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If $a,b,c\geq 1$ then:

$$\frac{(1+a+a^2)(1+b+b^2+b^3)(1+c+c^2+c^3+c^4)}{(1+a^2)(1+b^3)(1+c^4)}\leq\frac{15}{2}$$

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Solution 1 by Ravi Prakash-New Delhi-India, Solution 2 by Rajsekhar Azaad-India, Solution 3 by Soumava Chakraborty-Kolkata-India, Solution 4 by Soumitra Mandal-Chandar Nagore-India

Solution 1 by Ravi Prakash-New Delhi-India

$$\text{Let } f(x)=\frac{1+x+x^2}{1+x^2},x\geq 1$$

$$f'(x)=\frac{d}{dx}\left[1+\frac{x}{1+x^2}\right]=\frac{(1+x^2)-2x^2}{(1+x^2)^2}=\frac{1-x^2}{(1+x^2)^2}<0,\forall x>1$$

$$\Rightarrow f(x) \text{ decreases on } [1,\infty)\therefore f(x)\leq f(1)\ \forall x\geq 1$$

$$\Rightarrow\frac{1+a+a^2}{1+a^2}\leq\frac{3}{2}\ \forall a\geq 1\quad (1)$$

$$\text{Let } g(x)=\frac{1+x+x^2+x^3}{1+x^3}=1+\frac{x+x^2}{1+x^3}$$

$$g'(x)=\frac{(1+x^3)(1+2x)-3x^2(x+x^2)}{(1+x^3)^2}=\frac{1+2x+x^3+2x^4-3x^3-3x^4}{(1+x^3)^2}$$

$$=\frac{1+2x-2x^3-x^4}{(1+x^3)^2}$$

$$g'(x)=\frac{(1-x^4)-2x(1-x^2)}{(1+x^3)^2}=\frac{(1-x^2)(1+x^2-2x)}{(1+x^3)^2}=\frac{(1-x)^3(1+x)}{(1+x^3)^2}<0\ \forall x>1$$

$$\Rightarrow g(x) \text{ decreases on } [1,\infty)\therefore g(x)\leq g(1)$$

$$\Rightarrow\frac{1+b+b^2+b^3}{1+b^3}\leq\frac{4}{2}=2\ \forall b\geq 1\quad (2)$$

$$\text{Let } h(x)=\frac{1+x+x^2+x^3+x^4}{1+x^4}, x\ge 1$$

$$=1+\frac{x+x^2+x^3}{1+x^4}$$

$$h'(x)=\frac{(1+2x+3x^2)(1+x^4)-(x+x^2+x^3)(4x^3)}{(1+x^4)^2}$$

$$=\frac{1+2x+3x^2+x^4+2x^5+3x^6-4x^4-4x^5-4x^6}{(1+x^4)^2}$$

$$=\frac{1+2x+3x^2-3x^4-2x^5-x^6}{(1+x^4)^2}=\frac{(1-x^6)+2x(1-x^3)+3x^2(1-x^2)}{(1+x^4)^2}<0\ \forall x\ge 1$$

$$\Rightarrow h(x) \text{ decreases on } [1,\infty) \therefore h(x)\le h(1)\ \ \forall x\ge 1$$

$$\Rightarrow \frac{1+c+c^2+c^3+c^4}{1+c^4}\le\frac{5}{2}\ \ \forall c\ge 1\ \ (3)$$

Multiplyig (1), (2), (3) we get

$$\frac{(1+a+a^2)(1+b+b^2+b^3)(1+c+c^2+c^4)}{(1+a^2)(1+b^3)(1+c^4)}\le\frac{15}{2}$$

