$$A \in M_4(\mathbb{R}), det(A^2 + 3I_4) = det(A^2 + 2A + 2I_4) = 0$$

Find:

$$\Omega = det\ A$$

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Solution 1 by Serban George Florin-Romania, Solution 2 by Ravi Prakash-New Delhi-India

Solution 1 by Serban George Florin-Romania

$$P(x) = \det(A - xI_4) = x^4 + ax^3 + bx^2 + cx + d$$

$$\det(A + \sqrt{3}iI_4) \cdot \det(A - \sqrt{3}iI_4) = 0 \Rightarrow P(i\sqrt{3}) = 0$$

$$(i\sqrt{3})^4 + a(i\sqrt{3})^3 + b(i\sqrt{3})^2 + ci\sqrt{3} + d = 0$$

$$9 - 3a\sqrt{3}i - 3b + ci\sqrt{3} + d = 0, 9 - 3b + d + i\sqrt{3}(c - 3a) = 0$$

$$\Rightarrow c - 3a = 0, 9 - 3b + d = 0$$

$$x^2 + 2x + 2 = 0, \Delta = 4 - 8 = -4$$

$$x_1 = \frac{-2 + 2i}{2} = -1 + i, x_2 = -1 - i$$

$$\det(A - (-1 + i)I_4) \cdot \det(A - (-1 - i)I_4) = 0$$

$$\Rightarrow P(-1 + i) = (-1 + i)^4 + a(-1 + i)^3 + b(-1 + i)^2 + c(-1 + i)d = 0$$

$$(1 - 2i + i^2)^2 + a(-1 + 3i + 3(-1)i^2 + i^3) + b(1 - 2i + i^2) + c(-1 + i) + d = 0$$

$$-4 + a(-1 + 3i + 3 - i) - 2bi - c + ci + d = 0$$

$$-4 + 2a + 2ai - 2bi - c + ci + d = 0$$

$$(2a - c + d - 4) + i(2a - 2b + c) = 0$$

$$\begin{cases} c = 3a \\ d - 3b = -9 \\ 2a - c + d = 4 \\ 2a - 2b + c = 0 \end{cases} \Rightarrow \begin{cases} d - 3b = -9 \\ 2a - 3a + d = 4 \\ 2a - 2b + 3a = 0 \end{cases} \begin{cases} d - 3b = -9, d = 3b - 9 \\ -a + d = 4 \\ 5a - 2b = 0 \end{cases}$$

$$\begin{cases} -a + 3b - 9 = 4 \\ 5a - 2b = 0 \end{cases} \begin{cases} -a + 3b = 13 | \cdot 5 \\ 5a - 2b = 0 \end{cases} \Rightarrow 13b = 13 \cdot 5, b = 5$$

$$5a = 10, a = 2, d = 3b - 9 = 3 \cdot 5 - 9 = 6, c = 6$$

$$\Omega = \det A = P(0) = d = 6$$

Solution 2 by Ravi Prakash-New Delhi-India

Suppose $A \in M_4(\mathbb{R})$*. Let* $f(t) = \det(A - tI) = t^4 + at^3 + bt^2 + ct + d$ ***(1)***

$(a, b, c, d \in \mathbb{R})$ ***be a characteristic equation of*** A.

$d = f(0) = \det(A) =$ ***product of eigenvalues of*** A.

As $\det(A^2 + 3I) = 0$***, we get*** $\det\left((A - i\sqrt{3}I)(A + i\sqrt{3}I)\right) = 0 \Rightarrow$

$$\Rightarrow \det(A - i\sqrt{3}I)\det\left(\overline{A - i\sqrt{3}I}\right) = 0 \Rightarrow \det(A - i\sqrt{3}I)\,\overline{\det(A - i\sqrt{3}I)} = 0$$

$\Rightarrow |\det(A - i\sqrt{3}I)| = 0 \Rightarrow \det(A - i\sqrt{3}I) = 0 \Rightarrow \sqrt{3}i$ ***is an eigenvalue of*** A.

$\Rightarrow \sqrt{3}i, -\sqrt{3}i$ ***are zeros of*** $f(t)$***. Next,***

$$\det(A^2 + 2A + 2I) = 0 \Rightarrow \det\{(A + I)^2 + I\} = 0 \Rightarrow$$

$$\Rightarrow \det\left[\left((A + I) - iI\right)\left((A + I) + iI\right)\right] = 0 \Rightarrow$$

$$\Rightarrow \det(A + (1 - i)I)\det\overline{(A + (1 - i)I)} = 0 \Rightarrow |\det(A + (1 - i)I)|^2 = 0 \Rightarrow$$

$\Rightarrow \det(A + (1 - i)I) = 0 \Rightarrow -1 + i$ ***is an eigenvalue of*** A

$\therefore -1 + i, -1 - i$ ***are zeros of*** $f(t)$***. Thus,***

$$\det(A) = d = (\sqrt{3}i)(-\sqrt{3}i)(-1 + i)(-1 - i) = (3)(1 + 1) = 6$$