(t)>0\ \forall t>0$

$\Rightarrow$ ***(4) true*** $\Rightarrow$ ***(3)*** $\Rightarrow A<\frac{1}{3}\Rightarrow$ ***Q.E.D***

t	0		1		$+\infty$
$g(t)$	-6	↗	0	↗	$+\infty$