Solution 2 by Rajsekhar Azaad-India

$$\frac{1+a+a^2}{1+a^2}-\frac{3}{2}=\frac{-(a-1)^2}{2(1+a^2)}\le 0\Rightarrow\frac{1+a+a^2}{1+a^2}\le\frac{3}{2}\quad (i)$$

$$\text{Again, }\frac{1+b+b^2+b^3}{1+b^3}-2=\frac{1+b+b^2+b^3-2-2b^3}{1+b^3}=\frac{-(b^3+1)+b(b+1)}{1+b^3}=\frac{-(b+1)(b-1)^2}{1+b^3}\le 0$$

$$\Rightarrow\frac{1+b+b^2+b^3}{1+b^3}\le 2\quad (ii)$$

$$\text{Again, }\frac{1+c+c^2+c^3+c^4}{1+c^4}-\frac{5}{2}=\frac{-3(c^4+1)+2c(c^2+1)+2c^2}{2(1+c^4)}=-\frac{(c-1)^2(3c^2+4c+3)}{2(1+c^4)}\le 0$$

$$\Rightarrow\frac{1+c+c^2+c^3+c^4}{1+c^4}\le\frac{5}{2}\quad (iii)$$

on multiplying (i), (ii) and (iii)

$$\frac{(1+a+a^2)(1+b+b^2+b^3)(1+c+c^2+c^3+c^4)}{(1+a^2)(1+b^3)(1+c^4)}\le\frac{3}{2}\cdot 2\cdot\frac{5}{2}=\frac{15}{2}\quad \textit{(proved)}$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\frac{1+c+c^2+c^3+c^4}{1+c^4}=1+\frac{c+c^2+c^3}{1+c^4}\le 1+\frac{2c(1+c^2)+2c^2}{(1+c^2)^2}$$

$$\left(\because 1+c^4 \overset{Chebyshev}{\geq} \frac{1}{2}(1+c^2)^2\right)$$

$$= 1 + \frac{2c}{1+c^2} + \frac{2c^2}{(1+c^2)^2} \overset{(1)}{\leq} 1 + \frac{2c}{2c} + \frac{2c^2}{4c^2} \left(\because 1+c^2 \overset{A-G}{\geq} 2c\right)$$

$$= 1 + 1 + \frac{1}{2} = \frac{5}{2}$$

Again, $\frac{1+b+b^2+b^3}{1+b^3} = \frac{(1+b)(1+b^2)}{(b+1)(b^2-b+1)}$

$$= \frac{(b^2+1-b)+b}{(b^2+1-b)} = 1 + \frac{b}{b^2+1-b} \overset{(2)}{\leq} 1 + \frac{b}{b} = 2$$

$$\left(\because b^2+1-b \overset{A-G}{\geq} b\right)$$

Lastly, $\frac{1+a+a^2}{1+a^2} = 1 + \frac{a}{1+a^2} \overset{(3)}{\leq} 1 + \frac{a}{2a} = \frac{3}{2}\left(\because 1+a^2 \overset{A-G}{\geq} 2a\right)$

(1) $\times$ ***(2)*** $\times$ ***(3)*** $\Rightarrow LHS \leq \frac{5}{2}\cdot 2\cdot\frac{3}{2} = \frac{15}{2}$

(proved)

Solution 4 by Soumitra Mandal-Chandar Nagore-India

$$\frac{(1+a+a^2)(1+b+b^2+b^3)(1+c+c^2+c^3+c^4)}{(1+a^2)(1+b^3)(1+c^4)} =$$

$$= \left(1+\frac{a}{1+a^2}\right)\left(1+\frac{b(1+b)}{1+b^3}\right)\left(1+\frac{c(1+c+c^2)}{1+c^4}\right)$$

$$\overset{AM\geq GM}{\leq} \left(1+\frac{1}{2}\right)\left(1+\frac{b}{1-b+b^2}\right)\left(1+\frac{3}{2}\cdot\frac{c(1+c^2)}{1+c^4}\right)\left[\because \frac{3(1+x^2)}{2} \geq x^2+x+1\right]$$

$$\overset{AM\geq GM}{\leq} \frac{3}{2}\cdot 2\cdot\left(1+\frac{3}{2}\cdot\frac{2c}{1+c^2}\right)[\because 2(1+x^4) \geq (1+x^2)^2]$$

$$\overset{AM\geq GM}{\leq} \frac{3}{2}\cdot 2\cdot\frac{5}{2} = \frac{15}{2}$$

(proved)